Evaluation of some infinite sums

**Proposition 0.1** Let \( p \) and \( q \) be polynomials with \( \deg(q) \geq \deg(p) + 2 \) and let \( Q \) denote the (finite) set of roots of \( q \). Define

\[
f(z) = \frac{p(z) \cot(\pi z)}{q(z)} = \frac{p(z) \cos(\pi z)}{q(z) \sin(\pi z)}
\]

Then

\[
\sum_{n \in \mathbb{Z} \setminus Q} \frac{p(n)}{q(n)} = -\pi \sum_{w \in Q} \text{Res}[f; w]
\]

**Proof** The function \( f(z) \) has poles when \( z \in \mathbb{Z} \cup Q \). The poles at \( n \in \mathbb{Z} \setminus Q \) are simple and for these values of \( n \)

\[
\text{Res}[f; n] = \frac{p(n)}{\pi q(n)}.
\]

Let \( \Gamma_n \) be a square with corners \((n + 1/2)(\pm 1 \pm i)\). For \( n \) large enough, \( \Gamma_n \) will enclose all the zeros of \( q \).

\[
\begin{array}{c}
\bullet \quad z_i \in Q \\
\bullet \bullet \bullet \bullet \\
-3 & -2 & -1 & 0 & 1 & 2 & 3 \\
\end{array}
\]

The residue formula gives

\[
\int_{\Gamma_n} f(z) \, dz = 2\pi i \left\{ \frac{1}{\pi} \sum_{k \in \mathbb{Z} \setminus Q, |k| \leq n} \frac{p(k)}{q(k)} + \sum_{w \in Q} \text{Res}[f; w] \right\}
\]

The idea is to show that the integral on the left tends to zero as \( n \to \infty \). Standard bounds on polynomials give \( |p(z)/q(z)| \leq C|z|^{-2} \) for \( |z| \) large. This implies that for large \( n \)

\[
\sup_{z \in \Gamma_n} \left| \frac{p(z)}{q(z)} \right| \leq \frac{C_1}{n^2},
\]

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Next we need to show that 
\[
\sup_{z \in \Gamma_n} |\cot(\pi z)| \leq C_2.
\]
Let \(z = x + iy\) with \(y > 0\). Then
\[
|\cot(\pi z)| = \left| \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right| \leq \frac{|e^{i\pi z}| + |e^{-i\pi z}|}{|e^{i\pi z} - e^{-i\pi z}|} = \frac{e^{\pi y} + e^{-\pi y}}{e^{\pi y} - e^{-\pi y}} = \frac{1 + e^{-2\pi y}}{1 - e^{-2\pi y}}.
\]
So if \(y\) is large enough to ensure \(e^{-2y} \leq \frac{1}{2}\) then the right side is bounded by \((1 + 1/2)/(1 - 1/2) = 3\). Similarly, when \(y < 0\) we get
\[
|\cot(\pi z)| \leq 1 + \frac{e^{2\pi y}}{1 - e^{2\pi y}}.
\]
which is also bounded for \(y\) negative and large. This implies that \(|\cot(\pi z)|\) is bounded by a constant independent of \(n\) on the top and bottom of \(\Gamma_n\). We could also compute a bound for the left and right sides of \(\Gamma_n\) explicitly. Alternatively we can argue that \(|\cot(\pi(n+1/2)+iy)|\) is a continuous function that tends to 1 for \(y\) tending to \(\pm\infty\). Thus there must be a maximum value. But \(|\cot(\pi z)|\) is a periodic function so the value of \(|\cot(\pi(n+1/2)+iy)|\) is independent of \(n\). This implies that we have a uniform bound for \(|\cot(\pi z)|\) on the left and right sides of \(\Gamma_n\). Altogether, we have \(\sup_{z \in \Gamma_n} |\cot(\pi z)| \leq C_2\) as required. Finally
\[
\text{length}(\Gamma_n) \leq C_3 n.
\]
Thus
\[
\left| \int_{\Gamma_n} f(z) dz \right| = \left| \int_{\Gamma_n} \frac{p(z) \cot(\pi z)}{q(z)} dz \right| \leq \frac{C_1 C_2 C_3}{n} \to 0
\]
as \(n \to \infty\). This implies that
\[
\lim_{n \to \infty} \left\{ \frac{1}{\pi} \sum_{k \in \mathbb{Z} \backslash \{0\}, |k| \leq n} \frac{p(k)}{q(k)} + \sum_{w \in Q} \text{Res}[f; w] = 0 \right\}
\]
which gives the formula we wish to prove.

**Example**
\[
\sum_{n=-\infty}^{\infty} \frac{1}{1 + n^2} = -\pi \text{Res} \left[ \frac{\cot(\pi z)}{1 + z^2}; z = i \right] = -\pi \text{Res} \left[ \frac{\cot(\pi z)}{1 + z^2}; z = -i \right]
\]
\[
= -\pi \frac{\cot(-\pi i)}{-2i} = \pi \coth(\pi) \sim 3.153348095
\]

**Example**
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2} \sum_{n \in \mathbb{Z} \backslash \{0\}} \frac{1}{n^2} = -\frac{1}{2\pi} \text{Res} \left[ z^2 \cot(\pi z); z = 0 \right] = \frac{1}{6}
\]

**Remark** This method doesn’t work for \(\sum n^{-3}\) but we can get a formula for \(\sum \frac{(-1)^n p(n)}{q(n)}\) using \(f(z) = \frac{p(z)}{q(z) \sin(\pi z)}\).