1 Math 406, Project 1 Due Nov 15 th: Mold filling problems

2 Problem description

We consider a class of simple boundary value problems (BVP) of the form

$$\begin{aligned} Lp &= Dp'' = g(x) \quad \text{where } 0 < x < \ell(t) && (1) \\ \text{Left BC:} &\quad \begin{cases} \text{Specified pressure: } p(0) = p_0 \\ \text{Specified flux: } p'(0) = -Q_0/2D \end{cases} && (2) \\ \text{Right BC} &: \quad p(\ell(t)) = 0 && (3) \\ \text{Initial Condition} &: \quad p(x) = 0 \text{ at } t = 0 && (4) \end{aligned}$$

This class boundary value problem represents the pressure distribution within the fluid-filled region $0 < x < \ell(t)$ between two parallel plates a distance w_0 apart for which the flow velocity, according to Poiseuille's law, is given by

$$v = -\frac{w_0^2}{\mu'}\frac{dp}{dx} \tag{5}$$

Here $\mu' = 12\mu$ is the fluid viscosity and the velocity is obtained by integrating the parallel plate solution to the Navier Stokes equations across the gap between the plates. Associated with the Poiseuille velocity is the fluid flux within the channel, which is given by

$$q = w_0 v = -\frac{w_0^3}{\mu'}\frac{dp}{dx} = -D\frac{dp}{dx}, \text{ where } D = \frac{w_0^3}{\mu'}$$

The ODE (1) expresses the conservation of mass in which the flux gradient is balanced by the sources or sinks $g(x)$ distributed within the expanding domain $0 < x < \ell(t)$. Though the BVP is relatively simple to solve, the fact that the extent of the domain is unknown complicates the problem considerably. This type of problem is known as a "free boundary problem" or "moving boundary problem". At the moving front Poiseuille's law provides the so-called Stefan condition for the front velocity:

$$\dot{\ell}(t) = q(\ell(t))/w_0 \tag{6}$$

3 Simple solutions without distributed sources or sinks

Assuming no distributed source/sink term (i.e. $g(x) = 0$) determine the pressure distribution $p(x)$, the velocity $\dot{\ell}(t)$ of the moving front, and an expression for $\ell(t)$ for both boundary condition types given in (2). Now download the article:

V.R. Voller and Y.F. Chen, "Prediction of Filling Times of Porous Cavities," Int. J. Num. Meth. for Fluids, 23, 661-672, 1996.

Implement the VOF scheme for the prescribed pressure boundary condition case (2 I). Compare your results to the exact solution for the parameters: $w_0 = \mu' = C' = p_0 = 1$ over the time interval $0 < t < 100$. Provide a plot of your solutions.

4 Mold filling with fluid loss

Assume that fluid is being lost through the parallel plates via a diffusion process that leads to a sink term of the form

$$g(x) = \frac{C' H(t - t_0(x))}{\sqrt{t - t_0(x)}}$$

where $H(t)$ is the Heaviside function, C' is a given constant, and $t_0(x)$ is the time at which the fluid font arrives at the point x. Thus $t_0(x) = \ell^{-1}(x)$ the inverse function of $\ell(t)$.

1. For each boundary condition type given in (2) use the corresponding Green's function to determine an expression for $p(x)$ in terms of $\ell(t)$ and $g(x)$.

2. Use this expression to determine an expression for $p'(\ell)$. Now use the Stefan condition (6) to derive an expression for the front velocity $\dot{\ell}(t)$. In the integral that results use the transformation of variables $s = \ell(\tau)$, $ds = \dot{\ell}(\tau)d\tau$ to arrive at an Abel integral equation for $\dot{\ell}$ of the form:

$$\phi(\dot{\ell}(t)) = A + B \int_0^t \frac{\phi(\dot{\ell}(\tau))}{\sqrt{t - \tau}} d\tau \tag{7}$$

3. Since this integral equation is in the form of a Laplace Transform convolution take the Laplace transform of (7) to determine Laplace transform of $\phi(t)$. The Laplace transform $\mathcal{L}(\frac{1}{t^{1/2}}) = (\frac{\pi}{s})^{1/2}$ may prove useful.

4. Now invert the Laplace transform of ϕ to determine an expression for $\dot{\ell}(t)$ and thence an expression for $\ell(t)$. The inverse Laplace Transform $\mathcal{L}^{-1}(\frac{1}{s+\alpha s^{1/2}}) = e^{\alpha^2 t} \operatorname{erf} c(\alpha t^{1/2})$ may prove useful.

5. Assuming $w_0 = \mu' = C' = 1$, and $p_0 = 1$ and $Q_0 = 1$for the two different boundary condition types use the MATLAB function $erfcx(x) = e^{x^2} erfc(x)$ to plot $\ell(t)$ for $0 < t < 500$. Store the values of $\ell(t)$ in a vector for a relatively fine sampling of t values. Now use the MATLAB Piecewise Cubic Hermite Polynomial routine `pchip` to determine an approximation for the inverse function $t_0(x)$ by defining the following function: `t0 = @(s)ppval(pchip(Le,t),s);`where `Le` is the length vector and `t` is the vector of corresponding sample times. Now use `quadgk` to determine $p(x)$ and plot $p(x)$ at $t = 250$.

5 Asymptotics and scaling

5.1 Scaling

By dimensional analysis and scaling it is frequently possible to derive the fundamental power law relationships between the different variables in the model.

For the case in which the pressure p_0 is specified at the left endpoint of the domain, the boundary value problem is of the form:

$$Dp'' = \frac{C' H(t - t_0(x))}{\sqrt{t - t_0(x)}} \tag{8}$$

Introduce characteristic length, time, and pressure scales ℓ_*, t_* and $p_* = p_0$ and dimensionless variables

$$x = \ell_* \xi,\ t = t_* \tau,\ p(x) = p_0 \Pi(\xi),$$

to reduce (8) to the form

$$\mathcal{G}_\mu \Pi'' = \frac{H(\tau - \tau_0)}{\sqrt{\tau - \tau_0}}$$

By requiring $\mathcal{G}_\mu = 1$ determine γ in the power law relationship $\ell_* \tilde{}\ t_*^\gamma$.

Now consider the following unified form for our model problem in which the flux boundary condition is specified

$$Dp'' = \frac{C' H(t - t_0(x))}{\sqrt{t - t_0(x)}} - Q_0 \delta(x) \tag{9}$$

We may think of this problem as defined on $(-\ell, \ell)$ in which (1) and (2 II) define half of the problem (9) in which the solution is symmetric about $x = 0$. Introducing the characteristic length, time, and pressure scales ℓ_*, t_* and p_* as before re-write (9) in the following dimensionless form

$$\mathcal{G}_\mu \Pi'' = \frac{H(\tau - \tau_0)}{\sqrt{\tau - \tau_0}} - \mathcal{G}_v \delta(\xi)$$

By requiring that $\mathcal{G}_v = 1$ determine γ in the power law relationship $\ell_* \tilde{}\ t_*^\gamma$ and by requiring that $\mathcal{G}_\mu = 1$ determine θ in the power law relationship $p_* \tilde{}\ t_*^\theta$.

5.2 Asymptotics

Since the $erfc$ function is defined as $erfc(x) = \frac{2}{\pi^{1/2}} \int_x^\infty e^{-t^2} dt$ consider the asymptotics of the $erfcx$ function as we did in assignment 3. Define

$$J(x) = e^{x^2} \int_x^\infty e^{-t^2} dt$$

and differentiate $J(x)$ to obtain an ODE for J of the form

$$J(x) = \frac{1}{2x} + \frac{1}{2x} J'(x)$$

This forms a recursion that generates the following asymptotic series for $J(x)$ in the limit $x \to \infty$:

$$J(x) = \frac{1}{2x} - \frac{1}{4x^3} + \cdots$$

Thus

$$erfcx(x) \overset{x \to \infty}{\sim} \frac{1}{\pi^{1/2} x}$$

Use this asymptotic behavior in the expressions for $\ell(t)$ to determine the large time behavior of $\ell(t)$ for both boundary condition classes. Compare these asymptotic results to the power laws obtained by scaling. Plot the asymptotic solutions in both cases against the analytic solutions obtained in the previous section on a log-log scale.