MATH 257/316 MIDTERM 1 SECTION 102

Q1. $Ly = 8x^2y'' + 10xy' - (1-x)y = 0.$

a) $x = 0$ IS A SINGULAR POINT. ALL OTHER PTS $0<x<\infty$ ARE ORDINARY PTS

$x=0$: $\lim_{x\to 0} x \dfrac{10x}{8x^2} = \dfrac{5}{4} = p_0$ $\quad \lim_{x\to 0} x^2\left[\dfrac{-(1-x)}{8x^2}\right] = -\dfrac{1}{8} = q_0$ $\quad |q_0| \& |p_0| < \infty$ $\Rightarrow x=0$ IS A REGULAR SP

$r(r-1) + \frac{5}{4}r - \frac{1}{8} = 0 \Rightarrow 8r^2 + 2r - 1 = (4r-1)(2r+1) = 0 \Rightarrow \boxed{r = 1/4, -1/2}$

b) WE WOULD LET $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$ SINCE $x=1$ IS AN ORDINARY PT.

SINCE THE CLOSEST SINGULAR POINT TO $x=1$ IS AT $x=0$ $\rho \geq 1$.

c). LET $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, $y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$, $y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$

$$Ly = 8x^2y'' + 10xy' - y + xy = 0$$

$$= \underbrace{\sum_{n=0}^{\infty} 8a_n(n+r)(n+r-1)x^{n+r}}_{m=n} + \underbrace{\sum_{n=0}^{\infty} 10a_n(n+r)x^{n+r}}_{m=n} - \underbrace{\sum_{n=0} a_n x^{n+r}}_{m=n} + \underbrace{\sum_{n=0}^{\infty} a_n x^{n+r+1}}_{m=n+1,\ n=m-1,\ n=0 \Rightarrow m=1} = 0$$

$$= \sum_{m=0}^{\infty} a_m\left[(m+r)\{8(m+r-1)+10\} - 1\right]x^{m+r} + \sum_{m=1}^{\infty} a_{m-1}x^{m+r} = 0$$

$$= a_0[8r(r-1) + 10r - 1]x^r + \sum_{m=1}^{\infty}\left\{a_m\left[(m+r)\{8(m+r)+2\} - 1\right] + a_{m-1}\right\}x^{m+r} = 0$$

$x^r]$ $\quad 8r^2 + 2r - 1 = (4r-1)(2r+1) = 0 \quad r = 1/4, -1/2.$

$x^{m+r}_{m\geq 1}]$ $\quad a_m = -a_{m-1}/\left[(m+r)\{8(m+r)+2\} - 1\right] \quad m \geq 1.$

$r = -1/2$: $\quad a_m = \dfrac{-a_{m-1}}{(m-1/2)(8m-2)-1} = \dfrac{-a_{m-1}}{(2m-1)(4m-1)-1} = \dfrac{-a_{m-1}}{8m^2-6m+\cancel{1}-\cancel{1}} = \dfrac{-a_{m-1}}{m(8m-6)}$

$a_1 = \dfrac{-a_0}{1.2} = \dfrac{-a_0}{2} \qquad a_2 = \dfrac{-a_1}{20} = +\dfrac{a_0}{40}$

$\therefore y_1(x) = a_0 x^{-1/2}\left[1 - x/2 + x^2/40 - \cdots\right]$

$r = 1/4$: $\quad a_m = \dfrac{-a_{m-1}}{(m+1/4)(8m+4)-1} = \dfrac{-a_{m-1}}{(4m+1)(2m+1)-1} = \dfrac{-a_{m-1}}{8m^2+6m+\cancel{1}-\cancel{1}} = \dfrac{-a_{m-1}}{m(8m+6)}$

$a_1 = \dfrac{-a_0}{14} \qquad a_2 = \dfrac{-a_1}{44} = +\dfrac{a_0}{44.14}$

$y_2(x) = a_0 x^{1/4}\left[1 - \dfrac{x}{14} + \dfrac{x^2}{44.14} - \cdots\right]$

Q2 $\qquad u_t = u_{xx} - \gamma u \qquad 0<x<\pi/2 \quad t>0$

$u_x(0,t) = 0 = u(\pi/2, t)$

$u(x,0) = \cos(3x)$

LET $u(x,t) = X(x)\,T(t)$

$X\dot{T} = X''T - \gamma X T$

$\div XT]\quad \dfrac{\dot{T}}{T(t)} + \gamma = \dfrac{X''}{X(x)} = \lambda = \text{CONST}$

$T]\quad \dot{T} = (\lambda-\gamma)T \Rightarrow T(t) = Ce^{(\lambda-\gamma)t}.$

$X]\quad X'' = \lambda X \qquad X'(0) = 0 = X(\pi/2)$

$\lambda>0$: $\lambda = \mu^2$: $X'' - \mu^2 X = 0 \quad X = A\cosh\mu x + B\sinh\mu x$

$X'(x) = A\mu\sinh\mu x + B\mu\cosh\mu x$

$0 = X'(0) = B\mu \Rightarrow B = 0 \quad X(\pi/2) = A\cosh\mu\pi/2 = 0 \Rightarrow A = 0 \quad X\equiv 0$ TRIVIAL

$\lambda = 0$: $X'' = 0 \quad X' = A \quad X = Ax+B$

$0 = X'(0) = A \quad 0 = X(\pi/2) = B \quad \Rightarrow \quad X\equiv 0$ TRIVIAL SOL'N

$\lambda<0$: $\lambda = -\mu^2$. $X'' + \mu^2 X = 0 \quad X'(0) = 0 = X(\pi/2)$

$X(x) = A\cos\mu x + B\sin\mu x \qquad X'(x) = -A\mu\sin\mu x + B\mu\cos\mu x$

$0 = X'(0) = B\mu \Rightarrow B = 0 \qquad 0 = X(\pi/2) = A\cos\mu\pi/2 \Rightarrow \mu_n\frac{\pi}{2} = (2n+1)\frac{\pi}{2} \quad n=0,1,\ldots$

$\therefore\ \lambda_n = -(2n+1)^2 \quad n = 0,1,\ldots$ ARE EIGENVALUES &

$X_n = \cos(2n+1)x$ ARE THE CORRESPONDING EIGENFUNCTIONS

$\therefore\ u(x,t) = \sum_{n=0}^{\infty} a_n e^{-[(2n+1)^2+\gamma]t}\cos(2n+1)x.$

$\cos(3x) = u(x,0) = \sum_{n=0}^{\infty} a_n\cos(2n+1)x \Rightarrow \begin{array}{l} a_1 = 1 \\ a_n = 0 \quad n\neq 1 \end{array}$

$\therefore\ u(x,t) = e^{-(9+\gamma)t}\cos 3x.$