1. Consider the differential equation

\[ 2x^2 y'' - xy' + (1 + x^2)y = 0 \]  

(a) Classify the points \( 0 \leq x < \infty \) as ordinary points, regular singular points, or irregular singular points.

(b) Find two values of \( r \) such that there are solutions of the form \( y(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \).

(c) Use the series expansion in (b) to determine two independent solutions of (1). You only need to calculate the first three non-zero terms in each case.

[20 marks]

2. Consider a conducting metal bar of length \( \pi/2 \) whose initial temperature is \( u(x, 0) = x \) and which loses heat to its surroundings. Assume that the left end of the bar is maintained at a zero temperature while the right end is insulated. The temperature distribution in the bar \( u(x, t) \) is determined by the following initial boundary value problem for the heat equation:

\[ u_t = u_{xx} - u, \quad 0 < x < \pi/2, \quad t > 0 \]

\[ u(0, t) = 0, \quad u(x, \pi/2, t) = 0 \]

\[ u(x, 0) = x \]  

(a) Determine the solution to the boundary value problem (2) by separation of variables.  

[14 marks]

(b) Briefly describe how you would use the method of finite differences to obtain an approximate solution this boundary value problem that is accurate to \( O(\Delta x^2, \Delta t) \) terms. Use the notation \( u_n^k \approx u(x_n, t_k) \) to represent the nodal values on the finite difference mesh. Explain how you propose to approximate the boundary condition \( u_x(\pi/2, t) = 0 \) with \( O(\Delta x^2) \) accuracy.

Hint: Taylor’s expansion may prove useful: \( f(x + \Delta x) = f(x) + f'(x)\Delta x + \frac{f''(x)}{2!}\Delta x^2 + O(\Delta x^3) \).

[6 marks]

[total 20 marks]

3. The motion of a damped string subject to an imposed load satisfies the following initial-boundary value problem:

\[ u_{tt} + 2\gamma u_t = u_{xx} - 8 \sin x \cos x, \quad 0 < x < \pi, \quad t > 0 \]

\[ u(0, t) = 0, \quad u(\pi, t) = 0 \]

\[ u(x, 0) = 0, \quad u_t(x, 0) = \sin 3x. \]

(a) Determine the static deflection \( w(x) \) of the string (i.e., the steady solution), which is determined by solving (3) with \( u_{tt} = u_t = 0 \) and subject to the boundary conditions (4).

[5 marks]

(b) Let \( u(x, t) = w(x) + v(x, t) \) and determine the corresponding boundary value problem for \( v(x, t) \).

[5 marks]

(c) Assuming that \( \gamma < 1 \) use the method of separation of variables to solve for \( v(x, t) \) and therefore \( u(x, t) \).

[10 marks]

[total 20 marks]

4. Consider the eigenvalue problem

\[ x^2 y'' + xy' + \lambda y = 0 \]

\[ y'(1) = 0 = y(e^{\pi/2}) \]

(a) Reduce this problem to the form of a Sturm-Liouville eigenvalue problem. Determine the eigenvalues and corresponding eigenfunctions.  

[8 marks]

(b) Use the eigenfunctions in (a) to solve the following mixed boundary value problem for Laplace’s equation

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on the semi-annular region:

\[ u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \quad 1 < r < e^{\pi/2}, \quad 0 < \theta < \pi \]

\[ u(r, 0) = 0 \quad \text{and} \quad u(r, \pi) = f(r) \]

\[ \frac{\partial u(1, \theta)}{\partial r} = 0 \quad \text{and} \quad u(e^{\pi/2}, \theta) = 0 \]

[12 marks]

[total 20 marks]

5. Solve the inhomogeneous heat conduction problem subject to time dependent boundary conditions:

\[ u_t = \alpha^2 u_{xx} + 1 - xe^{-t}, \quad 0 < x < 1, \quad t > 0 \]

\[ u_x(0, t) = e^{-t}, \quad \text{and} \quad u(1, t) = t \]

\[ u(x, 0) = x. \]

[20 marks]