1. Show that $F(z) = \log(-z) + i\pi$ is a branch of $\log(z)$ with branch cut on the positive real axis. Is it true that $F(z) = \log_+(z)$? Here $\log_+$ denotes the branch of the log where the argument is chosen in $[0, 2\pi)$. (Hint: don’t forget to check values right on the cut)

**Solution:** A function $F(z)$ is a branch of $\log(z) \iff F(z) \in \log(z)$ for every $z \iff \exp(F(z)) = z$ for all $z$. For the given $F$,

$\exp(F(z)) = \exp(\log(-z) + i\pi) = \exp(\log(-z)) \exp(i\pi) = (-z)(-1) = z$

Thus $F(z)$ is a branch of the log.

Now we find the branch cut of $F$. The point $z$ is on branch cut for $F$ if $-z$ is on the cut for $\log_+$ (the principal branch). This means that $F$ has branch cut on $[0, \infty)$.

We now know that $F$ and $\log_+$ have the same branch cut. Moreover $F(-1) = \log(1) + i\pi = i\pi = \log_+(-1)$. Thus $F$ and $\log_+$ are branches with the same cut and the same value at a single point. This implies they are equal ... except possibly right on the cut where we have to decide from which side to take a limit. And in fact $F(1) = \log(-1) + \pi i = 2\pi i \neq \log_+(1) = 0$.

Recall that for $z \in \mathbb{C} \setminus \{0\}$ and $\alpha \in \mathbb{C}$, the complex power $z^\alpha$ is defined to be $z^\alpha = \exp(\alpha \log(z))$ (as a multivalued function).

2. Show that $(zw)^\alpha = z^\alpha w^\alpha$ as sets. (The set on the right is $\{a \cdot b : a \in z^\alpha, b \in w^\alpha\}$)

**Solution:** We have $\xi \in (zw)^\alpha \iff$

$\xi = \exp \left( \alpha \left( \ln(|zw| + i \arg(zw) + i2\pi k) \right) \right)$

for some $k \in \mathbb{Z}$. On the other hand, $\xi \in z^\alpha w^\alpha \iff$

$\xi = \exp \left( \alpha \left( \ln(|z|) + i \arg(z) + i2\pi l \right) \right) \cdot \exp \left( \alpha \left( \ln(|w|) + i \arg(w) + i2\pi m \right) \right)$

$= \exp \left( \alpha \left( \ln(|zw|) + i \arg(zw) + i2\pi (n + l + m) \right) \right)$

for some $l, m, n \in \mathbb{Z}$. Here the integer $n$ is chosen so that $\arg(z) + \arg(w) = \arg(zw) + i2\pi n$

Now the equality of the two sets follows from the fact that given $l, m, n \in \mathbb{Z}$, the sum $l + m + n$ is again in $\mathbb{Z}$, and conversely that for any choice of integer $n$, any integer $k$ can be written as $k = n + l + m$.
3. Show that $z^\alpha$

(a) is single valued if $\alpha \in \mathbb{Z}$,

(b) has $q$ values if $\alpha = p/q$, where $p, q \in \mathbb{Z}$ with no common factors and $q > 0$. (I really should have said $p$ and $q$ are relatively prime.)

(c) has infinitely many values if $\alpha$ is irrational.

Solution:

(a) If $\alpha \in \mathbb{Z}$ then $\exp(2\pi i \alpha k) = 1$ for $k \in \mathbb{Z}$ so

$$z^\alpha = \exp(\alpha \log(z)) = \{\exp(\alpha \Log(z) + i\alpha \Arg(z) + 2\pi i \alpha k) : k \in \mathbb{Z}\} = \{\exp(\alpha \Log(z) + 2\pi i \alpha \Arg(z)) \exp(2\pi i \alpha k) : k \in \mathbb{Z}\} = \{\exp(\alpha \Log(z) + 2\pi i \alpha \Arg(z)) \}. $$

So the set $z^\alpha$ contains a single value.

(b) From the calculation above, we see that the elements of the set $z^\alpha$ can be written as a fixed number $\exp(\alpha \Log(z))$ times the elements in the sequence \{w_k : k \in \mathbb{Z}\}, where $w_k = \exp(2\pi i p k/q)$. So we must determine how many distinct elements are in this sequence. The sequence is periodic with period $q$ so we need only consider \{w_k : k = 0, 1, \ldots, q - 1\}. We can always write $p = nq + r$ with $0 \leq r < q$. Since we are assuming that $q$ and $p$ are relatively prime we must have $r \neq 0$.

Now $w_k$ can also be written $w_k = \exp(2\pi ir k/q)$. If there are repetitions in the finite sequence above, that is, $w_k = w_l$ for $0 \leq k, l < q - 1$ with $k \neq l$ then $\exp(2\pi i r (k - l)/q) = 1$ which implies $r(k - l)/q = m \in \mathbb{Z}$ so that $r(k - l) = mq$. Since $r \neq 0$ and $k - l \neq 0$ we have that $m \neq 0$ so that $q$ divides $r(k - l)$. But $k - l \in \{-q + 1, \ldots, q - 2, q - 1\}$ so $q$ cannot divide $k - l$. Similarly $k$ cannot divide $r$ since $0 \leq r < q - 1$. This is a contradiction. Thus there are no repetitions.

(c) If $w^\alpha$ has only finitely many values then there must be repetitions in the sequence $\exp(2\pi i \alpha k)$ so that $\exp(2\pi i \alpha k) = \exp(2\pi i \alpha l)$ or $\exp(2\pi i \alpha (k - l)) = 1$ for some $k, l \in \mathbb{Z}$ with $k \neq l$. This implies $\alpha (k - l) = m \in \mathbb{Z}$ so that $\alpha = m/(k - l) \in \mathbb{Q}$.

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4. Identify the branch points of $f(z) = \log(z(z + 1)/(z - 1))$. (Don’t forget to check $z = \infty$.)

If we define a branch for $f(z)$ by choosing the principal branch of $\log(z)$, where are the branch cuts? (Note: this example illustrates that there may be a choice of branch cuts not obeying our “contractible loops” condition that still result in a single valued function.)

Solution:
Since \( f(z) = \log(z) + \log(z + 1) - \log(z - 1) \) there are branch points at \( z = 0, 1, -1 \). In addition, when \( z \) goes around a large circle, the first two terms each increase by \( 2\pi \) while the last term decreases by \( 2\pi \). So the total change in \( f \) is \( 2\pi \) and \( \infty \) is also a branch point.

If we define \( F(z) = \log(z(z+1)/(z-1)) \) then \( F \) is analytic near \( z \) unless \( z(z+1)/(z-1) \) is on the branch cut for \( \log \), i.e., \( z(z+1)/(z-1) = -p \) for some \( p \geq 0 \). For all \( z \) except \( z = 1 \) (which will turn out to be an endpoint of the branch cut) this is equivalent to \( z(z+1)+p(z-1) = z^2+(1+p)z-p = 0 \). This has solutions \( z = \pm \frac{-1+p}{2} \pm \sqrt{\left(\frac{1+p}{2}\right)^2 + p} \). As \( p \) ranges from 0 to \( \infty \), \( z_- \) ranges from \(-1\) to \(-\infty\) along the negative real axis while \( z_+ \) ranges from 0 to 1. (Here we can use that as \( p \to \infty \),
\[
z_+ = -\frac{1+p}{2} + \frac{1+p}{2} \sqrt{1 + \left(2 \frac{(1+p)}{1+p}\right)^2} \approx -\frac{1+p}{2} + \frac{1+p}{2} \left(2 + \frac{2}{1+p}\right)(p) = \frac{1}{2} \frac{2}{1+p} p \to 1.
\]
So we have shown that the branch cut for \( F \) is contained in \((-\infty, -1] \cup [0, 1]\). To be complete we should verify that when we cross these intervals, the value of \( F \) really does jump.

5. Find the branch points of \( f(z) = (z^3 + z^2 - 6z)^{1/2} \). Define a branch \( F(z) \) using the “range of angles” method that is continuous at \( z = -1 \) with \( F(-1) = -\sqrt{6} \).

**Solution:**

Since \( z^3 + z^2 - 6z = z(z+3)(z-2) \) there are branch points at \( z = 0, -3, 2 \). Also, since \( f(z) = z^{3/2} \left(\frac{1}{1 + \frac{6}{z^2}}\right)^{1/2} \) we see that \( \infty \) is also a branch point. To use the range of angles method we define \( r_i, \varphi_i \) for \( i = 1, 2, 3 \) by
\[
    z - 2 = r_1 e^{i\varphi_1} \quad z - 3 = r_2 e^{i\varphi_2} \quad z + 3 = r_3 e^{i\varphi_3}
\]
Then \( f(z) = (r_1 r_2 r_3)^{1/2} e^{i(\varphi_1 + \varphi_2 + \varphi_3)/2} \) and we can define a branch by choosing a range of angles for each \( \varphi_i \). Take \( \varphi_1 \in [0, 2\pi) \), \( \varphi_2 \in [0, 2\pi) \) and \( \varphi_3 \in (-\pi, \pi] \). Then for \( z \) close to \(-1 \), \( \varphi_1 \) is close to \( \pi \), \( \varphi_2 \) is close to \( \pi \), and \( \varphi_3 \) is close to \( 0 \), that is, they are away from the endpoints of their respective ranges. So there are no jumps near \( z = -1 \). The value of this branch at \(-1 \) is \((3 \cdot 1 \cdot 2)^{1/2} e^{i(\pi+\pi+0)/2} = \sqrt{6} e^{i\pi} = -\sqrt{6} \).

6. Construct a branch \( F(z) \) of \((z^2 + 1)^{1/2} \) that is
\[
    (i) \text{ analytic inside the unit circle, } \\
    (ii) \text{ analytic away from the imaginary axis, } \\
    (iii) \text{ equals } \sqrt{x^2 + 1} \text{ for } x \in \mathbb{R}. \\
    (iv) \text{ is continuous on the imaginary axis from the right.}
\]

Give an algorithm (i.e., a sequence of steps) that takes as input two real numbers \( x \) and \( y \) and computes \( F(x + iy) \)
Solution: Using the range of angles method we let \((z - i) = |z - i|e^{i\phi_1}\) and \((z + i) = |z + i|e^{i\phi_2}\) and define \(F(z) = |z - i|^{1/2}|z + i|^{1/2}e^{i(\phi_1 + \phi_2)/2} = \sqrt{|z^2 + 1|}e^{i(\phi_1 + \phi_2)/2}\) where \(\varphi_1 \in (-3\pi/2, \pi/2]\) and \(\varphi_2 \in [-\pi/2, 3\pi/2).\) With this choice the cuts are \([i, i\infty)\) and \((-i\infty, -i]\) on the imaginary axis so (i) and (ii) hold. We have \(\phi_1 + \phi_2 = 0\) when \(z \in \mathbb{R}\) which implies (iii). Finally, the open and closed endpoints have been chosen to make (iv) true.

The algorithm could be something like

```plaintext
# define the angles
phi1 = atan2(x, y-1);
phi2 = atan2(x, y+1);

# the angles will (probably) be in (-\pi, \pi] so we adjust
if (phi1 > Pi/2) then phi1 = phi1-2*Pi end;
if (phi2 < -Pi/2) then phi2 = phi2+2*Pi end;

# the function output would then be
F = sqrt(abs(z^2+1))*exp(i*(phi1+phi2)/2);
```

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