Math 257/316 Assignment 9 Solutions

Q4: \( \Delta u = 0 \quad 0 < x < \pi, \quad 0 < y \)

Let \( U(x, y) = X(x)Y(y) \) implies
\[
\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2 \quad u = 0 \quad \Delta x = 0 \quad U(x, y) = X(x)Y(y) \]

\[ X'' + X = 0 \quad X = A \cos \lambda x + B \sin \lambda x \]
\[ X(0) = X(\pi) = 0 \quad \lambda = \frac{n}{\pi}, \quad n = 0, 1, 2, \ldots \]

\[ Y'' - \lambda^2 Y = 0 \quad Y_n = A_n e^{\lambda y} + B_n e^{-\lambda y} \quad Y_n |_{y=0} = 0 \quad \lambda_n = \frac{n\pi}{2} \quad n = 0, 1, 2, \ldots \]

\[ U(x, y) = \sum_{n=0}^{\infty} B_n e^{\lambda_n y} \sin \lambda_n x \]

\[ U(x, 0) = \int_{0}^{\pi} u(x, y) \, dy = U(x, 0) = \sum_{n=0}^{\infty} B_n \sin \lambda_n x \]

\[ B_n = \frac{2}{\pi} \int_{0}^{\pi} u(x, 0) \sin \lambda_n x \, dx = \frac{2}{\pi} \int_{0}^{\pi} x \cos \left( \frac{2n-1}{2} \pi x \right) \, dx = \frac{2}{\pi} \int_{0}^{\pi} \frac{x}{2n-1} \cos \left( \frac{2n-1}{2} \pi x \right) \, dx \]

\[ = \frac{2}{(2n-1)^2 \pi^2} \left[ x \sin \left( \frac{2n-1}{2} \pi x \right) \right]_{0}^{\pi} + \frac{1}{(2n-1)^2 \pi^2} \int_{0}^{\pi} \sin \left( \frac{2n-1}{2} \pi x \right) \, dx \]

\[ = \frac{2}{(2n-1)^2 \pi^2} \left( \pi \right) + \frac{1}{(2n-1)^2 \pi^2} \left[ \frac{-\cos \left( \frac{2n-1}{2} \pi x \right)}{\frac{2n-1}{2}} \right]_{0}^{\pi} \]

\[ = \frac{2}{(2n-1)^2 \pi^2} \left( \pi \right) + \frac{2}{(2n-1)^2 \pi^2} \left( \frac{1}{2} \right) \int_{0}^{2n-1} \sin \left( \frac{2n-1}{2} \right) x \, dx \]

\[ = \frac{2}{(2n-1)^2 \pi^2} \left( \pi \right) + \frac{2}{(2n-1)^2 \pi^2} \left( \frac{1}{2} \right) \left[ -\cos \left( \frac{2n-1}{2} \pi x \right) \right]_{0}^{\pi} \]

\[ = \frac{2}{(2n-1)^2 \pi^2} \left( \pi \right) + \frac{2}{(2n-1)^2 \pi^2} \left( \frac{1}{2} \right) \left[ -\cos \left( \frac{2n-1}{2} \pi \right) \right]_{0}^{\pi} \]

\[ = \frac{2}{(2n-1)^2 \pi^2} \left( \pi \right) + \frac{2}{(2n-1)^2 \pi^2} \left( \frac{1}{2} \right) \left[ 0 \right]_{0}^{\pi} \]

\[ = \frac{2}{(2n-1)^2 \pi^2} \left( \pi \right) \]
\[ \begin{align*}
Q.2. \quad V_{rr} + \frac{1}{r} V_r + \frac{1}{r^2} V_{\theta \theta} &= 0 \\
V(r, 0) &= 0 \quad V_{\theta \theta}(r, \pi / 2) = 0 \\
V(\rho, \theta) &= \sin \theta \quad V(\rho, \theta) = 0
\end{align*} \]

Let \( V(r, \theta) = R(r) \Theta(\theta) \)

\[ \frac{x^2}{R} \frac{\Theta'' + \lambda^2 \Theta}{\Theta} = 0 \]

\[ \Theta'' + \lambda^2 \Theta = 0 \quad \Theta(0) = 0 = \Theta'(\pi / 2) \]

\[ \lambda_n = (2n + 1) / (2n + 3) \quad n = 0, 1, 2, \ldots \]

\[ \Theta_n = \sin \lambda_n \theta = \sin(\pi n / 2) \theta \]

\[ R(\lambda + 0) \frac{R'' + \lambda^2 R}{R} = 0 \]

\( R(\lambda + 0) = \theta n = 0 \)

\[ \begin{align*}
R(r) &= A r^{\lambda} + \alpha r^{-\lambda} \\
0 &= R(0) = A_0^{\lambda} + \alpha b^{-\lambda} \quad \Rightarrow \quad \alpha = -A b^{2 \lambda}
\end{align*} \]

\[ \begin{align*}
V(r, \theta) &= \sum_{n=0}^{\infty} A_n \left[ r^{\lambda_n} - b^{2 \lambda_n} r^{-\lambda_n} \right] \sin(\lambda_n \theta)
\end{align*} \]

\[ \sin \theta = V(\rho, 0) = \sum_{n=0}^{\infty} A_n \left[a^{\lambda_n} - b^{2 \lambda_n} a^{-\lambda_n} \right] \sin(2n + 1) \theta
\]

\[ \begin{align*}
A_0 &= 0 \quad n \neq 0 \quad A_2 (a^{\lambda_2} - b^{2 \lambda_2} a^{-\lambda_2}) = 1 \\
\alpha &= \frac{a^{\lambda_2} - b^{2 \lambda_2} a^{-\lambda_2}}{b^{\lambda_2} (a / b)^{2 \lambda_2} - (b / a)^{2 \lambda_2}}
\end{align*} \]

\[ \begin{align*}
V(r, \theta) &= \frac{r^{\lambda_2} - b^{2 \lambda_2} r^{-\lambda_2}}{b^{\lambda_2} (a / b)^{2 \lambda_2} - (b / a)^{2 \lambda_2}} \sin \lambda_2 \theta \\
&= \frac{(r / b)^{\lambda_2} - (b / r)^{\lambda_2}}{(a / b)^{2 \lambda_2} - (b / a)^{2 \lambda_2}} \sin \lambda_2 \theta \\
&= \frac{(r / b)^{\lambda_2} - (b / r)^{\lambda_2}}{(a / b)^{3} - (b / a)^{3}} \sin \lambda_2 \theta
\end{align*} \]
Problem 3 \[ \frac{d^2 \phi}{dx^2} - 6 \frac{d \phi}{dx} + \lambda \phi = 0 \quad 0 < x < L \]
\[ \phi(0) = 0 = \phi(L) \]

a) \[ F(x) = e^{\frac{6x}{2}} = e^{6x} \]

\[ \frac{d^2 \phi}{dx^2} - 6 \frac{d \phi}{dx} + \lambda \phi = 0 \]
\[ \frac{d^2 \phi}{dx^2} = (e^{6x})' = 6e^{6x} \phi \]

b) Homogeneous \[ \phi'' + 6 \phi' + \lambda \phi = 0 \]
\[ \phi = e^{-7x} \Rightarrow T^2 + 6T + \lambda = 0 \]
\[ T = \frac{-6 \pm \sqrt{36 - 4 \lambda}}{2} = -3 \pm \sqrt{9 - \lambda} = -3 \pm 3 \quad \lambda = 9 \]

\[ \lambda < 0 : \phi = A e^{(-3+\delta)x} + B e^{(-3-\delta)x} \]
\[ \phi(0) = A + B = 0 \Rightarrow B = -A \]
\[ \phi(L) = A e^{(-3+\delta)L} = 2A e^{-3L} \sinh(8L) = 0 \]

Since \( \delta \neq 0 \), the only solution is the trivial solution.

\[ \lambda = 9 : \phi = A e^{-3x} + B x e^{-3x} \]
\[ \phi(0) = A = 0 \]
\[ \phi(L) = B L e^{-3L} = 0 \Rightarrow B = 0 \text{ trivial solution} \]

\[ \lambda > 0: \quad \text{let} \quad T = -3 \pm \sqrt{9 - \lambda} = -3 \pm \mu \quad \text{where} \quad \mu = \sqrt{\lambda - 9} \]

Then \[ \phi(x) = C e^{-3x} [A \cos \mu x + B \sin \mu x] \quad \lambda = \mu^2 \]
\[ 0 = \phi(0) = A \Rightarrow A = 0 \]
\[ 0 = \phi(L) = C e^{-3L} B \sin \mu L = 0 \quad \mu L = n \pi \quad n = 1, 2, \ldots \]

Eigenvalues are \[ \lambda = 9 + (n \pi)^2 \quad n = 1, 2, \ldots \]

Eigenfunctions \[ \phi_n(x) = C e^{-3x} \sin \mu x \]

\[ \phi_n(x) = \sin(n \pi x) \]

\[ \int_0^L e^{6x} \phi_m(x) \phi_n(x) dx = \int_0^L e^{6x} [C e^{-3x} \sin(mx)] [C e^{-3x} \sin(nx)] dx \]
\[ = \int_0^L \sin((m-n) \pi x) \sin(n \pi x) dx = 0 \]
Problem 4:

(a) Reduce this problem to the form of a Sturm-Liouville eigenvalue problem. Determine the eigenvalues and corresponding eigenfunctions. [8 marks]

(b) Use the eigenfunctions in (a) to solve the following mixed boundary value problem for Laplace's equation on the quarter annular region:

\[ u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \quad 0 < r < 2, \quad 0 < \theta < \pi/2 \]

\[ u(r,0) = 0 \quad \text{and} \quad \frac{\partial u(r,\pi/2)}{\partial \theta} = f(r) \]

\[ u(1,\theta) = 0 \quad \text{and} \quad \frac{\partial u(2,\theta)}{\partial r} = 0 \]

\[ \lambda \in \mathbb{R} \]

[12 marks]

[total 20 marks]
Problem 5: \( u_t + x^2 u_{xx} + 4 x u_x \mid 0 < x < 2 \) \( t > 0 \)

\[ u(1, t) = 1 \quad u(2, t) = 1 \]
\[ u(x, 0) = 1 - 5 x^{-3/2} \]

Look for a steady solution \( w(x) \): \( x^2 w_{xx} + 4 x w_x = 0 \)

\[ w = x^T \Rightarrow T(t-1) + 4T = T^2 + 3T = 0 \Rightarrow T(t+3) = 0 \quad \sqrt{t} = 0, -3 \]

\[ w = A + B x^{-3} \]
\[ w(1) = A + B = 1 \quad A = 1 - B \]
\[ w(2) = 2^{-3} = \frac{1}{8} \Rightarrow B(1/8 - 1) = 0 \quad B = 0 \]

\[ \therefore w(x) = 1 \text{ is a steady soln -- this can be seen by inspection} \]

Let \( u(x, t) = w(x) + v(x, t) \)

\[ u_t = \left( \frac{v}{x^2} + v \right)_t = x^2 u_{xx} + 4 x u_x = \left( x^2 w_{xx} + 4 x w_x \right) + x^2 v_{xx} + 4 x v_x \]

\[ v_t = x^2 v_{xx} + 4 x v_x \]

\[ f = u(1, t) = w(1) + v(1, t) = 1 + v(1, t) \Rightarrow \sqrt{t} = 0 \]
\[ v(1, t) = 0 \]

\[ f = u(2, t) = w(2) + v(2, t) = 1 + v(2, t) \Rightarrow \sqrt{t} = 0 \]
\[ v(2, t) = 0 \]

\[ \therefore 5 x^{-3/2} = u(x, 0) = 1 + v(x, 0) \Rightarrow v(x, 0) = -5 x^{-3/2} \]

Now let \( u(x, t) = x \sqrt{2} x^2 4 x \sqrt{2} \sqrt{2} \sqrt{2} = -\lambda \]

\[ T = \sqrt{T} \Rightarrow T(0) = C e^{-\lambda t} \]

\[ x^2 x^2 + 4 x \lambda x + \lambda x = 0 \]
\[ x(1) = 0 = x(2) \]

Let's write in S-C form \( f(x) = \frac{e^{4 x x^2}}{2 x^4 / x^2} = e^{4 x x^2 / x^2} \lambda x^2 = x^2 \]

\[ x^4 x^2 + 4 x \lambda x^2 + \lambda x^2 x = (4 x^2 z) + \lambda x^2 x = 0 \]

homog. CE: \( f(x) = \lambda x^2 x = 0 \Rightarrow x(1) = 4 x + \lambda = \lambda^2 + 3 x + \lambda = 0 \]

\[ \lambda = -3 \pm \sqrt{9 - 4 \lambda^2} = -3 \pm \frac{1}{2} (9 - 4 \lambda^2) \]

\[ \lambda < \frac{\lambda}{4} \quad \lambda = x^{3/2} (x + C_1 x^3 + C_2 x^5) \quad \Rightarrow \theta = \frac{1}{2} \sqrt{9 - 4 \lambda^2} \]

\[ x(1) = C_1 + C_2 = 0 \quad C_2 = -C_1 \]

\[ x(2) = 2^{3/2} C_1 (2^2 - 2^3) = 0 \]

Either \( C_1 = 0 \) or \( 2^2 - 2^3 = 0 \Rightarrow 2^6 = 1 \Rightarrow 2 \theta = 0 \Rightarrow \theta = 0 \text{ which contradicts } \lambda < \frac{\lambda}{4} \]

\[ \therefore C_1 = 0 \text{ so the only soln is the trivial soln} \]
\[ \lambda = \frac{q^2}{4} : \sum X = C_2 x^{-3/2} + C_2^{-3/2} \]  
\[ X(1) = C_1 = 0 \]
\[ X(2) = C_2 2^{-3/2} \]  
\[ \Rightarrow C_2 = 0 \]

**Only solution is the trivial solution.**

\[ \lambda > \frac{3}{4} \]: Let \( Y = -\frac{3}{2} \pm \frac{1}{2} \sqrt{4\lambda - \frac{9}{4}} = -\frac{3}{2} \pm i \mu \) where \( \mu = \frac{1}{2} \sqrt{4\lambda - \frac{9}{4}} \)

\[ 4\mu^2 = 4\lambda - \frac{9}{4} \Rightarrow \mu = \frac{\sqrt{4\lambda - \frac{9}{4}}}{2} \]

\[ X = \frac{1}{2} \left[ A \cos(\mu \omega x) + B \sin(\mu \omega x) \right] \]

\[ X(1) = A \cos(\mu \omega x) = A = 0 \]
\[ X(2) = B 2^{-3/2} \sin(\mu \omega x) = 0 \Rightarrow \mu n = n \pi \quad n = 1, 2, \ldots \]

\[ \sum n = \frac{x^{-3/2} \sin(n \pi \omega x)}{\mu \omega x} \]

\[ Y(x,t) = \sum_{n=1}^{\infty} C_n e^{-\lambda n^2} x^{-3/2} \sin\left(n \pi \omega x \right) = \frac{9 + (\pi n)^2}{4} \frac{1}{(\omega x)^2} \]

\[ V(x,0) = -5 x^{-3/2} \sum_{n=1}^{\infty} C_n x^{-3/2} \sin\left(n \pi \omega x \right) \]

\[ -5 \int_{0}^{\frac{1}{2}} x^{-3/2} \sin(\mu \omega x) dx = \int_{1}^{2} x^{-1} \sin(\mu \omega x) dx = \sum_{n=1}^{\infty} C_n \int_{1}^{2} x^{-1} \sin(\mu \omega x) dx \]

\[ -5 \int_{0}^{\frac{1}{2}} \sin(\mu \omega x) dx = \sum_{n=1}^{\infty} C_n \int_{0}^{\frac{1}{2}} \sin(\mu \omega x) dx \]

\[ 5 \cos\left(\frac{n \pi x}{\omega x}\right) \chi_{2} = \frac{1}{n \pi} \sum_{n=1}^{\infty} C_n \sin\left(\frac{n \pi x}{\omega x}\right) \]  
\[ \Rightarrow 5(\cos(\pi n) - \frac{1}{2}) = C_n \frac{\omega x}{2} \]

\[ \Rightarrow C_n = \frac{10}{\omega x} \left[ \frac{(-1)^n}{1} + \frac{10}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n \pi} \frac{(\frac{n \pi x}{\omega x})^2}{\left(\frac{n \pi x}{\omega x}\right)^2} \right] x^{-3/2} \sin\left(n \pi \omega x \right) \]  

\[ \Rightarrow \]
Problem 6:

Consider the eigenvalue problem

\[ L\phi = x^2 \phi'' + x \phi' + \lambda \phi = 0 \]
\[ \phi'(1) = 0 = \phi'(e^\pi) \]

(a) Reduce this problem to the form of a Sturm-Liouville eigenvalue problem. Determine the eigenvalues and corresponding eigenfunctions. [8 marks]

(b) Use the eigenfunctions in (a) to solve the following mixed boundary value problem for Laplace's equation on the semi-annular region:

\[ u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \quad 1 < r < e^\pi, \quad 0 < \theta < \pi \]
\[ u(r, 0) = 0 \quad \text{and} \quad u(r, \pi) = f(r) \]
\[ \frac{\partial u(1, \theta)}{\partial r} = 0 \quad \text{and} \quad \frac{\partial u(e^\pi, \theta)}{\partial r} = 0 \]

(a) \[ F = \xi x + \eta = \frac{\xi x^3}{x^2} = \lambda \phi = \frac{\phi'' + \phi'}{x} + \lambda \phi = 0 \quad \text{which is } \phi_n = \sin(\kappa r) \quad \text{[12 marks]}

(b) \[ \phi_n = n \sin(n \theta) \quad \text{is a Cauchy Soln.} \quad \text{Let's look for solutions of the form } \phi = x^{\lambda} \quad \text{where } \lambda = \kappa > 0 \]

\[ \phi(\kappa) = x^{\kappa} \quad \Rightarrow \quad \lambda = \kappa > 0 \quad \text{and} \quad \lambda = n \pi \quad \text{when } \mu = \kappa > 0 \]

\[ \sin(n \theta) = 0 \quad \text{or} \quad e^{\pi r} = n \pi \quad \text{Thus for } \lambda = 0 \quad R = 1 \]

\[ \Theta(r) = A r + B \ln r \quad \Rightarrow \quad \Theta(0) = 0 \quad \Rightarrow \quad \Theta(r) = 0 \]

\[ R \equiv \cos(n \theta) \quad \text{where} \quad n = 0, 1, 2, \ldots \]

\[ u(r, \theta) = \sum_{n=0}^{\infty} B_n \sin(n \pi r) \cos(n \pi \theta) \]

\[ \Theta(\pi) = A = 0 \quad \Rightarrow \quad \Theta(0) = 0 \quad \Rightarrow \quad \Theta_r(0) = 0 \]

\[ \phi_n = n \sin(n \theta) \quad \text{[total 20 marks]} \]

Thus the eigenfunctions and eigenvalues are: \( \lambda_n = n \pi \quad \text{and} \quad \phi_n = \sin(n \theta) \)

\[ \alpha = 0 \quad \phi(0) = 0 \quad \Rightarrow \quad \Theta(0) = 0 \quad \Rightarrow \quad \Theta_r(0) = 0 \quad \Rightarrow \quad \Theta(0) = 0 \]

\[ R = \cos(n \theta) \quad \text{where} \quad n = 0, 1, 2, \ldots \]

\[ u(r, \theta) = \sum_{n=0}^{\infty} B_n \sin(n \pi r) \cos(n \pi \theta) \]

\[ \Theta(\pi) = A = 0 \quad \Rightarrow \quad \Theta(0) = 0 \quad \Rightarrow \quad \Theta_r(0) = 0 \]

\[ \phi_n = n \sin(n \theta) \quad \text{[total 20 marks]} \]
Problem 7:
We wish to determine how long a steel beam will take to lose its structural integrity when one end is subjected to a fire of increasing intensity. Consider the following one dimensional model in which the left boundary condition represents the heat flux due to the fire and the right boundary condition represents the heat lost to the environment. Solve the inhomogeneous heat conduction problem subject to time dependent boundary conditions:

\[ u_t = u_{xx} - x, \quad 0 < x < 1, \quad t > 0 \]
\[ u_x(0, t) = -t, \quad \text{and} \quad \frac{\partial u(1, t)}{\partial x} = -u(1, t) \]
\[ u(x, 0) = x^2 \]

(a) Determine a simple function \( w(x, t) \) that satisfies the inhomogeneous boundary conditions. [4 marks]

(b) Now let \( u(x, t) = w(x, t) + v(x, t) \) and determine the boundary value problem satisfied by \( v(x, t) \). [4 marks]

(c) Now determine a steady-state solution \( \omega(x) \) for the equation for \( v(x, t) \). Let \( v(x, t) = \omega(x) + \phi(x, t) \), and determine the boundary value problem satisfied by \( \phi(x, t) \). [4 marks]

(d) Complete the solution to the problem by using separation of variables to solve the boundary value problem for \( \phi(x, t) \). Determine the equation satisfied by the eigenvalues and illustrate the solutions graphically - you need not obtain an explicit expression for the eigenvalues. [8 marks]
Thus \( \phi(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A_n e^{-\mu_n^2 t} \cos \mu_n x \ \text{where} \ \frac{1}{\mu_n} = \tan \frac{\mu_n}{2} \quad n=0,1,2,\ldots \)

Now \( \Phi(x,0) = \sum_{n=0}^{\infty} A_n \cos \mu_n x \)

\[
\int_0^1 3 \cos \mu_n x \, dx = \sum_{n=1}^{\infty} A_n \int_0^1 \cos \mu_n x \cos \mu_n x \, dx
\]

Now by S-L theory \( \int_0^1 \cos \mu_n x \cos \mu_m x \, dx = 0 \) if \( m \neq n \) and if \( m = n \) we have

\[
\int_0^1 \cos^2 \mu_n x \, dx = \frac{1}{2} \int_0^1 (1 + \cos 2\mu_n x) \, dx = \frac{1}{2} \left[ \frac{1 + \sin 2\mu_n}{2\mu_n} \right] = \frac{1}{2} \left[ 1 + \frac{\sin \mu_n \cos \mu_n}{\mu_n} \right]
\]

\[
= \frac{1}{2} \left[ 1 + \frac{\sin^2 \mu_n}{\mu_n} \right] \ \text{since} \ \frac{\cos \mu_m}{\mu_m} = \frac{\sin \mu_m}{\mu_m}
\]

\[
\therefore \ A_n = \frac{2.3}{1 + \sin^2 \mu_m} \int_0^1 \cos \mu_n x \, dx = \frac{6}{\mu_n(1 + \sin^2 \mu_m)} \sin \mu_n x \bigg|_0^1 = \frac{6\sin \mu_m}{\mu_m(1 + \sin^2 \mu_m)}
\]

\[
\therefore \ u(x,t) = (x^2 - 3) + 6 \sum_{n=0}^{\infty} \frac{\sin \mu_n}{\mu_n(1 + \sin^2 \mu_m)} e^{-\mu_n^2 t} \cos \mu_n x
\]