CONFORMAL MAPPING AND PDE'S

Let \( w = f(z) \) be a conformal map, then if \( \phi = \phi(x, y) \) the transformation produces

\[ \phi_{xx} + \phi_{yy} = \left| f'(z) \right|^2 (\phi_{uu} + \phi_{vv}) \]

where \( w = u + iv \).

On the right side of (x) we have \( \phi = \phi(u, v) \).

Thus if \( \phi_{xx} + \phi_{yy} = 0 \) in \( D \) \( \rightarrow \) \( \phi_{uu} + \phi_{vv} = 0 \) in image of \( D \) under \( f(z) \).

Note: As for boundary condition we have that if \( \phi \) constant on \( \partial D \) then \( \phi \) constant on image of \( \partial D \) under \( f(z) \).

Also if \( \nabla \phi \cdot n = 0 \) on \( \partial D \) then in the \( w \)-plane we also get \( \nabla \phi \cdot n = 0 \) on \( \partial D' \).

I will list below many problems involving conformal map and solution to Laplace's equation. The idea with mapping is to map the problem to one where we can spot the form of the solution essentially by inspection.

The following problems are ones where we obtain "easy" solutions.

**Easy #1** Find \( T(x, y) \) such that \( T_{xx} + T_{yy} = 0 \) in \( y > 0 \)

with boundary data given below

\[ T_{xx} + T_{yy} = 0 \]

\[ T = T_1, \quad \phi_2, \quad T = 0 \]

\[ T = T_0, \quad \phi_1, \quad x = 1 \]

We get 3 equations for \( A, B \) and \( C \).

\[ Q_1 = 0, \quad Q_1 = 0 \quad \Rightarrow \quad T = A = T_0 \]

\[ Q_1 = \pi, \quad Q_2 = 0 \quad \Rightarrow \quad T = A + B \pi = 0 \quad B = -T_0/\pi \]

\[ Q_1 = \phi_1 = \pi \quad \Rightarrow \quad T = A + \pi (B + C) = T_1 \quad C = T_1/\pi \]

Let's try

\[ T = A + B \phi + C \phi_2 \]

Clearly this is harmonic

\[ T = T_0 - T_0 \phi_1 + T_1 \phi_2 \]

with

\[ Q_1 = \tan^{-1} \left( \frac{y}{x-1} \right) \]

\[ Q_1 = \tan^{-1} \left( \frac{y}{x+1} \right) \]
EASY #2 (CONCENTRIC CIRCLES)

\[ T = T_0 \quad \text{in} \quad \Gamma_0 < r < \Gamma_0 \]
\[ T = T_1 \quad \text{on} \quad r = \Gamma_1 \]
\[ T = T_0 \quad \text{on} \quad r = \Gamma_0 \]

\[ T = A + B \log r \]
\[ A = T_1 - B \log \Gamma_1 \]
\[ B = \frac{T_1 - T_0}{\log(\Gamma_1) - \log(\Gamma_0)} \]

\[ T = T_1 + \frac{(T_1 - T_0)}{\log(\Gamma_1/\Gamma_0)} \log(\Gamma/r) \]

EASY #3

\[ T = T_0 \quad \text{in} \quad \Gamma_0 < r < \Gamma_0 \]
\[ T = T_1 + (T_1 - T_0) \frac{r}{\Gamma_1} \]
\[ A + B \log r = T_1 \quad \Rightarrow \quad B = \frac{T_1 - T_0}{\log(\Gamma_1) - \log(\Gamma_0)} \]

MAPPING PROB 1

SOLVE \( T_{xx} + T_{yy} = 0 \) IN CIRCLE \( x^2 + y^2 = 1 \)
WITH \( T = T_1 \) ON UPPER PART OF CIRCLE
\( T = T_2 \) ON LOWER PART OF CIRCLE

W = \( e^{i\phi} \left( \frac{z-1}{1+z} \right) \)

TAKE \( z = i \) INTO \( W = 1 \) THEN

\[ i = e^{i\phi} \left( \frac{i-1}{1+i} \right) = e^{i\phi} \left[ i \right] \]

\( e^{i\phi} = -i \)

\( \phi = 3\pi/2 \).

NOTICE AS WE TRAVERSE CURVE IN Z-PLANE THE INSIDE OF CIRCLE IS ON OUR LEFT.
THIS IS PRESERVED BY THE MAP AND HENCE WE GET UPPER 1/4 PLANE.
NOW \[ Q = \tan^{-1}(\frac{V}{U}) = \arg(W) \].

THEN \[ T = \frac{(T_2 - T_1)}{\pi} Q + T_1 \]

\[ T = \frac{(T_2 - T_1)}{\pi} \tan^{-1}(\frac{V}{U}) + T_1 \]

NOW

\[ U + iV = i\left(\frac{1-Z}{1+Z}\right) = i\left[\frac{(1-x) - iy}{(1+x) + iy}\right] = \frac{i[(1-x) - iy][1+x] - iy}{(1+x)^2 + y^2} \]

\[ U = \frac{2y}{(1+x)^2 + y^2} \quad V = \frac{(1-x^2 - y^2)}{(1+x)^2 + y^2} \]

\[ \tan^{-1}(\frac{V}{U}) = \tan^{-1}\left(\frac{\frac{1-x^2 - y^2}{2y}}{\frac{1}{2}}\right) \]

Thus \[ T = \frac{(T_2 - T_1)}{\pi} \tan^{-1}\left(\frac{\frac{1-x^2 - y^2}{2y}}{\frac{1}{2}}\right) + T_1 \]

**Mapping Problem 2**

NOW RECALL THE JOUKOWSKI MAP

\[ W = \frac{1}{2}(Z + \frac{R^2}{Z}) \]

Let \[ Z = r e^{i\theta} \]. Then

\[ W = \frac{1}{2} \left( r e^{i\theta} + \frac{R^2}{r} e^{-i\theta} \right) \]

\[ W = \frac{1}{2} \left( r + \frac{R^2}{r} \right) \cos \theta + i \left( r - \frac{R^2}{r} \right) \sin \theta \]

Let \[ r = R \rightarrow U = R \cos \theta, \quad V = 0 \] As \( \theta \) ranges from \(-\pi\) to \(\pi\) we get the line \( U = R \) for \( V = 0 \).

Let \( \theta = 0 \) or \( \theta = \pi \rightarrow W = \frac{1}{2} \left( r + \frac{R^2}{r} \right) \) for \( r > R \) we get \( W < -R \) or \( U > R \) with \( V = 0 \).
Notice for \( r \geq R \) fixed we have
\[
\frac{u^1}{4} (r - R_1 \gamma)^2 + \frac{v^1}{4} (r - R_2 \gamma)^2 = 1
\]

This will generate the entire upper \( 1/4 \) plane. Hence,
\[
T_{xx} + T_{yy} = 0 \quad w = \frac{1}{2} (z + R^2 \gamma)
\]

Now
\[
T = C_0 + C_1 \varphi_1 + C_2 \varphi_2
\]

\( \varphi_1 : \varphi_1 = 0 \rightarrow T = 0 \rightarrow C_0 = 0 \)

\( \varphi_2 : \varphi_2 = 0 \rightarrow T = T_0 \rightarrow C_1 = 0 \)

\( \varphi_3 = \pi \rightarrow T = T_0 \rightarrow C_2 = -T_0 \)

Therefore
\[
T = \frac{T_0}{\pi} \left( \varphi_1 - \varphi_2 \right) \quad \varphi_1 = \tan^{-1} \left( \frac{v}{u - R} \right) \quad \varphi_2 = \tan^{-1} \left( \frac{v}{u + R} \right)
\]

Recall \( \tan^{-1} A - \tan^{-1} B = \tan^{-1} \left( \frac{A - B}{1 + AB} \right) \)
\[A = \frac{v}{u - R} \quad B = \frac{v}{u + R} \]

This gives
\[
T = \frac{T_0}{\pi} \tan^{-1} \left( \frac{2vR}{u^2 + v^2 - R^2} \right)
\]

Now
\[
u = \frac{1}{2} (r - R_1 \gamma \cos \varphi) \quad v = \frac{1}{2} (r - R_2 \gamma \sin \varphi)
\]

\[
u^2 + v^2 = \frac{1}{4} r^2 + \frac{1}{4} \frac{R_1^4}{r^2} + \frac{R_1^4}{2} \left( \cos^2 \varphi - \sin^2 \varphi \right) = \frac{1}{4} r^2 + \frac{1}{4} \frac{R_1^4}{r^2} + \frac{R_1^4}{2} \sin 2 \varphi
\]

\[-R^2 + u^2 + v^2 = \frac{1}{4} (r - R_1 \gamma)^2 + \frac{R_1^4}{2} \left( \cos 2 \varphi - 1 \right)
\]

\[
u^2 + v^2 - R_1^2 = \frac{1}{4} (r - R_1 \gamma)^2 + R_1^2 \sin^2 \varphi
\]

\[
T = \frac{T_0}{\pi} \tan^{-1} \left[ \frac{R (r - R_1 \gamma \sin \varphi)}{\frac{1}{4} (r - R_1 \gamma)^2 + R^2 \sin^2 \varphi} \right]
\]
For $r > R$ fixed we have

$$\frac{u^3}{4 (r + R^2/r)} + \frac{v^3}{4 (r - R^2/r)} = 1$$

equations. This gives upper 1/2 plane.

$$T_x x + T_y y = 0$$

$$W = \frac{1}{2} (z + R^2/z)$$

$T = 0$  $\eta \Phi_1$  $T = T_0$  $\eta \Phi_2$  $T = 0$

Let $T = c_0 + c_1 \Phi_1 + c_2 \Phi_2$

Linear system for $c_i$'s give

$$T = \frac{T_0}{\pi} (\Phi_1 - \Phi_2)$$

$$T = \frac{T_0}{\pi} \left( \tan^{-1} \left( \frac{V}{U + R} \right) - \tan^{-1} \left( \frac{V}{U - R} \right) \right)$$

with $U = \frac{1}{2} (r + R^2/r) \cos \phi$  $V = \frac{1}{2} (r - R^2/r) \sin \phi$

Using $\tan^{-1} A - \tan^{-1} B = \tan^{-1} \left( \frac{A - B}{1 + AB} \right)$ we can write

$$T = \frac{T_0}{\pi} \tan^{-1} \left( \frac{2 VR}{U^2 + V^2 - R^2} \right)$$

**Mapping Problem 3**

Solve $T_{xx} + T_{yy} = 0$

in semi-circle as shown

Now map $-1$ to $0$

and $0$ to $\infty$

Then image of the diameter of semi-circle $|x| \leq 1$ gives $0 \leq u \leq R$, $v = 0$

(i.e. positive real axis). Since mapping is conformal at $z=-1$ image of semi-circle

will give a line perpendicular to real axis in $w$-plane and thus line will go

through the origin.

To show that semi-circle yields the entire imaginary axis let $z = e^{i \phi}$  $\phi \in (0, \pi)$.

Then $W = \frac{1 + e^{i \phi}}{1 - e^{-i \phi}} = i \cot (\phi/2)$ this ranges from $0$ to $\infty$ as

$\phi$ ranges from $\pi$ to $0$. 
Now take a point inside at $z = i\sqrt{2}$.

Then $w = \frac{2 + i}{2 - i} = \frac{3 - i}{4}$ (first quadrant).

Therefore

Let $T = C_0 + C_1\phi$

Then

$T = T_1 + 2\left(\frac{T_2 - T_1}{\pi}\right)\phi$

$T = 2\left(\frac{T_2 - T_1}{\pi}\right)\tan^{-1}\left(\frac{\sqrt{u}}{u}\right) + T_1$

$u + i\nu = \frac{(1 + X + iY)}{(1 - X - iY)} = \frac{(1 - X^2 - Y^2) + 2iY}{(1 - X)^2 + Y^2}$

$\frac{\nu}{u} = \frac{2y}{1 - X^2 - Y^2}$

Thus

$T = 2\left(\frac{T_2 - T_1}{\pi}\right)\tan^{-1}\left(\frac{2y}{1 - X^2 - Y^2}\right) + T_1$

Mapping Problem 4

Recall $w = \sin z$ map

Therefore take

$W = \sin\left(\frac{\pi z}{4}\right)$

Mapping not conformal at $z = \pm\pi/2$. 
\[ T = T_0 - T_1 \frac{q_2}{q_1} + T_0 \frac{q_3}{q_1} \]
\[ Q_2 = \tan^{-1} \left( \frac{V}{U-1} \right) \]
\[ Q_1 = \tan^{-1} \left( \frac{V}{U+1} \right) \]

Now \[ W = \sin \left( \frac{\pi z}{q} \right) \] gives:

\[ U + iV = \sin \left( \frac{\pi x}{q} \right) \cosh \left( \frac{\pi y}{q} \right) + i \cos \left( \frac{\pi x}{q} \right) \sinh \left( \frac{\pi y}{q} \right) \]

\[ U = \sin \left( \frac{\pi x}{q} \right) \cosh \left( \frac{\pi y}{q} \right) \]
\[ V = \cos \left( \frac{\pi x}{q} \right) \sinh \left( \frac{\pi y}{q} \right) \]
\[ T = T_1 - T_0 \frac{\tan^{-1} \left( \frac{V}{U-1} \right)}{\pi} + \frac{T_0}{\pi} \tan^{-1} \left( \frac{V}{U+1} \right) \]

**Mapping Problem 5**

\[ T_{xx} + T_{yy} = 0 \text{ between circles} \]

\[ T=1 \text{ on big circle } x^2 + y^2 = 4 \]
\[ T=0 \text{ on small circle } (x-1)^2 + y^2 = 1 \]

Want to choose \( z = 2 \) to be pole of map. Then images of each circle is a line. Let \( z = 0 \) get mapped to origin. Let diameter of circle \((x-0xu)\) get mapped to a real line segment.

Thus \[ W = \frac{yz}{z-2} \rightarrow \text{inner circle is a line through the origin.} \]

Which line? Let \( z = 1 + i \). We get

\[ W = \frac{y(z^2)}{z^2} = -i \] \( y \) choose \( y = -i \) to get real axis.
If \( Y = \frac{x}{2} \), the image of inner circle is the real axis. The diameter \( 1 \times 1 \leq 2 \) with \( y = 0 \) gets mapped to a segment of imaginary axis. Thus, the image of big circle must be a line \( \perp \) to imaginary axis since map \( u \) is conformal at point \( D \). To get this line we let \( z = -2 \).

\[ W = \frac{Z}{2} \quad \text{Choose} \quad Y = i \rightarrow W = i/z. \quad \text{The line runs parallel to x-axis} \text{ and has } y = \frac{1}{z}. \]

The solution in \( v \)-plane is

\[ T = 2v. \]

Now \( u + iv = \frac{i(x + iy)}{(x-2) + iy} = i \frac{(x+iy)}{(x-2)^2 + y^2} \)

\[ v = \frac{x(x-2) + y^2}{(x-2)^2 + y^2} \]

Hence \( T = 2 \left[ \frac{(x-1)^2 + y^2 - 1}{(x-2)^2 + y^2} \right] \)

Can check that \( T = 0 \) on \((x-1)^2 + y^2 = 1\)

And \( T = 1 \) on \((x^2 + y^2) = 4\).
Solve Laplace's equation between nonconcentric circles.

It was proved in class that the map, with \(|d| < 1\),
\[
W = \frac{Z - k}{\alpha Z - 1}
\]
takes unit circle \(|Z| < 1\) into itself (i.e. \(|W| < 1\)).

We then try \(\alpha\) real to get concentric circles.
Take \(-1 < \alpha < 1\), then real \(\alpha\) is mapped to real
axis, and map is conformal at \(A\) and \(B\). Thus
center of image of small circle lies on real
axis in \(W\) plane.

To get the center at the origin we need that
\[
W(X_0 + p) = -W(X_0 - p)
\]
I.e.
\[
\frac{X_0 + p - d}{\alpha(X_0 + p) - 1} = \frac{d - (X_0 - p)}{\alpha(X_0 - p) - 1}
\]
get quadratic equation
\[
\alpha^2 X_0 - \alpha \left(1 + X_0^2 - p^2\right) + X_0 = 0
\]
\[
\alpha' = -\frac{1}{2X_0} \left(p^2 - X_0^2 - 1\right) = \frac{1}{2} \left[ \frac{1}{X_0^3} \left(p^2 - X_0^2 - 1\right)^2 - 4 \right]^{1/2}
\]
Roots are real if
\[
(p^2 - X_0^2 - 1)^2 > 4X_0^2, \quad |p^2 - X_0^2 - 1| > 2X_0
\]
This is satisfied when \(p + X_0 < 1\) (not touching)

Since the product of the two roots \(d_1\) and \(d_2\) are 1
I.e. \(d_1d_2 = 1\)
\[
\left(\alpha - d_1\right)\left(\alpha - d_2\right) = 0
\]
\[
\rightarrow \alpha^2 - (d_1 + d_2)\alpha + d_1d_2 = 0
\]
\[
d_1 + d_2 = \frac{(1 + X_0^2 - p^2)}{X_0}
\]
\[
d_1d_2 = 1
\]

Hence choose \(\alpha\) to be the smaller root.

\[
|W| < 1
\]
\[
\gamma = \left| \frac{X_0 + p - \alpha}{\alpha(X_0 + p) - 1} \right|
\]
As an example let \( p = 0.30 \) and \( x_0 = 0.30 \). Then we get \( d = \frac{1}{3} \).

\[
W = \frac{Z - \frac{1}{3}}{\frac{1}{3}Z - 1} = \frac{3Z - 1}{Z - 3} = f(z)
\]

\[
d^2 = \frac{10}{3} \pm \frac{8}{3} = \frac{1}{3}
\]

Since \( z = 0 \) is on small circle we get that radius of small circle in image plane is \( |f(0)| = \frac{1}{3} \).

Thus \( \Gamma = |W| \)

\[
\Gamma = \left| \frac{3Z - 1}{Z - 3} \right| = \frac{3Z - 1}{|Z - 3|} = \left( \frac{(3x - 1)^2 + 9y^2}{(x - 3)^2 + y^2} \right)^{1/2}
\]

Thus \( T_{uu} + T_{vv} = 0 \) between \( |W| = \frac{1}{3} \) and \( |W| = 1 \).

Thus \( T = T_0 \) on \( |W| = 1 \)

\[
T = \begin{cases} 
T_0 & \text{on } \Gamma = 1 \\
T_1 & \text{on } \Gamma = \frac{1}{3}
\end{cases}
\]

Now we result on page 2 with \( T_0 = 1, \quad \Gamma = \frac{1}{3} \). This gives

\[
T = T_1 + \left( T_1 - T_0 \right) \frac{\log(3\Gamma)}{\log\left(\frac{1}{3}\right)}
\]

But \( \Gamma = |f(z)| \)

\[
\Gamma = \left| \frac{3Z - 1}{Z - 3} \right| = \frac{3Z - 1}{|Z - 3|} = \left( \frac{(3x - 1)^2 + 9y^2}{(x - 3)^2 + y^2} \right)^{1/2}
\]

\[
T = T_1 + \left( T_1 - T_0 \right) \frac{\log\left[ 3 \left( \frac{(3x - 1)^2 + 9y^2}{(x - 3)^2 + y^2} \right)^{1/2} \right]}{\log\left(\frac{1}{3}\right)}
\]
\[ |z - i a| = R \]
\[ T = T_i \]
\[ T_{xx} + T_{yy} = 0 \]
\[ \text{Assume } R < a \]

\[
T = T_{ij} 
\]

\[ \bar{z}_p \]

We try to map to concentric circles.

Let \( z_p \) be a point inside circle

\( \bar{z}_p \) is the symmetric point with respect to x-axis.

Now \( z^0 = z_p \rightarrow w = 0 \)

\( w = 0 \) and \( w = \phi \) are symmetric points w.r.t. image of the x-axis under the map.

Hence image of x-axis must be a circle centered at origin (it is only object with \( w = 0, w = \phi \) as symmetric point).

Take \( w = \frac{z - z_p}{z - \bar{z}_p} \) then if \( z \) is real we have \( |w| = 1 \rightarrow \text{circle of radius 1} \)

\( w = |z| \) \( \rightarrow \) image of x-axis.

Now we want \( z_p \) and \( \bar{z}_p \) to be symmetric point w.r.t. the circle. This will yield that image of circle is a circle centered at the origin.

Now \( z_p \) and \( \bar{z}_p \) are symmetric w.r.t. small circle if

\[
\bar{z}_p = a \bar{z} + \frac{R_1}{\bar{z}_p + a} \]

\[
\bar{z}_p (\bar{z}_p + a) = (\bar{z}_p + a) (a \bar{z} + \frac{R_1}{a}) + R_1 \]

\[
\rightarrow (\bar{z}_p)^2 = R_1 - a^2 \]

\[
\rightarrow z_p = i (a^2 - R_1) \frac{1}{i} \]

Claim that \( z_p \) lies inside \( |z - i a| = R \) since

\[
z_p = i (a - R) \frac{1}{i} (a + R) \frac{1}{i} (a - R) \frac{1}{i} (a - R) \frac{1}{i} \]

\[
= i (a - R) \left( \frac{a + R}{a - R} \right)^{1/i} \cdot i (a - R) \quad \text{since } R < a \]

\[
= z_p - i a < -i R \]

\[
|z_p - i a| < R \quad \checkmark \]
Therefore take the map

\[ W = \frac{Z - Z_p}{Z - \bar{Z}_p} \text{ with } Z_p = \frac{i}{(a^2 - R^2)^{1/2}} \]

Then we will get concentric circles in W-plane.

So find \( r \) take a point \( Z = i(a - R) \) and then calculate \(|W|\).

**Example**

**Let** \( a = 5 \) \( R = 4 \) **then** \( Z_p = 3i \)

\[ W = \frac{Z - 3i}{Z + 3i} \]

**\( |Z - 3i| = 4 \)**

\[ |W| = \left| \frac{i - 3i}{1 + 4i} \right| = \frac{1}{2} \]

\[ T_{uu} + T_{vv} = 0 \quad \frac{1}{2} \leq \Gamma \leq 1 \]

\[ T = T_0 \quad \text{on} \quad \Gamma = 1 \]

\[ T = T_1 \quad \text{on} \quad \Gamma = \frac{1}{2} \]

Now we use eq. 2 with \( \Gamma_0 = 1, \Gamma_1 = \frac{1}{2} \)

\[ T = T_1 + (T_1 - T_0) \log \left( \frac{2 \Gamma}{\log \left( \frac{1}{2} \right)} \right) \]

\[ \Gamma = \left[ \frac{x^2 + (y-3)^2}{x^2 + (y+3)^2} \right]^{1/2} \]

\[ r = \frac{|Z - 3i|}{|Z + 3i|} \]

\[ \Gamma = \left[ \frac{x^2 + (y-3)^2}{x^2 + (y+3)^2} \right]^{1/2} \]

\[ \Rightarrow \quad T = T_1 + \left( T_1 - T_0 \right) \log \left[ 2 \left( \frac{x^2 + (y-3)^2}{x^2 + (y+3)^2} \right)^{1/2} \right] \]