M257/316 MIDTERM 2 SECTION 101 2017

Q1. $u_t = u_{xx} + e^{-2t}\cos x + x \qquad 0<x<\pi/2,\ t>0$

BC: $u_x(0,t) = t \qquad u(\frac{\pi}{2},t) = \frac{\pi}{2}t$

IC: $u(x,0) = 0$.

FIND A FUNCTION $w(x,t) = A(t)x + B(t)$ THAT SATISFIES THE BC

$t = w_x = A(t)$; $\frac{\pi}{2}t = w(\frac{\pi}{2},t) = \frac{\pi}{2}A + B(t) \Rightarrow B(t) = 0 \quad \therefore \boxed{w(x,t) = xt}$

NOW LET $u(x,t) = w(x,t) + v(x,t)$

PDE: $u_t = w_t + v_t = \cancel{x} + v_t = (\underbrace{\cancel{w_{xx}} + v_{xx}}_{u_{xx}}) + e^{-2t}\cos x + \cancel{x} \Rightarrow v_t = v_{xx} + e^{-2t}\cos x$

BC $\begin{cases} \cancel{t} = u_x(0,t) = w_x(0,t) + v_x(0,t) = \cancel{t} + v_x(0,t) & \Rightarrow v_x(0,t) = 0 \\ \cancel{\frac{\pi}{2}t} = u(\pi/2,t) = w(\frac{\pi}{2},t) + v(\frac{\pi}{2},t) = \cancel{\frac{\pi}{2}t} + v(\pi/2,t) & \Rightarrow v(\frac{\pi}{2},t) = 0 \end{cases}$

IC: $0 = u(x,0) = w(x,0) + v(x,0) = x.0 + v(x,0) \Rightarrow v(x,0) = 0.$

THE EIGENVALUES AND EIGENFUNCTIONS ASSOCIATED WITH THESE MIXED HOMOGENEOUS BC ON V

ARE: $\lambda_n = -\mu_n^2 = -\left[\frac{(2n-1)\pi/2}{\pi/2}\right]^2 = -(2n-1)^2 \quad n=1,2,\ldots \quad X_n(x) = \cos(2n-1)x.$

LET $S(x,t) = e^{-2t}\cos x = \sum_{n=1}^{\infty} S_n(t)\cos(2n-1)x \Rightarrow S_n(t) = e^{-2t}\delta_{n1}$

NOW LET $v(x,t) = \sum_{n=1}^{\infty} v_n(t)\cos\mu_n x \qquad v_t = \sum_{n=1}^{\infty}\frac{dv_n}{dt}\cos\mu_n x \qquad v_{xx} = \sum_{n=1}^{\infty} v_n(-\mu_n^2)\cos\mu_n x.$

$\therefore 0 = v_t - v_{xx} - e^{-2t}\cos x = \sum_{n=1}^{\infty}\left\{\frac{dv_n}{dt} + \mu_n^2 v_n - e^{-2t}\delta_{n1}\right\}\cos\mu_n x = 0$

SINCE THE $\cos\mu_n x$ ARE LI $\quad \frac{dv_n}{dt} + \mu_n^2 v_n = e^{-2t}\delta_{n1}$

USING THE INTEGRATING FACTOR $e^{\mu_n^2 t} \Rightarrow \frac{d}{dt}\left\{e^{\mu_n^2 t}v_n\right\} = e^{(\mu_n^2-2)t}\delta_{n1}$

$$\therefore \quad e^{\mu_n^2 t}v_n = \frac{e^{(\mu_n^2-2)t}\delta_{n1}}{(\mu_n^2-2)} + C_n$$

$$\therefore \quad v_n = \frac{e^{-2t}}{(\mu_n^2-2)}\delta_{n1} + C_n e^{-\mu_n^2 t}.$$

THUS $\quad v(x,t) = \sum_{n=1}^{\infty}\left\{\frac{e^{-2t}\delta_{n1}}{\mu_n^2-2} + C_n e^{-\mu_n^2 t}\right\}\cos\mu_n x$

NOW $\quad 0 = v(x,0) = \sum_{n=1}^{\infty}\left(\frac{\delta_{n1}}{\mu_n^2-2} + C_n\right)\cos\mu_n x \Rightarrow C_n = \frac{-\delta_{n1}}{\mu_n^2-2}$

$$\therefore u(x,t) = xt + \sum_{n=1}^{\infty}\frac{\delta_{n1}}{\mu_n^2-2}\left(e^{-2t} - e^{-\mu_n^2 t}\right)\cos\mu_n x \qquad n=1 \Rightarrow \mu_n = 1$$

$$u(x,t) = xt + \left(e^{-t} - e^{-2t}\right)\cos x$$

Q2. $u_{tt} = u_{xx} \quad 0 < x < \pi/2, \; t > 0$

$u(0,t) = 0 \quad u_x(\frac{\pi}{2}, t) = t$

$u(x,0) = 0 \quad u_t(x,0) = \sin 3x + x$

CONSTRUCT $W(x,t) = A(t)x + B(t)$ TO MATCH THE BC

$0 = W(0,t) = B(t) \qquad t = W_x(\frac{\pi}{2}, t) = A(t) \Rightarrow \boxed{W(x,t) = xt}$

NOW LET $u(x,t) = W(x,t) + V(x,t)$ AND LOOK FOR THE BVP SATISFIED BY $V(x,t)$

$u_{tt} = \cancel{W_{tt}} + V_{tt} = \cancel{W_{xx}} + V_{xx} \qquad \Rightarrow \quad V_{tt} = V_{xx}$

BC: $0 = u(0,t) = w(0,t) + v(0,t) = 0 + v(0,t) \qquad \Rightarrow_{BC} \begin{cases} V(0,t) = 0 \\ V_x(\frac{\pi}{2}, t) = 0 \end{cases}$

$t = u_x(\frac{\pi}{2}, t) = W_x(\frac{\pi}{2}, t) + V_x(\frac{\pi}{2}, t) = t + V_x(\frac{\pi}{2}, t)$

IC: $0 = u(x,0) = x.0 + V(x,0) \qquad \Rightarrow_{IC} \begin{cases} V(x,0) = 0 \\ V_t(x,0) = \sin 3x. \end{cases}$

$\sin 3x + \cancel{x} = u_t(x,0) = W_t(x,0) + V_t(x,0) = \cancel{x} + V_t(x,0) \Rightarrow$

METHOD 1: SEPARATION OF VARIABLES – LET $V(x,t) = X(x)T(t)$

$X\ddot{T} = X''T$

$\div XT \Rightarrow \quad \frac{\ddot{T}(t)}{T(t)} = \frac{X''}{X(x)} = \text{CONST} = \lambda = -\mu^2$

$X]\quad \left.\begin{array}{l} X'' + \mu^2 X = 0 \\ X(0) = 0 = X'(\frac{\pi}{2}) \end{array}\right\} \quad \mu_n = (2n-1) \quad n = 1, 2, \ldots \quad X_n = \sin \mu_n x.$

$T]\quad \ddot{T} + \mu^2 T = 0 \Rightarrow T_n(t) = A_n \cos \mu_n t + B_n \sin \mu_n t.$

$V(x,t) = \sum_{n=1}^{\infty} [A_n \cos \mu_n t + B_n \sin \mu_n t] \sin \mu_n x$

$V_t(x,t) = \sum_{n=1}^{\infty} [-A_n \mu_n \sin \mu_n t + B_n \mu_n \cos \mu_n t] \sin \mu_n x.$

IC: $0 = V(x,0) = \sum_{n=1}^{\infty} A_n \sin \mu_n x \Rightarrow A_n = 0$

$\sin 3x = V_t(x,0) = \sum_{n=1}^{\infty} B_n \mu_n \sin \mu_n x \Rightarrow B_n = \delta_{n2} / \mu_n$

$\therefore u(x,t) = xt + \sum_{n=1}^{\infty} \frac{\delta_{n2}}{\mu_n} \sin \mu_n t \sin \mu_n x \qquad n = 2 \Rightarrow \mu_n = 3$

$= xt + \frac{1}{3} \sin 3t \sin 3x = xt + \frac{1}{6} [\cos 3(x-t) - \cos 3(x+t)]$

METHOD 2: USING D'ALEMBERT'S SOLUTION

$u(x,t) = xt + \frac{1}{2} \int_{x-t}^{x+t} \sin 3s \, ds = xt - \frac{1}{6} \cos 3s \Big|_{x-t}^{x+t}$

$= xt + \frac{1}{6} [\cos 3(x-t) - \cos 3(x+t)] \qquad$ AS ABOVE.