**Proposition** Let \( f(z) \) be analytic inside and on a closed counterclockwise curve \( C \), then,

\[
\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} \, dz = \mathcal{N}_0(f)
\]

\( \mathcal{N}_0(f) = \# \) zeros of \( f(z) \) inside \( C \)

(Counting multiplicity: i.e., a zero of order \( m \) is counted \( m \) times.)

**Proof** Define \( g(z) = \frac{f'(z)}{f(z)} \). Then \( g(z) \) is analytic inside \( C \) except at \( \# \) poles that occur when \( f(z) = 0 \).

Therefore,

\[
\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} \, dz = \sum_{j=1}^{K} \text{Res} \left[ \frac{f'(z)}{f(z)} ; z_j \right]
\]

\( K = \# \) zeroes of \( f(z) \)

Now \( \text{Res} \left[ \frac{f'(z)}{f(z)} ; z_j \right] \) we must calculate.

Near \( z = z_j \) we have \( f(z) = (z-z_j)^{m_j} h(z) \), where \( h(z) \) is analytic and \( h(z_j) \neq 0 \). Here \( m_j \) is the order of the zero at \( z_j \).

Now

\[
\frac{f'(z)}{f(z)} = \frac{m_j (z-z_j)^{m_j-1} h(z) + (z-z_j)^{m_j} h'(z)}{(z-z_j)^m h(z)} = \frac{m_j + h'(z)}{(z-z_j) h(z)}
\]

Now let \( z \to z_j \), we have

\[
\frac{f'(z)}{f(z)} \to \frac{m_j}{(z-z_j) h(z)}
\]

We calculate

\[
\text{Res} \left[ \frac{f'(z)}{f(z)} ; z_j \right] = m_j.
\]

Hence, we calculate

\[
\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} \, dz = \sum_{j=1}^{K} m_j = \mathcal{N}_0(f).
\]

For simple zeroes, \( m_j = 1 \) \( \forall j \), and \( \mathcal{N}_0(f) \leq K \).

Now we let \( z = z(t) \) parametrize \( C \), then,

\[
I = \frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} \, dz = \frac{1}{2\pi i} \int_a^b \frac{f'(z(t))}{f(z(t))} z'(t) \, dt = \frac{1}{2\pi i} \int_a^b \frac{d}{dt} \left[ \log(f(z(t))) \right] \, dt
\]
Integrating once, we get

\[ I = \frac{1}{2\pi i} \left[ \log(F(z)) - \log(F(z(a))) \right] \]

Hence

\[ I = \frac{1}{2\pi i} \left[ \Delta(\log|F(z)|) + i \Delta(\arg(F(z))) \right] \]

\[ \Delta = \text{change over the circuit.} \]

However, \( F(z(a)) = F(z(b)) \) since \( z(a) = z(b) \).

Thus

\[ I = \frac{1}{2\pi} \Delta(\arg(F(z))). \]

Therefore, we have the final result,

\[ \frac{1}{2\pi} \Delta(\arg(F(z))) = N_0(F). \]

**Example**

<table>
<thead>
<tr>
<th>z-plane</th>
<th>w-plane</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Diagram" /></td>
<td><img src="image" alt="Diagram" /></td>
</tr>
</tbody>
</table>

\[ w = f(z) \]

\[ \Delta(\arg(w)) = 6\pi \]

Hence \( N_0(F) = 3 \) inside \( C \).

**Application (Nyquist)**

Let \( F(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n \) \( a_i \)'s are real.

Assume \( F(z) \) has no zeros on imaginary axis. Then, the claim is that there are no zeros in right half-plane when

\[ \Delta \Pi \left[ \arg(F(iy)) \right] = -n\pi \]

where \( n = \text{degree}(F) \).

\( \Pi \) = imaginary axis directed downwards.
**Proof** Consider the contour as shown below.

In the z-plane,

\[
\lim_{R \to \infty} \left[ \frac{1}{2\pi i} \sum_{C_R} \arg(f(z)) + \frac{1}{2\pi} \Delta_n \arg(f(z)) \right] = N_0(f).
\]

Now \( f(z) = a_0 z^n \left( 1 + \frac{a_1}{a_0 z} + \frac{a_2}{a_0 z^2} + \cdots \right) \)

\[ \arg(f(z)) \approx n\pi \text{ as } R \to \infty. \]

Therefore, \( \lim_{R \to \infty} \Delta_{C_R} \arg(f(z)) = n\pi. \)

This gives

\[
N_0(f) = \frac{1}{2\pi} \left[ n\pi + \Delta_n \arg(f(iy)) \right] \leftarrow \text{Argument Principle}.
\]

Therefore, if \( \Delta_n \arg(f(iy)) = -n\pi, \) then \( N_0(f) = 0. \)

**Example 1** Show that there are no zeroes of \( f(z) = z^3 + 2z^2 + z + 1 \)

lie in the left \( \frac{1}{2} \) plane.

We have \( \lim_{R \to \infty} \Delta_{C_R} \arg(f(z)) = 3\pi. \)

Hence,

\[
N_0(f) = \frac{1}{2\pi} \left( 3\pi + \Delta_n \arg(f(iy)) \right).
\]

We calculate \( f(iy) = (iy)^3 + 2(iy)^2 + iy + 1 \)

\( \text{Re } f = 0 \rightarrow y_{R_2} = \pm \sqrt[3]{2} \)

We write \( f(iy) = 1 - 2y^2 + i(y - y^3) \)

\( \text{Im } f = 0 \rightarrow y = 0, \ y_{x_2} = \pm 1. \)

<table>
<thead>
<tr>
<th>Y</th>
<th>Re(f)</th>
<th>Im(f)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>-∞</td>
<td>&lt;0</td>
</tr>
<tr>
<td>B</td>
<td>y_{x_+}</td>
<td>&lt;0</td>
</tr>
<tr>
<td>C</td>
<td>y_{R_+}</td>
<td>=0</td>
</tr>
<tr>
<td>D</td>
<td>0</td>
<td>&gt;0</td>
</tr>
<tr>
<td>E</td>
<td>y_{R_-}</td>
<td>=0</td>
</tr>
<tr>
<td>F</td>
<td>y_{x_-}</td>
<td>&lt;0</td>
</tr>
<tr>
<td>G</td>
<td>-∞</td>
<td>&lt;0</td>
</tr>
</tbody>
</table>

Now at A: \( \text{im}[f]/\text{re}[f] \to \infty \)

Now at G: \( \text{im}[f]/\text{re}[f] \to -\infty \)

We get \( \Delta_n \arg(f(iy)) = -3\pi. \)

Therefore, we get \( N_0(f) = 0 \)

→ no zeroes in right half-plane.
EXAMPLE 2

**Find the number of zeroes of**

\[ f(z) = z^3 - 2z^2 + 4 \] **in right half-plane.**

**Now**

\[ \lim_{R \to \infty} \frac{1}{2\pi i} \oint_{C_R} \frac{f'(z)}{f(z)} \, dz = \# \text{zeroes}. \]

**Now we calculate**

\[ \lim_{R \to \infty} \Delta_C \left( \arg(f(z)) \right) = 3\pi. \]

**Now**

\[ f(iy) = 2y^3 + 4 - iy^3 \]

\[ \text{RE} \to 0 \quad \text{as} \quad y \to \pm \infty. \]

\[ \text{IM} = 0 \quad \text{when} \quad y = 0 \]

\[ \text{RE} = 2y^3 + 4 \quad \text{IM} = -y^3. \]

\[
\begin{array}{c|c|c|c}
  y & \text{RE}(f) & \text{IM}(f) \\
  \hline
  A & > 0 & < 0 \\
  B & > 0 & = 0 \\
  C & > 0 & > 0 \\
  \hline
\end{array}
\]

**Therefore,**

\[ \# \text{zeroes} = \frac{1}{2\pi} (3\pi + \pi) = 2. \quad \Rightarrow \quad \# \text{zeroes} = 2, \text{in the right half-plane}. \]

**Application**

**Consider the differential equation**

\[ d(y) = f(t) \] **with initial conditions.**

**We take Laplace transform to get**

\[ Y(s) = \frac{P(s)}{Q(s)} \]

**where, suppose, Q(s) is a polynomial in s, and that P(s) is analytic in s.**

**Suppose, that we can use the Argument Principle to show that there are no zeroes of Q(s) in the right half-plane. Then since**

\[ Y(t) = \sum_{j=1}^{K} \text{Res} \left[ \frac{P(s)}{Q(s)}; s_j \right] e^{s_j t}. \]

**We have Y bounded as t \to \infty.**

\[
\begin{array}{c|c|c|c|c|c}
  \text{zeros of } Q & \text{S-plane} \\
  \hline
  x & \times & x & \times & \times & \times \\
  \hline
\end{array}
\]
Example: Find the number of zeroes of \( p(z) \) in right-half-plane

\[
p(z) = z^4 + 2z^3 + 3z^2 + z + 2.
\]

Recall

\[
\text{No}(p) = \frac{1}{2\pi} \left[ 4\pi + \Delta_p \arg(p(iy)) \right].
\]

Now

\[
p(iy) = y^4 - 2iy^3 - 3y^2 + iy + 2 = (y^4 - 3y^2 + 2) + i(y - y^3)
\]

so

\[
p(iy) = (y^2 - 1)(y^2 - 2) + i(y(1 - 2y^2))
\]

\[
\text{RE}[p(iy)] = 0 \quad \text{when} \quad y_{R_1} = \pm \sqrt{2}, \quad y_{R_2} = \pm 1
\]

\[
\text{IM}[p(iy)] = 0 \quad \text{when} \quad y = 0, \quad y_{I_2} = \pm 1/\sqrt{2}
\]

As \( y \to +\infty \), \( \text{RE} / \text{IM} \to -\infty \) or \( \text{IM} / \text{RE} \to 0^- \)

As \( y \to -\infty \), \( \text{IM} / \text{RE} \to 0^+ \)

<table>
<thead>
<tr>
<th>( y )</th>
<th>( \text{RE}(p) )</th>
<th>( \text{IM}(p) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \infty )</td>
<td>&gt; 0</td>
<td>&lt; 0</td>
</tr>
<tr>
<td>( y_{R_1} )</td>
<td>= 0</td>
<td>&lt; 0</td>
</tr>
<tr>
<td>( y_{R_2} )</td>
<td>&lt; 0</td>
<td>&lt; 0</td>
</tr>
<tr>
<td>( y_{I_1} )</td>
<td>&gt; 0</td>
<td>= 0</td>
</tr>
<tr>
<td>( y_{I_2} )</td>
<td>&gt; 0</td>
<td>= 0</td>
</tr>
<tr>
<td>( y_{I_3} )</td>
<td>&gt; 0</td>
<td>&lt; 0</td>
</tr>
<tr>
<td>( y_{R_2} )</td>
<td>= 0</td>
<td>&lt; 0</td>
</tr>
<tr>
<td>( y_{I_1} )</td>
<td>&lt; 0</td>
<td>&gt; 0</td>
</tr>
<tr>
<td>( -\infty )</td>
<td>&gt; 0</td>
<td>&gt; 0</td>
</tr>
</tbody>
</table>

Notice it is does not circle the origin.

Hence \( \Delta_p \arg(p(iy)) = 0 \)

so

\[
\text{No} = \frac{1}{2\pi} (4\pi + 0) = 2
\]

\[\rightarrow\] There are two zeroes in RHP.