M257/316 MIDTERM 2 SECTION 102 2017

Q1 $$u_t = u_{xx} + e^{-4t}\sin(x) + 1 \qquad 0<x<\pi/2,\ t>0$$

BC: $u(0,t) = t \qquad u_x(\frac{\pi}{2},t) = 1$

IC: $u(x,0) = x$

LOOK FOR A FUNCTION $W(x,t) = A(t)x + B(t)$ THAT SATISFIES THE BC

$t = W(0,t) = B(t) \qquad 1 = W_x(\frac{\pi}{2},t) = A(t) \quad \Rightarrow \quad \boxed{W(x,t) = x+t}$

NOW LET $u(x,t) = W(x,t) + V(x,t)$ AND LOOK FOR THE B.V.P. SATISFIED BY V.

PDE: $u_t = W_t + V_t = \cancel{1} + V_t = \underbrace{\overbrace{W_{xx}}^{0} + V_{xx}}_{u_{xx}} + e^{-4t}\sin x + \cancel{1} \Rightarrow V_t = V_{xx} + \underbrace{e^{-4t}\sin x}_{S(x,t)}$

BC: $\cancel{t} = u(0,t) = W(0,t) + V(0,t) = \cancel{t} + V(0,t) \quad \Rightarrow \quad V(0,t) = 0$

$\cancel{1} = u_x(\frac{\pi}{2},t) = W_x(\frac{\pi}{2},t) + V_x(\frac{\pi}{2},t) = \cancel{1} + V_x(\frac{\pi}{2},t) \quad \Rightarrow \quad V_x(\frac{\pi}{2},t) = 0$

IC: $x = u(x,0) = W(x,0) + V(x,0) = x + V(x,0) \Rightarrow V(x,0) = 0$

THE EIGENVALUES AND EIGENFUNCTION ASSOCIATED WITH THE HOMOG. MIXED BC ON V

ARE: $\lambda_n = -\mu_n^2 = -\left[\frac{(2n-1)^2\pi/2}{\pi/2}\right]^2 = -(2n-1)^2 \quad n=1,2,.. \qquad X_n = \sin(2n-1)x.$

NOW LET $S(x,t) = e^{-4t}\sin x = \sum_{n=1}^{\infty} S_n(t)\sin(2n-1)x \Rightarrow S_n(t) = \delta_{n1}e^{-4t}.$

LET $V(x,t) = \sum_{n=1}^{\infty} V_n(t)\sin\mu_n x \qquad V_t = \sum_{n=1}^{\infty}\frac{dV_n}{dt}\sin\mu_n x \qquad V_{xx} = \sum_{n=1}^{\infty} -\mu_n^2 V_n \sin\mu_n x$

$\therefore V_t - V_{xx} - e^{-4t}\sin x = \sum_{n=1}^{\infty}\left\{\frac{dV_n}{dt} + \mu_n^2 V_n - e^{-4t}\delta_{n1}\right\}\sin(2n-1)x = 0$

THUS, SINCE $\sin\mu_n x$ ARE L.I., $\frac{dV_n}{dt} + \mu_n^2 V_n = e^{-4t}\delta_{n1} \qquad n=1,2,...$

MULTIPLYING BY THE INTEGRATING FACTOR $e^{\mu_n^2 t}$ YIELDS

$$\frac{d}{dt}\left\{e^{\mu_n^2 t}V_n\right\} = e^{(\mu_n^2-4)t}\delta_{n1}$$

$$\therefore \quad e^{\mu_n^2 t}V_n = \frac{e^{(\mu_n^2-4)t}}{(\mu_n^2-4)}\delta_{n1} + C_n.$$

THUS $$V_n(t) = \frac{e^{-4t}\delta_{n1}}{(\mu_n^2-4)} + C_n e^{-\mu_n^2 t}$$

AND

$$V(x,t) = \sum_{n=1}^{\infty}\left(\frac{e^{-4t}\delta_{n1}}{\mu_n^2-4} + C_n e^{-\mu_n^2 t}\right)\sin\mu_n x$$

$$0 = V(x,0) = \sum_{n=1}^{\infty}\left(\frac{\delta_{n1}}{\mu_n^2-4} + C_n\right)\sin\mu_n x \Rightarrow C_n = -\frac{\delta_{n1}}{\mu_n^2-4}$$

$$\therefore u(x,t) = x+t + \sum_{n=1}^{\infty}\left(\frac{\delta_{n1}}{\mu_n^2-4}\right)\left[e^{-4t} - e^{-\mu_n^2 t}\right]\sin\mu_n x \qquad n=1 \Rightarrow \mu_n = 1$$

$$\therefore \quad u(x,t) = x+t + \frac{(e^{-t} - e^{-4t})}{3}\sin x$$

Q2. $u_{tt} + 2\gamma u_t = u_{xx} \qquad 0 < x < \pi/2, \; t > 0 \qquad 0 < \gamma < 1$

$u_x(0,t) = 1 \qquad u(\frac{\pi}{2}, t) = \pi/2$

$u(x,0) = x \qquad u_t(x,0) = \cos(5x)$

a) STEADY SOL'N $\quad W_t, W_{tt} = 0 \qquad 0 = W_{xx} \Rightarrow W(x) = Ax + B$

$1 = W_x = A, \; \frac{\pi}{2} = W(\frac{\pi}{2}, t) = 1 . \frac{\pi}{2} + B \Rightarrow B = 0 \quad \boxed{W(x) = x}$ IS THE STEADY SOLN.

b) $u(x,t) = W(x) + V(x,t)$

PDE: $u_{tt} + 2\gamma u_t = (\underbrace{W_{tt}}_{0} + V_{tt}) + 2\gamma(\underbrace{W_t}_{0} + V_t) = \cancel{W_{xx}} + V_{xx} \Rightarrow V_{tt} + 2\gamma V_t = V_{xx}$

BC: $\cancel{1} = u_x(0,t) = W_x(0,t) + V_x(0,t) = \cancel{1} + V_x(0,t) \quad \Rightarrow V_x(0,t) = 0$

$\pi/2 = u(\frac{\pi}{2}, t) = W(\frac{\pi}{2}, t) + V(\frac{\pi}{2}, t) = \frac{\pi}{2} + V(\frac{\pi}{2}, t) \quad \Rightarrow V(\frac{\pi}{2}, t) = 0$

IC: $\cancel{x} = u(x,0) = W(x) + V(x,0) = \cancel{x} + V(x,0) \quad \Rightarrow V(x,0) = 0$

$\cos 5x = u_t(x,0) = \cancel{W_t(x)} + V_t(x,0) = V_t(x,0) \quad \Rightarrow V_t(x,0) = \cos 5x$

c) LET $V(x,t) = X(x)T(t)$

$X\ddot{T} + 2\gamma X\dot{T} = X''T$

$\div XT] \quad \dfrac{\ddot{T} + 2\gamma\dot{T}}{T(t)} = \dfrac{X''}{X(x)} = \lambda = -\mu^2$

X] $\left.\begin{array}{l} X'' + \mu^2 X = 0 \\ X'(0) = 0 = X(\pi/2) \end{array}\right\} \quad \mu_n = (2n-1) \quad n = 1, 2, \ldots \quad X_n = \cos(2n-1)x.$

T] $\ddot{T} + 2\gamma\dot{T} + \mu^2 T = 0$ CONST COEFF ODE $\Rightarrow T(t) = e^{rt} \Rightarrow r^2 + 2\gamma r + \mu^2 = 0$

$\therefore \quad r_n = -\gamma \pm \sqrt{\gamma^2 - \mu_n^2} = -\gamma \pm i\sqrt{\mu_n^2 - \gamma^2} = -\gamma \pm i\theta_n \qquad \theta_n = \sqrt{\mu_n^2 - \gamma^2} \quad \mu_n \geq 1 > \gamma$

$\therefore \; T_n(t) = [A_n \cos\theta_n t + B_n \sin\theta_n t] e^{-\gamma t}$

$V(x,t) = \sum_{n=1}^{\infty} [A_n \cos\theta_n t + B_n \sin\theta_n t] e^{-\gamma t} \cos\mu_n x$

$0 = V(x,0) = \sum_{n=1}^{\infty} A_n \cos\mu_n x \Rightarrow A_n = 0$

$V_t(x,t) = \sum_{n=1}^{\infty} B_n [\theta_n \cos\theta_n t - \gamma \sin\theta_n t] e^{-\gamma t} \cos\mu_n x.$

$\cos 5x = V_t(x,0) = \sum_{n=1}^{\infty} B_n \theta_n \cos(2n-1)x \Rightarrow B_n = \delta_{n3}/\theta_n$

$\therefore u(x,t) = x + \sum_{n=1}^{\infty} \dfrac{\delta_{n3}(\sin\theta_n t)}{\theta_n} e^{-\gamma t} \cos\mu_n x \qquad n = 3 \Rightarrow \mu_n = 5$

$$u = x + \frac{\sin(\sqrt{25 - \gamma^2}\, t)}{(25 - \gamma^2)^{1/2}} e^{-\gamma t} \cos(5x)$$