\[ I = \int_{0}^{\pi} \frac{\sin \theta}{5 + 2 \cos \theta} \, d\theta = -\frac{1}{2} \int_{-\pi}^{\pi} \frac{\sin \theta}{5 + 2 \cos \theta} \, d\theta \]  
(since \( \cos \) is even)

\[ = -\frac{1}{2i} \oint_{|z|=1} \frac{8}{5 + 2 \left(z + \frac{1}{2}\right)} \, \frac{dz}{2i} \]

\[ = \frac{8}{2i} \oint_{|z|=1} \frac{1}{z^2 + 5z + 1} \, dz \]

**Singularities at**  
\[ z = -\frac{5}{2} \pm \frac{1}{2} \sqrt{25 - 4} = -\frac{5}{2} \pm \frac{1}{2} \sqrt{21} \]

Since \( z_+ \cdot z_- = \text{the constant term} = 1 \), \( |z_+|/|z_-| = 1 \) so one root is inside, one outside. \( z_+ \) is inside.

\[ \text{Res} \left[ \frac{1}{z^2 + 5z + 1}, z_+ \right] = \text{Res} \left[ \frac{1}{(z-z_+)(z-z_-)}, z_+ \right] = \frac{1}{z_+ - z_-} \]

\[ = \frac{1}{\sqrt{21}} \]

\[ I = \frac{8}{2i} \cdot \pi i \cdot \frac{1}{\sqrt{21}} = \frac{8 \pi}{\sqrt{21}} \]
\[ I = \int_0^{2\pi} (\cos \Theta)^{2n} \ d\Theta = \int_0^{2\pi} \frac{1}{2} \sum_{k=0}^{2n} \binom{2n}{k} \cos^k \Theta \sin^{2n-k} \Theta \ dz \]
\[ = \frac{1}{2} \sum_{k=0}^{2n} \binom{2n}{k} \int_0^{2\pi} \cos^k \Theta \sin^{2n-k} \Theta \ dz \]
\[ = \frac{1}{2} \sum_{k=0}^{2n} \binom{2n}{k} \frac{\sin^{2n-k+1} \Theta}{2n-k+1} \bigg|_0^{2\pi} \]
\[ = \frac{2n}{2^n} \binom{2n}{n} = \frac{2n}{2^{2n-1} (n!)^2} \]

\[ I = \int_0^{2\pi} e^{i \cos \Theta} \cos (n \Theta - \sin \Theta) \ d\Theta = \int_0^{2\pi} \frac{1}{2} e^{i \cos \Theta} \left( e^{i n \Theta - i \sin \Theta} + e^{-i n \Theta + i \sin \Theta} \right) \ d\Theta \]
\[ = \frac{1}{2} \int_0^{2\pi} \left[ e^{i \cos \Theta - i \sin \Theta} + e^{i n \Theta - i \sin \Theta} \right] \ d\Theta \]
\[ = \frac{1}{2} \int_0^{2\pi} \left[ e^{-i n \Theta} e^{i \sin \Theta} + e^{i \sin \Theta} e^{-i n \Theta} \right] \ d\Theta \]
\[ = \frac{1}{2i} \sum_{k=1}^{2n} \left( e^{i \frac{k}{2}} z^n + e^{-i \frac{k}{2}} z^{-n} \right) \frac{dz}{z} \]

**Singularity at** \( z = 0 \). To evaluate residue, expand

\[ e^{i \frac{k}{2} z^{n-1}} = z^{n-1} \sum_{k=0}^{\infty} \frac{k!}{k!} z^{n-1-k} = \sum_{k=0}^{\infty} \frac{k!}{k!} z^{n-1-k} \]

\[ \text{Res} \left[ e^{i \frac{k}{2} z^{n-1}} ; 0 \right] = \begin{cases} 0 & \text{if } n < 0 \\ \frac{1}{n!} & \text{if } n \geq 0 \end{cases} \]

\[ e^{\frac{z}{2} z^{-h-1}} = z^{-h-1} \sum_{k=0}^{\infty} \frac{1}{k!} z^k = \sum_{k=0}^{\infty} \frac{1}{k!} z^{-h-1+k} \]

\[ \text{Res} \left[ e^{\frac{z}{2} z^{-h-1}} ; 0 \right] = \begin{cases} 0 & \text{if } h \leq 0 \\ \frac{1}{h!} & \text{if } h > 0 \end{cases} \]

\[ I = \begin{cases} \frac{2\pi i}{2i} \cdot \left( \frac{1}{h!} + \frac{1}{h!} \right) = \frac{2\pi}{h!} & \text{if } h > 0 \\ 0 & \text{if } h < 0 \end{cases} \]
$$I = \int_0^\infty \frac{x^2 + 1}{x^4 + 1} \, dx = \frac{1}{2} \pi \int_0^\infty \frac{x^2 + 1}{x^4 + 1} \, dx$$

Singlarities in UHP are \( z = e^{in/4}, e^{i3n/4} \) (simple poles) so

$$I = \frac{2ni}{2} \left\{ \text{Res} \left[ \frac{x^2 + 1}{x^4 + 1}; e^{in/4} \right] + \text{Res} \left[ \frac{x^2 + 1}{x^4 + 1}; e^{i3n/4} \right] \right\}$$

$$= \pi i \left\{ \frac{e^{in/2} + 4}{4} e^{i3n/4} + \frac{e^{i5n/4} + 1}{4} e^{i3n/4} \right\}$$

$$= \frac{i\pi}{4} \left\{ e^{-i\pi/4} + e^{-i3\pi/4} + e^{-i5\pi/4} + e^{-i7\pi/4} \right\}$$

$$= \frac{i\pi}{2} \left\{ \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} - \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right\} = \frac{\pi}{\sqrt{2}}$$

Since \( e^{-z^2} \) is analytic everywhere

$$\int_{\gamma_1 + \gamma_2 + \gamma_3 + \delta_1} e^{-z^2} \, dz = 0$$

$$\int_{\gamma_1} e^{-z^2} \, dz = \int_0^\infty e^{-x\lambda} \, dx \quad \text{as} \quad \lambda \to \infty$$

$$\int_{\gamma_2} e^{-z^2} \, dz = \int_{\epsilon} e^{-(x+\lambda)^2} \, dx = -\int_0^\infty e^{-(x^2 + 2i\lambda x + \lambda^2)} = -e^{\lambda^2} \int_0^\infty e^{-x^2 - 2\lambda x} \, dx$$

$$\int_{\gamma_3} e^{-z^2} \, dz = \int_0^\lambda e^{-(x+y)^2} \, dy = \int_0^\lambda e^{-x^2 - 2iyx + y^2} \, dy$$

$$\left| \int_{\gamma_2} e^{-z^2} \, dz \right| \leq \max_{\gamma \in \gamma_2} e^{-x^2 + y^2} \lambda \leq e^{\lambda^2 - y^2} \lambda \to 0 \quad \text{as} \quad \lambda \to \infty$$
\[
\int e^{-z^2} \, dz = -i \int_0^\lambda e^{y^2} \, dy
\]

Therefore, taking the limit \( p \to \infty \) we get:

\[
\frac{\sqrt{n}}{2} + 0 - e^{\lambda^2} \int_0^\infty e^{-x^2 - 2i\lambda x} \, dx - i \int_0^\lambda e^{y^2} \, dy
\]

Take the real part:

\[
\frac{\sqrt{n}}{2} = e^{\lambda^2} \int_0^\infty e^{-x^2 \cos(2\lambda x)} \, dx \quad \text{which } \Rightarrow
\]

\[
\int_0^\infty e^{-x^2 \cos(2\lambda x)} \, dx = \frac{e^{\lambda^2 \sqrt{n}}}{2} \quad \text{since } \cos \text{ is even}.
\]

Take the imaginary part:

\[
- e^{\lambda^2} \int_0^\infty e^{-x^2 \sin(-2\lambda x)} = - e^{\lambda^2} \int_0^\lambda e^{y^2} \, dy \quad \text{which } \Rightarrow
\]

\[
\int_0^\infty e^{-x^2 \sin(2\lambda x)} \, dx = e^{-\lambda^2} \int_0^\lambda e^{y^2} \, dy.
\]

\[
[11] \quad I = \int_0^\infty \frac{dx}{x^3 + 1}. \quad \text{Following the hint, we use the contour}
\]

\[
\text{Singularity inside contour is } e^{im/3} \text{ so}
\]

\[
\int + \int - \int_y = 2\pi i \text{ Res} \left[ \frac{1}{z^3+1} ; e^{im/3} \right] = 2\pi i \frac{1}{3} e^{im/3}
\]

\[
\int_{\gamma_1} \frac{dz}{z^3+1} \rightarrow I \quad \text{as } \gamma \to \infty
\]

\[
\left| \int_{\gamma_2} \frac{dz}{z^3+1} \right| \leq \frac{1}{\sqrt{s^3+1}} \cdot 2\pi s \rho \rightarrow 0 \quad \text{as } \gamma \to \infty
\]

\[
\left( \int_{\gamma_3} \frac{dz}{z^3+1} \right) = \int_0^\rho \frac{e^{i2\pi/3} \, dx}{(e^{i2\pi/3} x)^3 + 1} = e^{i2\pi/3} \int_0^\rho \frac{1}{x^3 + 1} \, dx
\]

\[
\to e^{i2\pi/3} I
\]
So \((1 - e^{i2\pi/3}) I = \frac{2\pi i e^{-i2\pi/3}}{3}\)

\[ I = \left(\frac{2\pi i}{3}\right) \frac{e^{-i2\pi/3}}{1 - e^{i2\pi/3}} = \left(\frac{2\pi i}{3}\right) \frac{e^{-i\pi/3}}{e^{-i\pi/3}} \frac{e^{-i2\pi/3}}{1 - e^{i2\pi/3}} \]

\[ = \frac{2\pi i}{3} \frac{e^{-i\pi/3}}{e^{-i\pi/3} - e^{i\pi/3}} = \frac{\pi}{3} \left(-1\right) = \frac{\pi}{3} \frac{2}{\sqrt{3}} = \frac{2\pi \sqrt{3}}{9}. \]
\[ 15(\text{b}) \quad \sum_{k=-\infty}^{\infty} \frac{1}{(k-\frac{1}{2})^2} = -\pi \left\{ \text{Res} \left[ \frac{\cos(nz)}{(k-\frac{1}{2})^2 \sin(nz)} ; \frac{1}{2} \right] \right\} \]

\[ = -\pi \lim_{z \to \frac{1}{2}} \frac{d}{dz} \left[ \frac{\cos(nz)}{\sin(nz)} \right] = -\pi \lim_{z \to \frac{1}{2}} \frac{-n^3 \sin(nz)^2 - \pi \cos(nz)^2}{\sin(nz)^2} = \pi^2 \]

\[ 15(\text{c}) \quad \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{2} \sum_{k=-\infty}^{\infty} \frac{1}{k^2} = -\frac{\pi^2}{2} \text{Res} \left[ \frac{\cos(nz)}{z^2 \sin(nz)} ; 0 \right] \]

\[ \frac{\cos(nz)}{z^2 \sin(nz)} = \frac{1}{n z^3} \left( 1 - \frac{1}{2!} \frac{n^2 z^2}{1} + \frac{1}{4!} \frac{n^4 z^4}{4} - \cdots \right) \]

\[ \frac{1}{n z^3} \left( 1 - \frac{1}{2!} \frac{n^2 z^2}{1} + \frac{1}{4!} \frac{n^4 z^4}{4} - \cdots \right) \left( 1 + \left( \frac{1}{3!} \frac{n^2 z^2}{2} - \frac{1}{5!} \frac{n^4 z^4}{4} - \cdots \right) \right. \]

\[ \left. + \left( \frac{1}{3!} \frac{n^4 z^4}{2} - \cdots \right)^2 \right) + \cdots \]

Find the coef of the \( \frac{1}{2} \) term \[ \Rightarrow \frac{1}{\pi} \left( \frac{1}{3!} n^2 - \frac{1}{2!} n^4 \right) = \frac{\pi^2}{3} \]

\[ \text{Res} \left[ \frac{\cos(nz)}{z^2 \sin(nz)} ; 0 \right] = \frac{-\pi}{3} \quad \text{so} \quad \sum_{k=1}^{\infty} \frac{1}{k^2} = -\frac{\pi^2}{2} \left( -\frac{\pi}{3} \right) = \frac{\pi^2}{6} . \]