ASSIGNMENT 5: MATH 257-316

1. $$u_t = \alpha^2 u_{xx} \qquad (1)$$

BC: $\dfrac{\partial u}{\partial x}(0,t) = 0 \qquad \dfrac{\partial u}{\partial x}(\pi,t) = 0$

IC: $u(x,0) = \cos\gamma x. \qquad 0 < x < \pi.$

LET $u(x,t) = X(x)T(t) \Rightarrow \dot{T}/\alpha T(t) = X''(x)/X(x) = \text{CONST} = -\lambda^2$

$T>$ $\dot{T} = -\alpha^2\lambda^2 T \Rightarrow T(t) = Ce^{-\alpha^2\lambda^2 t}$

$X>$ $\left.\begin{array}{l} X'' + \lambda^2 X = 0 \\ X'(0) = 0 = X'(\pi) \end{array}\right\} \Rightarrow \boxed{\lambda \neq 0}$: $X = A\cos\lambda x + B\sin\lambda x$

$X' = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$

$X'(0) = B\lambda = 0 \overset{\lambda\neq 0}{\Longrightarrow} B = 0$

$X'(\pi) = -A\lambda\sin\lambda\pi = 0 \xrightarrow[\text{NONTRIVIAL SOLN}]{\lambda \neq 0} \lambda = n,\ n = 1, 2, \ldots$

THUS $\lambda_n = n \quad n = 1, 2, \ldots$ ARE THE EIGENVALUES AND

$X_n = \cos(nx)$ ARE THE CORRESPONDING EIGENFUNCTIONS.

$\boxed{\lambda = 0}$ $X = Ax + B \qquad X' = A$

$X'(0) = A = 0$ SO THAT THE BC $X'(\pi)$ HOLDS AUTOMATICALLY.

$\therefore X_0(x) = B\cdot 1$ IS THE NONTRIVIAL EIGENFUNCTION

CORRESPONDING TO $\lambda_0 = 0$

BY SUPERPOSITION (BECAUSE (1) IS LINEAR

$$u(x,t) = A_0 1 + \sum_{n=1}^{\infty} A_n\cos(nx)\, e^{-\alpha^2 n^2 t}$$

NOW $\cos\gamma x = u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n\cos(nx)$

WHICH IS JUST A HALF-RANGE FOURIER COS SERIES $A_0 = \frac{a_0}{2}$, $A_n = a_n$, $L = \pi$.

$$a_0 = \frac{2}{\pi}\int_0^\pi \cos\gamma x\,dx = \frac{2}{\pi\gamma}\sin\gamma x\Big|_0^\pi = \frac{2\sin\gamma\pi}{\gamma\pi} \qquad \gamma \notin \mathbb{Z} = \{\ldots -1, 0, 1, 2, \ldots\}$$

$$= 0 \qquad \gamma \in \mathbb{Z}\setminus\{0\}$$

$$= 2 \qquad \gamma = 0.$$

$\gamma \notin \mathbb{Z}$ $\quad a_n = \dfrac{2}{\pi}\displaystyle\int_0^\pi \cos\gamma x\cos(nx)\,dx \overset{\gamma\notin\mathbb{Z}}{=} \dfrac{1}{\pi}\int_0^\pi \cos(\gamma+n)x + \cos(\gamma-n)x\,dx$

$$= \frac{2}{\pi}\left\{\frac{\sin(\gamma+n)\pi}{(\gamma+n)} + \frac{\sin(\gamma-n)\pi}{(\gamma-n)}\right\}$$

$\gamma \in \mathbb{Z}$ $\quad a_n = \dfrac{2}{\pi}\displaystyle\int_0^\pi \cos\gamma x\cos(nx)\,dx = \begin{cases} 0 & \gamma \neq n \\ 1 & \gamma = n \end{cases}$

$\therefore \gamma = 0 \in \mathbb{Z}$: $\quad u(x,t) = 1$

$\gamma = n \in \mathbb{Z}$: $\quad u(x,t) = e^{-\alpha^2 n^2 t}\cos(nx)$

$\gamma \notin \mathbb{Z}$: $\quad u(x,t) = \dfrac{\sin(\gamma\pi)}{(\gamma\pi)} + \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\left\{\frac{\sin(\gamma+n)\pi}{(\gamma+n)} + \frac{\sin(\gamma-n)\pi}{(\gamma-n)}\right\}\cos nx\, e^{-\alpha^2 n^2 t}$

2/

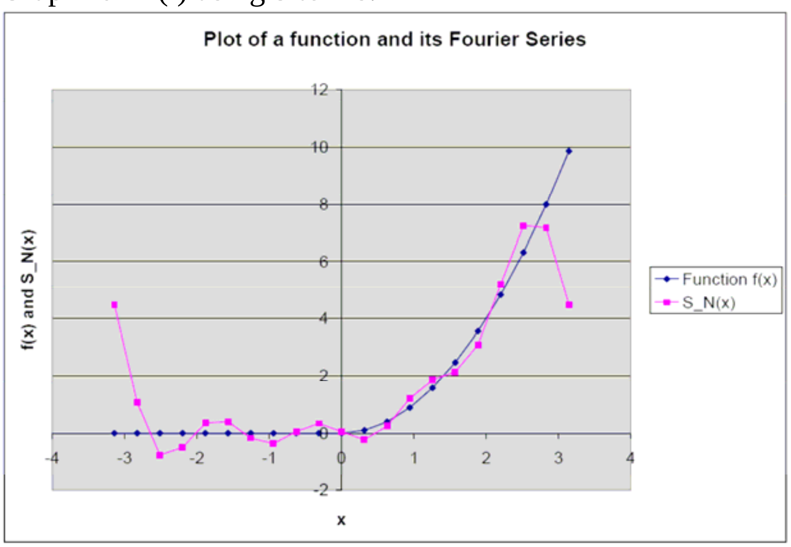
$$u(\pi,0) = \cos\gamma\pi = \frac{\sin\gamma\pi}{\gamma\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} (\gamma-n)\frac{\{\sin\gamma\pi\cos n\pi + \cos\gamma\pi\sin(n\pi)\}\cos(n\pi)}{\gamma^2-n^2}$$

$$+ (\gamma+n)\frac{\{\sin\gamma\pi\cos n\pi - \cos\gamma\pi\sin(n\pi)\}\cos(n\pi)}{\gamma^2-n^2}$$

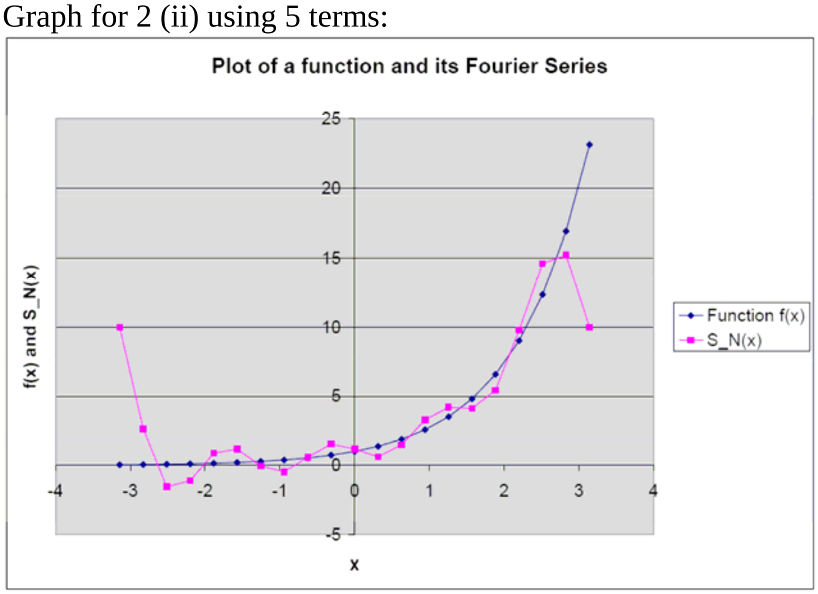
$$= \frac{\sin\gamma\pi}{\gamma\pi} + \frac{2\sin\gamma\pi}{\pi}\sum_{n=1}^{\infty}\frac{\{\gamma-n+\gamma+n\}(\cos(n\pi))^2}{\gamma^2-n^2}$$

$$\therefore \quad \cot\gamma\pi = \frac{1}{\pi}\left\{\frac{1}{\gamma} - \sum_{n=1}^{\infty}\frac{2\gamma}{n^2-\gamma^2}\right\}$$

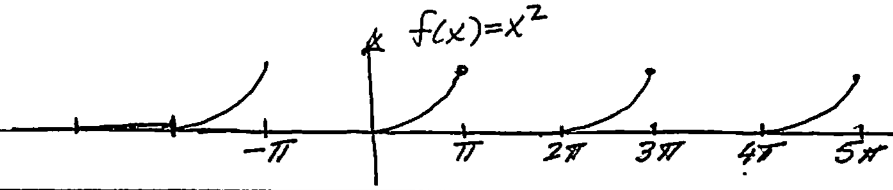
Graph for 2 (i) using 5 terms:



Graph for 2 (ii) using 5 terms:



2. (i)



$L=\pi$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{1}{\pi}\frac{\pi^3}{3} = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx = \frac{1}{\pi}\left\{ \frac{x^2\sin(nx)}{n}\Big|_0^{\pi} - \frac{2}{n}\int_0^{\pi} x\sin nx\,dx \right\}$$

$$= \frac{1}{\pi}\left\{ \frac{2x}{n^2}\cos(nx)\Big|_0^{\pi} - \frac{2}{n^2}\int_0^{\pi}\cos(nx)\,dx \right\}$$

$$= \frac{1}{\pi}\left\{ \frac{2}{n^2}\left[\pi(-1)^n - 0.1\right] - \frac{2}{n^3}\sin(nx)\Big|_0^{\pi} \right\} = 2(-1)^n/n^2$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x^2\sin(nx)\,dx = \frac{1}{\pi}\left\{ -\frac{x^2\cos(nx)}{n}\Big|_0^{\pi} + \frac{2}{n}\int_0^{\pi} x\cos(nx)\,dx \right\}$$

$$= \frac{\pi(-1)^{n+1}}{n} + \frac{2}{\pi n^2}\left\{ \frac{x\sin(nx)}{n}\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nx)\,dx \right\}$$

$$= \frac{\pi(-1)^{n+1}}{n} + \frac{2}{\pi n^3}\cos(nx)\Big|_0^{\pi}$$

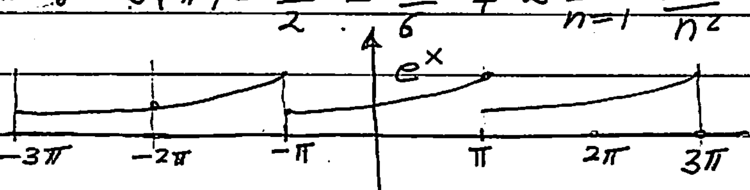
$$= \frac{\pi(-1)^{n+1}}{n} + \frac{2}{\pi n^3}\left[(-1)^n - 1\right]$$

$$\therefore\ f(x) \sim \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx) + \sum_{n=1}^{\infty}\left\{\frac{(-1)^{n+1}\pi}{n} + \frac{2}{\pi n^3}\left[(-1)^n-1\right]\right\}\sin nx = S(x)$$

NOW SET $x=0 \Rightarrow 0 = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} + 0 \Rightarrow \frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$

SET $x=\pi$: $S(\pi) = \frac{\pi^2}{2} = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2} + 0 \Rightarrow \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

(ii)



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{\pi}\left(e^{\pi} - e^{-\pi}\right) = 2\sinh\pi/\pi$$

NOW $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\cos nx\,dx = \frac{Re}{\pi}\int_{-\pi}^{\pi} e^{(1+in)x}\,dx$

AND $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\sin(nx)\,dx = \frac{Im}{\pi}\int_{-\pi}^{\pi} e^{(1+in)x}\,dx$

NOW $\int_{-\pi}^{\pi} e^{(1+in)x}\,dx = \frac{e^{(1+in)x}}{(1+in)}\Big|_{-\pi}^{\pi} = \frac{\left(e^{(1+in)\pi} - e^{-(1+in)\pi}\right)(1-in)}{(1+in)\times(1-in)}$

$$= \left[e^{\pi}(-1)^n - e^{-\pi}(-1)^n\right](1-in)/(1+n^2) = 2\sinh\pi\,(-1)^n(1-in)/(1+n^2)$$

$$\therefore\ f(x) = \frac{\sinh\pi}{\pi}\left\{1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\left(\cos nx - n\sin nx\right)\right\}$$

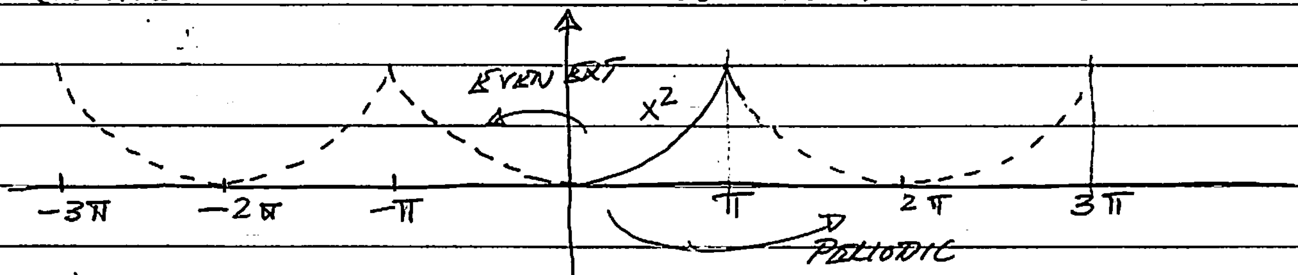
3. (a) $f(-x) = (-x)^2 + |-x| = x^2 + |x| = f(x)$ f IS EVEN.

(b) $f(-x) = e^{\sin^2(-x)} = e^{\sin^2 x} = f(x)$ EVEN.

(c) $f(-x) = \cosh(-x) + \sinh(-x) = \cosh x - \sinh x \neq f(x)$ NEITHER ODD OR EVEN

4. (a) $f(x) = x^2 \quad 0 < x < \pi$

(i) HALF-RANGE COSINE SERIES: WE WANT THE EVEN EXTENSION OF f:



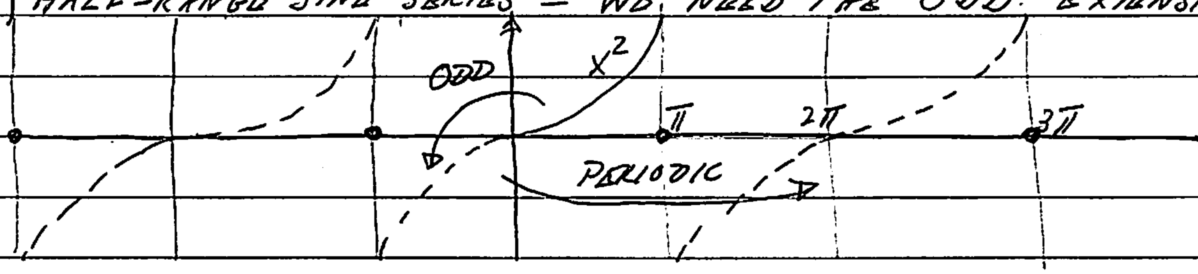
$b_n = 0$

$a_0 = \frac{2}{\pi}\int_0^\pi x^2 dx = \frac{2}{\pi}\frac{x^3}{3}\Big|_0^\pi = 2\pi^2/3$

$a_n = 2\left\{2(-1)^n/n^2\right\} = 4(-1)^n/n^2$ USING INTEGRAL IN 2(i)

$f_{EVEN}(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)$.

(ii) HALF-RANGE SINE SERIES - WE NEED THE ODD EXTENSION OF f:

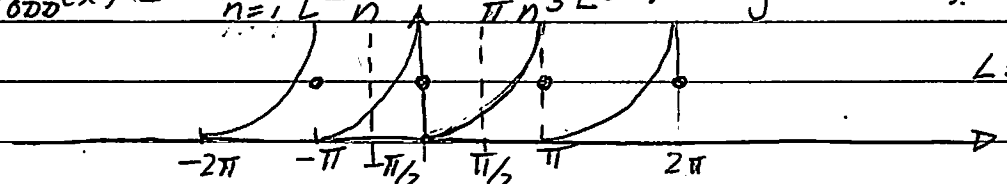


$a_n = 0$

$b_n = \frac{2}{\pi}\int_0^\pi x^2 \sin(nx)\,dx = 2\left\{\frac{\pi(-1)^{n+1}}{n} + \frac{2}{\pi n^3}\left[(-1)^n - 1\right]\right\}$.

$f_{ODD}(x) = 2\sum_{n=1}^{\infty}\left\{\frac{\pi(-1)^{n+1}}{n} + \frac{2}{\pi n^3}\left[(-1)^n - 1\right]\right\}\sin(nx)$.

(b)



$L = \pi/2$

$a_0 = \frac{1}{\pi/2}\int_{-\pi/2}^{\pi/2} f(x)dx = \frac{2}{\pi}\int_0^\pi x^2 dx = \frac{2}{\pi}\pi^3/3 = 2\pi^2/3$

$a_n = 2/\pi \int_0^\pi x^2 \cos\left(\frac{n\pi x}{\pi/2}\right)dx = \frac{2}{\pi}\int_0^\pi x^2 \cos(2nx)dx$

$= \frac{2}{\pi}\left[\frac{2x}{(2n)^2}\cos(2nx)\Big|_0^\pi - \frac{2}{(2n)^3}\sin(2nx)\Big|_0^\pi\right] = \frac{1}{n^2}$ (USING 2(i))

$b_n = \frac{2}{\pi}\int_0^\pi x^2 \sin(2nx)dx$

$$b_n = 2\left\{ \frac{-\pi}{(2n)} + \frac{2}{\pi(2n)^3} \cancel{\cos(2nx)} \Big|_0^{\pi} \right\} = -\frac{\pi}{n}$$ (USING PREVIOUS INTEGRALS FROM 2(i))

$$\therefore f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} - \pi \frac{\sin(nx)}{n}$$