Math 257/316 PDE Assignment 4
Due in class on Wednesday October 3, 2018

1. **Separation of variables**: Determine whether the method of separation of variables can be used to replace the following PDE’s by a pair of ODE’s. If so, find the equations.
   (a) $x^2u_{xx} = tu_t$.
   (b) $u_{xx} + (x + y)u_{yy} = 0$.

2. **Eigenvalue Problems**: Find all eigenvalues and corresponding eigenfunctions for the following problems
   (a) $y'' + \lambda y = 0$ \hspace{1cm} (0 < x < 1), \hspace{1cm} y'(0) = 0, \hspace{1cm} y(1) = 0.
   (b) $y'' + 2y' + \lambda y = 0$, \hspace{1cm} (0 < x < \pi), \hspace{1cm} y(0) = 0, \hspace{1cm} y(\pi) = 0.$

3. **Finite Difference Approximations**: Use Taylor’s series about the point $x$ for $f(x + \Delta x)$, and $f(x + 2\Delta x)$ to determine the order $p$ in each of the following finite difference approximations
   (a) $\frac{-3f(x) + 4f(x + \Delta x) - f(x + 2\Delta x)}{2\Delta x} = f'(x) + O(\Delta x^p)$
   (b) $\frac{f(x) - 2f(x + \Delta x) + f(x + 2\Delta x)}{\Delta x^2} = f''(x) + O(\Delta x^p)$

4. **(Will be marked) Modes of vibration of a cantilever beam.** The vibrations of an elastic bar of length $L$ are governed by the forth order partial differential equation
   \[ \alpha^2 \frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} = 0 \]
   (a) Assuming a sinusoidal time variation of the solution of the form $y(x, t) = \exp(i\omega t)w(x)$ show that the spatial component $w(x)$ satisfies
   \[ w'''' - \left( \frac{\omega}{\alpha} \right)^2 w = 0 \]  \hspace{1cm} (1)
   (b) Show that the general solution of the 4th order ODE (1) is of the form
   \[ w(x) = A \cos \mu x + B \sin \mu x + C \cosh \mu x + D \sinh \mu x \]
   (c) If the beam is clamped at the left endpoint $x = 0$ and can move freely at the other endpoint $x = L$, the cantilever beam is subject to the boundary conditions
   \[ w(0) = 0 = w'(0) \] \hspace{1cm} (2)
   \[ w''(L) = 0 = w'''(L) \]
   Determine the eigenvalues associated with the eigenvalue problem comprising (1) subject to the boundary conditions (2).
   **Hint:** First use the conditions $w(0) = 0 = w'(0)$ to show that $A = -C$ and that $B = -D$. Use these conditions to eliminate $C$ and $D$ and the remaining two conditions
$w''(L) = 0 = w'''(L)$ to arrive at a 2×2 system of equations for the remaining constants $A$ and $B$. Using the condition for a non-trivial solution to this system derive the following transcendental equation for the eigenvalues

$$\cos \mu L = -\frac{1}{\cosh \mu L} \quad (3)$$

(d) Now plot the left and right hand sides of (3) over the range $0 \leq \mu L \leq 5\pi$. Provide a graphical estimate for the smallest crossing point $\mu_1L$ say. Interpret this in terms of the lowest frequency of vibration of the beam. From the graph can you give an estimate of the larger frequencies of vibration, i.e. when $\mu_nL \gg 1$.

5. (Will be marked) Find all eigenvalues and corresponding eigenfunctions for the following boundary value problem $x^2y'' + xy' + \lambda y = 0, (1 < x < 2), y(1) = 0 = y'(2)$. Only consider the case $\lambda > 0$.

6. (Will be marked) Flux boundary condition: Consider the following boundary value problem for the heat equation

$$u_t = \alpha^2 u_{xx}, \quad 0 < x < 1, \quad t > 0, \quad \alpha^2 = 0.2$$

$BC : u(0,t) = 0, \quad u_x(1,t) = 0$

$IC : u(x,0) = \sin (\pi x/2)$

(a) Use the method of separation of variables to solve the above boundary value problem.

(b) EXCEL EXERCISE: As shown in class the insulated boundary condition at the right endpoint of the bar $x = 1$ can be approximated by the following difference quotient:

$$\frac{\partial u(1,t)}{\partial x} = \frac{u(1 + \Delta x, t) - u(1 - \Delta x, t)}{2\Delta x} = 0$$

This equation reduces to the condition: $u(1 + \Delta x, t) = u(1 - \Delta x, t)$. Now if $x_N = 1$ is the right endpoint of the bar, then $x = 1 + \Delta x$ which falls outside the bar! However, we can trick the finite difference scheme into imposing this boundary condition by introducing a fictitious meshpoint $x_{N+1} = x_N + \Delta x$ and forcing the value of the solution $u(x_N - \Delta x, t)$ at this point to be the same as $u(x_N + \Delta x, t)$ in accordance with the condition above. Implement this in the spreadsheet: Heat0.xls posted on the web site by placing these fictitious values in column W. Plot $u(x, t = 0.5)$ obtained using the numerical solution and that obtained by separating variables on the same plot, print it out and hand it in with your assignment.