University of British Columbia  
Math 301, Section 201 

Midterm 1 

Date: February 13, 2013  
Time: 11:00 - 11:50pm 

Name (print): 
Student ID Number: 
Signature: 

Instructor: Richard Froese 

Instructions: 

1. No notes, books or calculators are allowed. 

2. Read the questions carefully and make sure you provide all the information that is asked for in the question. 

3. Show all your work. Answers without any explanation or without the correct accompanying work could receive no credit, even if they are correct. 

4. Answer the questions in the space provided. Continue on the back of the page if necessary. 

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1. (a) Find and classify the singularities of \( f(z) = \frac{\cot(\pi z)}{z^2} \)

(b) Calculate the residue of \( f(z) = \frac{\cot(\pi z)}{z^2} \) at each of its singularities. (Hint: Sometimes the easiest way to find the residue is to compute the Laurent series directly by manipulating series.)
(c) The sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$ can be evaluated by integrating $f(z) = \frac{\cot(\pi z)}{z^2}$ over a suitable contour $\Gamma_N$ and taking $N \to \infty$. Draw $\Gamma_N$ and mark the singularities on your diagram. What does the Cauchy residue theorem say when applied to $\Gamma_N$?

(d) State what estimates are required to perform the evaluation of $\sum_{n=1}^{\infty} \frac{1}{n^2}$. (You need not prove them.) What is the value of the infinite sum?
2. When we use the range of angles method to construct a branch of $f(z) = (4 - z^2)^{-1/2}$, we write $(z - 2) = |z - 2| e^{i\theta_1}$, $(z + 2) = |z + 2| e^{i\theta_2}$, and use an expression for $f(z)$ in terms of these quantities. Branches can then be specified by choosing ranges for the angles $\theta_1$ and $\theta_2$.

(a) Write down the expression for $f(z)$

(b) What range of angles results in a branch cut on the interval $[-2, 2]$ and positive values of $f(z)$ on the top lip of the cut? Does this branch have a residue at infinity? If so, compute it.
(c) Evaluate the integral \( I = \int_{-2}^{2} \frac{1}{\sqrt{4-x^2}} \, dx \) by integrating one of the branches from the previous parts around a suitable contour and taking a limit. Draw the contour and indicate which parts of the integral vanish in the limit. You need not prove the needed estimates.
3. (a) Draw the contour and the branch cut you would use to evaluate \( I = \int_0^\infty \frac{x^\alpha}{1+x^4} \, dx \). Where are the singularites enclosed by the contour located?

(b) The procedure for evaluating \( I \) relies on the integral over some portions of the contour tending to zero in the limit. Provide the needed estimates and give range of \( \alpha \) for which each estimate will work.
4. Let $D$ be the half strip

$$ D = \{ x + iy : x \leq 0, 0 \leq y \leq 1 \} $$

(a) What is the image of $D$ under $f(z) = z^2$?