Math 257/316 Assignment 9 Due Friday November 23 in class

Problem 1 (don’t hand in): Find the solution of Laplace’s equation in the semi-infinite strip \( \{(x, y); 0 \leq x \leq 2, \ y \geq 0\} \) satisfying the following mixed boundary conditions:

\[
\begin{cases}
    u(0, y) = 0, & \text{for all } y \geq 0 \\
    u(x, 0) = 2\sin(3\pi x/4) - 3\sin(7\pi x/4) & \text{for all } 0 \leq x \leq 2, \\
    \lim_{y \to +\infty} u(x, y) = 0
\end{cases}
\]

Problem 2 (don’t hand in): A metal plate occupies a quarter-annular region \( 0 < a \leq r \leq b \) and \( 0 \leq \theta \leq \pi/2 \). The vertical face is insulated while the horizontal face and the outer hoop are maintained at 0 degrees. The inner hoop is maintained at a temperature of \( \sin(3\theta) \). Determine the steady state temperature by solving the following BVP in \( \Omega \):

\[
\begin{cases}
    v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = 0 & \text{in } \Omega \\
    v(r, 0) = 0 & \text{for } a < r < b, \quad v_{\theta}(r, \pi/2) = 0 & \text{for } a < r < b \\
    v(a, \theta) = \sin 3\theta & \text{for } 0 < \theta < \pi/2, \quad v(b, \theta) = 0 & \text{for } 0 < \theta < \pi/2
\end{cases}
\]

Problem 3 (don’t hand in): Consider the BVP:

\[
\phi'' + 6\phi' + \lambda \phi = 0, \quad 0 < x < L \\
\phi(0) = 0 \\
\phi(L) = 0
\]

(a) Put this BVP into Sturm-Liouville form.

(b) Compute all eigenvalues and eigenfunctions.

(c) Show explicitly that the eigenfunctions are mutually orthogonal.

(Don’t forget to include the weight function inside the integral.)

Problem 4 (don’t hand in): Consider the eigenvalue problem

\[
x^2 y'' + xy' + \lambda y = 0 \\
y(1) = 0 = y'(2)
\]

a. Reduce this problem to the form of a Sturm-Liouville eigenvalue problem. Determine the eigenvalues and corresponding eigenfunctions.

b. Use the eigenfunctions in (a) to solve the following mixed boundary value problem for Laplace’s equation on the quarter-annular region:

\[
\begin{cases}
    u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, & 1 < r < 2, \quad 0 < \theta < \pi/2 \\
    u(r, 0) = 0 \text{ and } \frac{\partial u(r, \pi/2)}{\partial \theta} = f(r) \\
    u(1, \theta) = 0 \text{ and } \frac{\partial u(2, \theta)}{\partial r} = 0
\end{cases}
\]

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Problem 5 (don’t hand in):
Solve the following heat conduction problem:

\[ u_t = x^2 u_{xx} + 4x u_x \quad \text{for } x \in (1, 2), \ t > 0 \]
\[ u(1, t) = 1 \quad u(2, t) = 1 \]
\[ u(x, 0) = 1 - 5x^{-3/2} \]

Problem 6 (hand in):
Consider the eigenvalue problem

\[ x^2 y'' + xy' + \lambda y = 0 \]
\[ y'(1) = 0 = y'(e^\pi) \]

1. Reduce this problem to the form of a Sturm-Liouville eigenvalue problem. Determine the eigenvalues and corresponding eigenfunctions.

2. Use the eigenfunctions in (a) to solve the following mixed boundary value problem for Laplace’s equation on the semi-annular region:

\[ u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \quad 1 < r < e^\pi, \quad 0 < \theta < \pi \]
\[ u(r, 0) = 0 \quad \text{and} \quad u(r, \pi) = f(r) \]
\[ \frac{\partial u(1, \theta)}{\partial r} = 0 \quad \text{and} \quad \frac{\partial u(e^\pi, \theta)}{\partial r} = 0 \]

Problem 7 (hand in):
We wish to determine how long a steel beam will take to lose its structural integrity when one end is subjected to a fire of increasing intensity. Consider the following one dimensional model in which the left boundary condition represents the heat flux due to the fire and the right boundary condition represents the heat lost to the environment. Solve the inhomogeneous heat conduction problem subject to time dependent boundary conditions:

\[ u_t = u_{xx} - x, \quad 0 < x < 1, \ t > 0 \]
\[ u_x(0, t) = -t, \quad \text{and} \quad \frac{\partial u(1, t)}{\partial x} = -u(1, t) \]
\[ u(x, 0) = x^2. \]

1. Determine a simple function \( w(x, t) \) that satisfies the inhomogeneous boundary conditions.
2. Now let \( u(x, t) = w(x, t) + v(x, t) \) and determine the boundary value problem satisfied by \( v(x, t) \).

3. Now determine a steady-state solution \( \omega(x) \) for the equation for \( v(x, t) \). Let \( v(x, t) = \omega(x) + \phi(x, t) \), and determine the boundary value problem satisfied by \( \phi(x, t) \).

4. Complete the solution to the problem by using separation of variables to solve the boundary value problem for \( \phi(x, t) \). Determine the equation satisfied by the eigenvalues and illustrate the solutions graphically - you need not obtain an explicit expression for the eigenvalues.