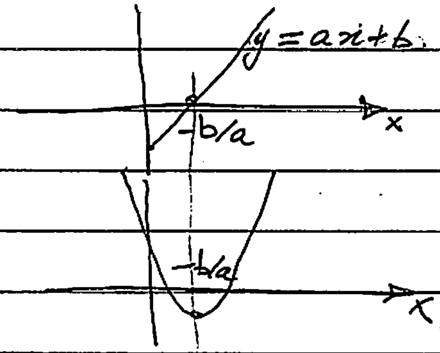
VARIATIONAL METHODS

IDEA: OFTEN A PROBLEM CAN BE CAST AS A MINIMIZATION PROBLEM



EG1: $ax + b \qquad x = b/a$

EQUIVALENT MINIMIZATION PROBLEM

$$E(x) = \frac{1}{2}ax^2 + bx$$

$$0 = E'(x) = ax + b$$

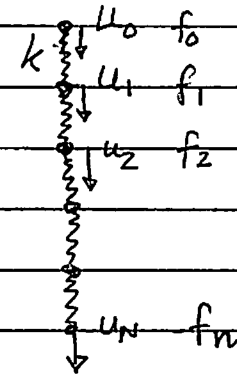
EG2: ASSUME A IS SYMMETRIC

$$Ax = b$$

$$E(x) = \frac{1}{2}x^T A x - x^T b$$

$$0 = \frac{\partial E}{\partial x} = Ax - b = 0.$$

EG3: EQUILIBRIUM OF A MASS SPRING SYSTEM



$$PE = E = \sum_{n=1}^{N} \frac{1}{2}k(u_n - u_{n-1})^2 + f_i u_i$$

$$0 = \frac{\partial E}{\partial x_m} = k(u_m - u_{m-1}) - k(u_{m+1} - u_m) + f_i$$

$$= -k\{u_{m+1} - 2u_m + u_{m-1}\} + f_i = 0$$

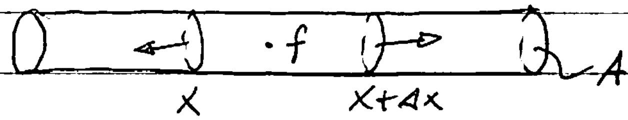
$$\left(\frac{k\Delta x}{A}\right) u'' = \frac{f_i}{A} \qquad\qquad F = ku$$

$$\sigma = \frac{F}{A} = \frac{k\,\Delta u}{A} = \left(\frac{k\,\Delta x}{A}\right)\frac{\Delta u}{\Delta x}$$

$$E u'' = p \qquad \text{EQUILIBRIUM EQ}$$

$$\sigma = E u_x$$

EG4 EQUILIBRIUM OF A BAR

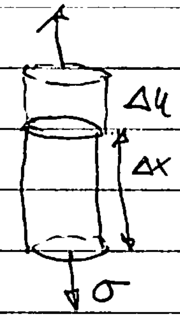


$$\sigma(x+\Delta x)A - \sigma(x)A = fA\Delta x$$

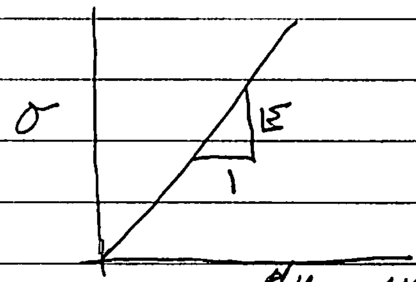
f BODY FORCE PER UNIT VOLUME

$$\frac{\sigma(x+\Delta x)-\sigma(x)}{\Delta x} = f$$

$$\frac{\partial \sigma}{\delta x} = f.$$



HOOKE'S LAW: $\sigma = E\dfrac{\partial u}{\partial x}$



$\therefore\; E u_{xx} = f.$ $\quad E$ CONSTANT

$(E u_x)_x = f \quad E(x)$

$$\frac{du}{dx} \simeq \frac{\Delta u}{\Delta x}$$

POTENTIAL ENERGY IN THE BAR:

$$V[u] = \int_0^l \tfrac{1}{2}E(u')^2 + fu\,dx \qquad u(0) = 0 = u(l)$$

V IS A FUNCTIONAL

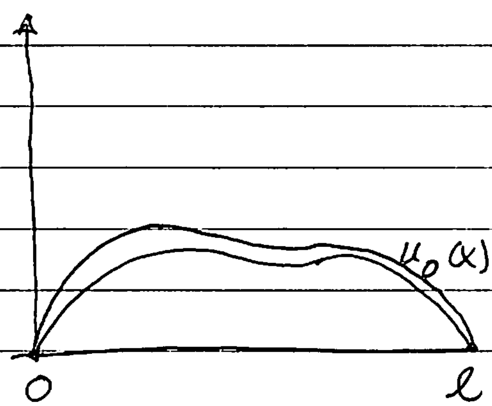
$$V[\cdot]: H^1 \longrightarrow \mathbb{R} \qquad H^1 = \{u;\ \int (u')^2 dx < \infty\}$$

HOW CAN WE MINIMIZE THE FUNCTIONAL $V[\cdot]$ WITH RESPECT TO u?

- EULER INTRODUCED A WONDERFUL DEVICE TO REDUCE THIS FUNCTIONAL MINIMIZATION PROBLEM TO A STANDARD CALCULUS PROBLEM
- ASSUME u_0 IS THE MINIMIZER THAT WE WANT.
- CONSIDER A 1-PARAMETER FAMILY OF FUNCTIONS DEFINED BY

$$u(x;\varepsilon) = u_0(x) + \varepsilon\eta(x)$$

WHERE $\eta(0) = 0 = \eta(l)$ AND IS SUFFICIENTLY SMOOTH FOR THE INTEGRALS TO EXIST BUT OTHERWISE η IS ARBITRARY.



NOW

$$V[u(x;\varepsilon)] = V[u_0+\varepsilon\eta] = \int_0^\ell \frac{E}{2}(u_0'+\varepsilon\eta')^2 + f(u_0+\varepsilon\eta)\,dx = V(\varepsilon)$$

WHERE $V(\varepsilon)$ IS A FUNCTION OF THE PARAMETER ε RATHER THAN A FUNCTIONAL $V[u]$.

TO LOCATE THE MINIMUM WE DIFFERENTIATE W.R.T. ε:

$$0 = \left.\frac{\partial V(\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} = \left[\int_0^\ell \frac{2E}{2}(u_0'+\varepsilon\eta')\cdot\eta' + f\,\eta\,dx\right]_{\varepsilon=0}$$

$$= \int_0^\ell E u_0'(x)\,\eta'(x) + f(x)\,\eta(x)\,dx$$

$$= \cancel{\left[E u_0'\,\eta\right]_0^\ell} - \int_0^\ell \eta\{(E u_0')' - f\}\,dx.$$

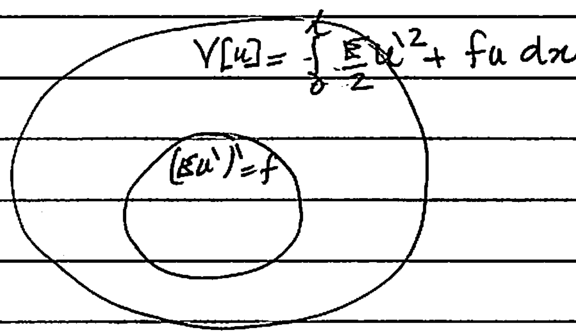
SINCE $\eta(0)=0=\eta(\ell)$ THE BOUNDARY TERMS VANISH
AND SINCE $\eta(x)$ IS ARBITRARY, WE MAY CHOOSE $\eta = \{(E u_0')' - f\}\,dx$
SO THAT

$$\int_0^\ell \{(E u_0')' - f\}^2\,dx = 0$$

THUS THE ONLY POSSIBILITY IS THAT

$$\boxed{(E u_0')' = f} \qquad (1)$$

- (1) IS A NECESSARY CONDITION FOR A MINIMUM i.e., IF $u_0(x)$ IS A MINIMIZER OF $V[u]$ THEN u_0 MUST SATISFY $(E u_0')' = f$.
- (1) IS KNOWN AS THE EULER-LAGRANGE EQUATION ASSOCIATED WITH MINIMIZING THE FUNCTIONAL $V[u]$



- DOES $u_0 : (Eu')' = f$ REALLY MINIMIZE $V[u]$?

CONSIDER ANY FUNCTION $u(x)$ SATISFYING THE BC $u(0) = 0 = u(\ell)$.

$$V[u] - V[u_0] = \int_0^\ell \frac{E}{2} u'^2 - \frac{E}{2} u_0'^2 + f(u - u_0)\, dx$$

NOW
$$\frac{1}{2}(u' - u_0')^2 = \frac{u'^2}{2} - \frac{2}{2} u' u_0' + \frac{u_0'^2}{2}$$
$$= \frac{u'^2}{2} - u_0'(u' - u_0') - \frac{u_0'^2}{2}$$

$$\therefore \frac{1}{2}(u' - u_0')^2 + u_0'(u' - u_0') = \frac{u'^2}{2} - \frac{u_0'^2}{2}$$

$$\therefore V[u] - V[u_0] = \int_0^\ell \frac{E}{2}(u' - u_0')^2 + E u_0'(u' - u_0') + f(u - u_0)\, dx$$
$$= \int_0^\ell \frac{E}{2}(u' - u_0')^2 - \cancel{\{(E u_0')' - f\}}(u - u_0)\, dx + E u_0' \cancel{(u - u_0)}\Big|_0^\ell$$
$$= \int_0^\ell \frac{E}{2}(u' - u_0')^2 dx \geq 0.$$

- WHY COMPLICATE THE DIFFERENTIAL EQUATION PROBLEM BY CONVERTING IT TO A MINIMIZATION PROBLEM?
  - USEFUL FOR APPROXIMATION → RAYLEIGH–RITZ METHOD
  - NOTICE THAT $V[u]$ IS DEFINED ON A BROADER CLASS OF FUNCTIONS THAN THE ODE WHICH REQUIRES THAT u BE TWICE DIFFERENTIABLE.

APPROXIMATION – THE RAYLEIGH-RITZ METHOD

CONSIDER $Lu = u'' = f$

$$u(0)=\alpha \qquad u(1)=\beta$$

VARIATIONAL CALCULUS, CONSIDER VARIATIONS $\delta I, \delta u$.

$$I[u] = \int_0^1 \tfrac{1}{2} u'^2 + f u \, dx$$

$$0 = \delta I[u] = \int_0^1 u' \cdot \delta u' + f \, \delta u \, dx \qquad \delta u(0) = 0 = \delta(1)$$

$$0 = \cancelto{0}{u' \delta u \Big|_0^1} - \int_0^1 (u'' - f) \, \delta u \, dx$$

δu ARBITRARY $\Rightarrow$ $u'' = f$.

RAYLEIGH-RITZ APPROXIMATION

LOOK FOR AN APPROXIMATION OF THE FORM

$$\bar{u}(x) = \sum_{n=1}^{N} \alpha_n \varphi_n(x)$$

$$I[\underset{\sim}{\alpha}] = I[\bar{u}] = \int_0^1 \tfrac{1}{2} \Big(\sum_{n=1}^{N} \alpha_n \varphi_n' \Big)^2 + f \Big(\sum_{n=1}^{N} \alpha_n \varphi_n \Big)$$

$$0 = \frac{\partial I}{\partial \alpha_m} = \int_0^1 \Big(\sum_{n=1}^{N} \alpha_n \varphi_n' \Big) \cdot \varphi_m' + f \varphi_m$$

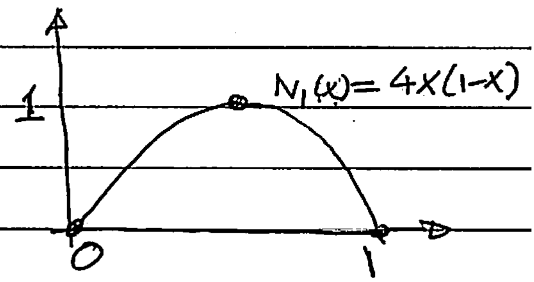
$$= \sum_{n=1}^{N} \alpha_n \int_0^1 \varphi_n' \varphi_m' \, dx + \int_0^1 f \varphi_m(x) \, dx$$

$$A \alpha = b$$

$$A_{mn} = \int_0^1 \varphi_n' \varphi_m' \, dx = A_{nm} \qquad b_m = - \int_0^1 f \varphi_m \, dx$$

EG: $u'' = -x^2$, $u(0) = 0 = u(1)$ $\Big\}$ $u_{EX}(x) = \dfrac{x}{12}(1-x^3)$

USE A SINGLE QUADRATIC LAGRANGE BASIS FUNCTION



$$\bar{u}(x) = \alpha\, 4x(1-x) = 4\alpha x - 4\alpha x^2$$

$$\bar{u}'(x) = 4\alpha - 8\alpha x = 4\alpha(1-2x) = \alpha N_1'(x)$$

$$I[\alpha] = \int_0^1 \tfrac{1}{2}(\bar{u}')^2 + f\,\bar{u}\; dx = -\tfrac{1}{5}\alpha + \tfrac{8}{3}\alpha^2$$

$$A_{11} = \int_0^1 N_1'(x)\, N_1'(x)\, dx = \int_0^1 16(1-2x)^2\, dx = 16/3$$

$$b_1 = -\int_0^1 (-x^2)(4x - 4x^2)\, dx = +1/5$$

$$\alpha = \tfrac{1}{5} \big/ (16/3) = \frac{3}{80}$$

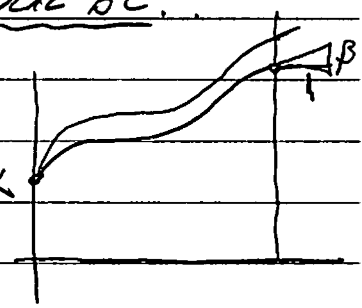
NATURAL VS ESSENTIAL BOUNDARY CONDITIONS

ESSENTIAL BOUNDARY CONDITIONS:

NOTICE THAT THE BASIS FUNCTION $N_1(x) = 4x(1-x)$ HAD TO SATISFY BOTH BOUNDARY CONDITIONS $u(0)=0=u(1)$. TYPICALLY THESE INVOLVE THE SOLUTION u AT THE BOUNDARY POINTS NOT DERIVATIVES OF u. BOUNDARY CONDITIONS THAT NEED TO BE IMPOSED ON THE TRIAL SOLUTION ARE CALLED ESSENTIAL BC.

NATURAL BOUNDARY CONDITIONS

IN THE CASE OF DERIVATIVE BC IT IS POSSIBLE TO BUILD THE BOUNDARY CONDITION INTO THE ENERGY FUNCTIONAL TO BE MINIMIZED. IN THIS CASE ALL THAT IS REQUIRED IS THAT THE TRIAL SOLUTION HAVE SUFFICIENT FREEDOM TO BE ABLE TO SATISFY THIS BOUNDARY CONDITION. SUCH A BOUNDARY CONDITION IS CALLED A NATURAL BC.



CONSIDER
$$u'' = f$$
$$u(0) = \alpha \quad u'(1) = \beta.$$

AND CONSIDER THE ENERGY FUNCTIONAL
$$E[u] = \int_0^1 \tfrac{1}{2}(u')^2 + fu\,dx - \beta u(1)$$

$$0 = \delta E = \int_0^1 u'\,\delta u' + f\,\delta u\,dx - \beta\,\delta u(1)$$
$$= u'\delta u\Big|_0^1 - \int_0^1 \delta u\,[u'' - f]\,dx - \beta\,\delta u(1)$$
$$= [u'(1) - \beta]\,\delta u(1) - u'(0)\,\delta u(0) - \int_0^1 \delta u\,[u'' - f]\,dx$$

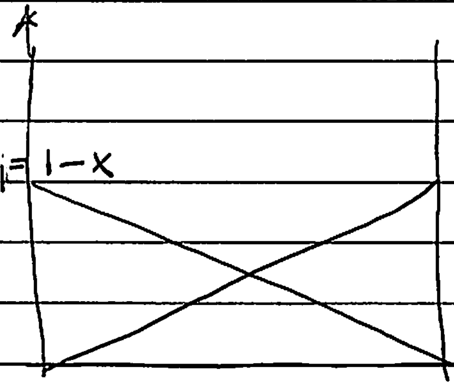
SINCE $u(0) = \alpha$ IS AN ESSENTIAL BC WE REQUIRE $\delta u(0) = 0$.
SINCE δu IS ARBITRARY WE OBTAIN THE FOLLOWING NECESSARY CONDITIONS
$$u'' = f \quad \text{AND} \quad u'(1) = \beta.$$

EG. $u'' = -x = f$ $\qquad u_{EX} = 2 + \frac{7x}{2} - \frac{x^3}{6}$

$u(0) = 2 \quad u'(1) = 3$

ESSENTIAL $\qquad$ NATURAL

LINEAR BASIS FUNCTIONS:



$$\bar{u}(x) = 2N_1(x) + aN_2(x)$$
$$= 2(1-x) + ax$$
$$\bar{u}'(x) = a - 2$$
$$E(a) = \int_0^1 \frac{1}{2}(\bar{u}')^2 + f\bar{u}\,dx - 3\bar{u}(1)$$
$$= -\frac{1}{3} - \frac{10a}{3} + \frac{(-2+a)^2}{2}$$

$$0 = \frac{dE}{da} = -\frac{10}{3} + (-2+a) \Rightarrow a = 16/3$$

$$\bar{u}(x) = 2(1-x) + \frac{11}{2}x$$

ALTERNATIVELY

$$E[a] = \int_0^1 \frac{1}{2}[2N_1'(x) + aN_2']^2 + (-x)[2N_1(x) + aN_2(x)]\,dx - 3a$$

$$0 = \frac{\partial E}{\partial a} = \frac{\partial}{\partial a}\int_0^1 [2N_1'(x) + aN_2'](N_2') - x[2N_1(x) + aN_2(x)]\,dx - 3$$

$$= 2\int_0^1 N_1'(x)N_2'(x)\,dx + a\int_0^1 [N_2'(x)]^2 dx - \int_0^1 x^2 dx - 3$$

$$A_{12} = \int_0^1 (-1).(1)\,dx = -1 \qquad A_{22} = \int_0^1 (1)(1)\,dx = 1$$

$$\therefore\ 0 = 2(-1) + a.1 - 1/3 - 3$$

$$a = 16/3$$

$$\bar{u}(x) = 2(1-x) + \frac{16}{3}x$$

HOW DO WE ARRIVE AT A COST FUNCTIONAL?

CONSIDER $u'' = f$

$u(0) = \alpha \quad u'(1) = \beta$

$$0 = \int_0^1 v(u'' - f)\,dx$$

$$= v u' \Big|_0^1 - \int_0^1 u'v' + f v\,dx$$

$$= v(1)\beta - v(0)u'(0) - \int_0^1 u'v' + f v\,dx$$

$v(0) = 0$

$$0 = \int_0^1 u'v' + f v\,dx - v(1)\beta \qquad (*)$$

NOW WHAT FUNCTIONAL IS $(*)$ THE FIRST VARIATION OF?

$$E[u] = \int_0^1 \tfrac{1}{2}(u')^2 + f u\,dx - u(1)\beta$$

ROBIN BOUNDARY CONDITION

$$u'' = f$$
$$u(0) = \alpha \qquad u'(1) + \beta u = \gamma$$

CONSIDER THE ENERGY FUNCTIONAL

$$E[u] = \int_0^1 \tfrac{1}{2}(u')^2 + f u \, dx + u(1)\left[\beta \frac{u(1)}{2} - \gamma\right]$$

$$0 = \delta E[u] = \int_0^1 u' \delta u' + f \delta u \, dx + \{\beta u(1) - \gamma\} \delta u(1)$$

$$0 = u' \delta u \Big|_0^1 - \int_0^1 [u'' - f] \delta u \, dx + \{\beta u(1) - \gamma\} \delta u(1)$$

SINCE $u(0) = \alpha$ IS AN ESSENTIAL BC $\delta u(0) = 0$

$$0 = \{u'(1) + \beta u(1) - \gamma\} \delta u(1) - \int_0^1 [u'' - f] \, \delta u \, dx$$

SINCE δu IS ARBITRARY THE NECESSARY CONDITIONS FOR A MINIMUM ARE THAT u SHOULD SATISFY

$$u'' = f$$
$$u'(1) + \beta u(1) = \gamma.$$

EIGENVALUE PROBLEMS:

$$-u'' = \lambda u \qquad u(0) = 0 = u(1)$$

$$-\int_0^1 u'' u \, dx = \lambda \int_0^1 u^2 dx$$

$$\lambda \int_0^1 u^2 dx = \cancel{u' u \Big|_0^1} + \int_0^1 (u')^2 dx$$

$$\therefore \quad \lambda = \frac{\int_0^1 (u')^2 dx}{\int_0^1 u^2 dx} = I[u] \quad \text{THE RAYLEIGH QUOTIENT.}$$

$$\lambda_1 = \min_{u|_0^1 = 0} I[u] = \min_{u|_0^1 = 0} \frac{\int_0^1 (u')^2 dx}{\int_0^1 u^2 dx} = \min_{\substack{u|_0^1 = 0 \\ \int_0^1 u^2 dx = 1}} \int_0^1 (u')^2 dx$$

$$0 = \delta I = \frac{2\left(\int_0^1 u' \delta u' dx\right) \int_0^1 u^2 dx - \int_0^1 (u')^2 dx \, 2\int_0^1 u \delta u \, dx}{\int_0^1 u^2 dx}$$

$$= 2\left\{ \int_0^1 u' \delta u' - \lambda_1 \int_0^1 u \, \delta u \, dx \right\}$$

$$\therefore \quad 0 = \cancel{u' \delta u \Big|_0^1} - \int_0^1 \left\{ u'' + \lambda_1 u \right\} \delta u \, dx$$

δu ARBITRARY $\Rightarrow -u'' = \lambda_1 u$.

THE EIGENVALUE PROBLEM AS A CONSTRAINED MINIMIZATION PROBLEM

RECALL $\lambda_1 = \min_{u|_0^1=0} \dfrac{\int_0^1 u'^2 dx}{\int_0^1 u^2 dx} = \min_{\substack{u|_0^1=0 \\ \int_0^1 u^2 dx = 1}} \int_0^1 u'^2 dx$

MINIMIZE $\int_0^1 u'^2 dx$ $\qquad u|_0^1 = 0$

SUBJECT TO $\int_0^1 u^2 dx = 0$

CONSIDER $J[u] = \int_0^1 (u')^2 dx + \mu \int_0^1 u^2 dx$ $\qquad \mu$ = LAGRANGE MULT.

$$0 = \delta J = 2\int_0^1 u' . \delta u' d + 2\mu \int_0^1 u \, \delta u \, dx$$

$$0 = u' \delta u \big|_0^1 - \int_0^1 [u'' - \mu u] \delta u \, dx$$

SINCE δu IS ARBITRARY WE OBTAIN THE NECESSARY CONDITION

$\therefore \quad -u'' = -\mu u \qquad \therefore -\mu$ IS AN EIGENVALUE $-u''$

ASSUMING $\int_0^1 u_0^2 dx = 1$ AND $-u_0'' = -\mu u_0$

$$-\mu 1 = -\mu \int_0^1 u_0^2 dx = -\int_0^1 u_0'' u_0 dx = -u_0' u_0 \big|_0^1 + \int_0^1 (u_0')^2 dx = \lambda_1$$

$\therefore$ 1) $\therefore$ THE LAGRANGE MULTIPLIER $\mu = -\lambda_1$

2) THE MINIMIZER OF $\int_0^1 (u_0')^2 dx$ SUBJECT TO $\int_0^1 u_0^2 dx = 1$

IS AN EIGENFUNCTION OF $Lu = -u''$ WITH EIGENVALUE λ_1

EXAMPLE: DETERMINE THE LOWEST EIGENVALUE & EIGENFUNCTION OF

$$Lu = -u'' = \lambda u$$

SUBJECT TO $u(0) = 0 = u(1)$

EXACT SOL'N: $u'' + \beta^2 u = 0 \qquad \beta^2 = \lambda$

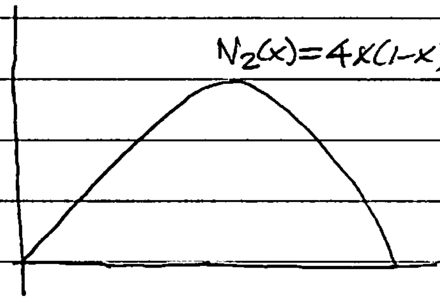
$$u = A\cos\beta x + B\sin\beta x$$

$$0 = u(0) = A \quad 0 = u(1) = B\sin\beta \Rightarrow \beta_n = n\pi \quad n = 1, 2 \ldots$$

THE LOWEST EIGENVALUE IS $\lambda_1 = \beta_1^2 = \pi^2 = 9.869$

APPROXIMATION: $\bar{u}(x) = a\,4x(1-x) = aN_2(x)$

$$\bar{u}' = 4a - 8ax$$



$$I[u] = \frac{\int_0^1 (\bar{u}')^2 dx}{\int_0^1 \bar{u}^2 dx} = \frac{16a^2/3}{8a^2/15} = \frac{30}{3} = 10$$

OR

$$\min \int_0^1 (u')^2 dx = \min 16a^2/3$$

SUBJECT TO $\int_0^1 u^2 dx = 8a^2/15 = 1 \quad$ OR $\quad a = \sqrt{\frac{15}{8}}$

$$\therefore \lambda_1 = \frac{16a^2}{3} = \frac{16}{3}\left(\sqrt{\frac{15}{8}}\right)^2 = \frac{30}{3} = 10.$$

OR: $\quad 0 = \delta I \Rightarrow \int_0^1 u'\,\delta u'\,dx = \lambda_1 \int_0^1 u\,\delta u\,dx$

LET $\bar{u} = aN_2(x) = a\,4(x - x^2) \cdot \quad \bar{u}' = 4a - 8ax$

$$\delta\bar{u} = bN_2(x) \qquad \delta u' = b(4 - 8x)$$

$$\therefore ba\int_0^1 (N_2')^2 dx = \lambda_1\, ab \int_0^1 N_2^2\, dx \qquad \forall\, b$$

$$\therefore \quad a\left\{\frac{16}{3} - \lambda_1 \frac{8}{15}\right\} = 0 \Rightarrow \lambda_1 = \frac{30}{3} = 10.$$

THE METHOD OF WEIGHTED RESIDUALS AND THE WEAK FORM

WHAT DO WE DO IF THE PROBLEM IS DISSIPATIVE SO THAT IT IS NOT POSSIBLE TO CONSTRUCT A POTENTIAL ENERGY FUNCTIONAL?

EG: $u'' - u' = 0$ $\qquad$ $u(x) = \dfrac{\sinh(1-x)}{\sinh(1)}$

$u(0) = 1 \quad u(1) = 0$

MORE GENERALLY, GIVEN A DIFFERENTIAL OPERATOR

$$Lu = f$$

$$u(0) = \alpha \quad u'(1) = \beta$$

CONSIDER THE RESIDUAL

$$R(u) = Lu - f$$

AND REQUIRE THAT

$$\int_0 R(u)\, v\, dx = 0$$

FOR ARBITRARY V IN SOME SPACE OF FUNCTIONS.

EG: 1) $v(x) = \delta(x - x_k) \quad k = 1, 2, \ldots, N$

$R(u(x_k)) = 0$ $\qquad$ THE COLLOCATION METHOD

2) $v(x) = x^0, x^1, x^2, \ldots, x^k, \ldots \qquad v(x)$

$\int_0 R(u(x))\, x^k\, dx = 0$ $\qquad$ THE METHOD OF MOMENTS.

THE WEAK FORM AND THE FINITE ELEMENT METHOD

CONSIDER THE ONE DIMENSIONAL POISSON PROBLEM

$$\left.\begin{array}{ll} -u'' = f & u'' + f = 0 \\ u(a) = G & u'(b) = H \end{array}\right\} (S)$$

WR: $\displaystyle\int_a^b v\{u'' + f\}\,dx = 0 \quad \forall v$

$$v u' \Big|_a^b - \int_a^b u'v' - fv\,dx = 0$$

$$v(b)H - v(a)u'(a) - \int_a^b u'v' - fv\,dx = 0$$

$$\left.\begin{array}{l} \text{FIND } u \in H^1_G = \{u : \int_a^b (u')^2\,dx < \infty,\ u(a) = G\} \\ \text{SUCH THAT} \\ \quad a(u,v) = \int_a^b u'v'\,dx = v(b)H + \int_a^b fv\,dx \\ \text{FOR ALL } v \in H^1_0 = \{v : \int_a^b (v')^2\,dx < \infty,\ v(a) = 0\} \end{array}\right\} \text{WEAK FORM (W)}$$

NOTE:
WE HAVE JUST SHOWN THAT IF u SATISFIES (S) THEN u MUST ALSO SATISFY (W).

CLAIM: (CONVERSE) IF u IS TWICE DIFFERENTIABLE AND u SATISFIES (W) THEN u IS A SOLUTION OF (S).

PROOF: LET u SOLVE (W). THEN $\int_a^b u'v'\,dx = v(b)H + \int_a^b fv\,dx \ \forall v \in H^1_0$

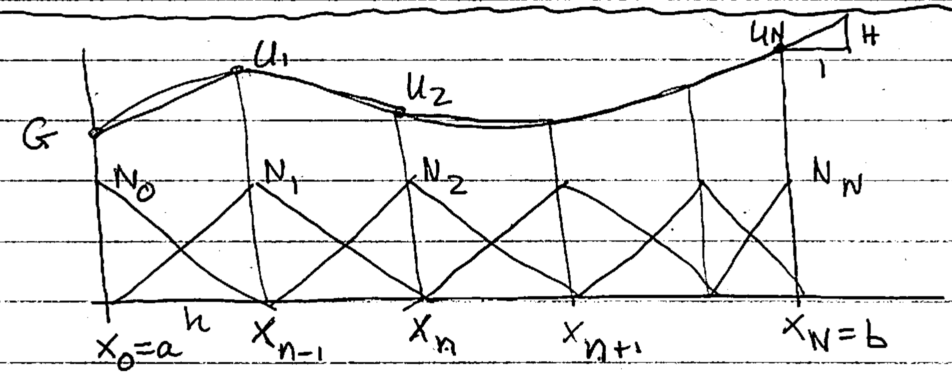
THEN $\quad u'v\Big|_{\cancel{a}}^b - \int_a^b v u''\,dx = v(b)H + \int_a^b fv\,dx$

$$\{u'(b) - H\}v(b) - \int_a^b v\{u'' + f\}\,dx = 0 \quad \text{FOR ARBITRARY } v \in H^1_0$$

$\therefore \quad u'' + f = 0 \quad$ AND $\quad u'(b) = H$

AND SINCE $u \in H^1_G \Rightarrow u(a) = G$ IT FOLLOWS THAT u SATISFIES (S)

FINITE DIMENSIONAL APPROXIMATION USING FINITE ELEMENTS



LET $\bar{u}_h(x) = G N_0(x) + \sum_{n=1}^{N} u_n N_n(x) = \mathcal{U}_G^h \subset H_G^1$

FOR THE GALERKIN APPROXIMATION WE ASSUME THE SAME BASIS FUNCTIONS FOR $\bar{v}_h(x)$ EXCLUDING THE BASIS FUNCTION $N_0(x)$ SINCE $\bar{v}_h(0) = 0$ SO THAT $\bar{v}_h(x) \in \mathcal{U}_0^h \subset H_0^1$

MAKING USE OF THE WEAK FORM WE MUST DETERMINE $\bar{u}^h(x) \in \mathcal{U}_G^h \subset H_G^1$ SUCH THAT

$$a(\bar{u}^h, \bar{v}^h) = \int_a^b \left\{ G N_0' + \sum_{n=1}^{N} u_n N_n'(x) \right\} \left\{ \sum_{m=1}^{N} v_m N_m'(x) \right\}$$

$$= v^h(b) H + \int_a^b f(x) \bar{v}^h(x)\, dx$$

$$= \sum_{m=1}^{N} v_m N_m(b) H + \sum_{m=1}^{N} v_m \int_a^b f(x) N_m(x) dx$$

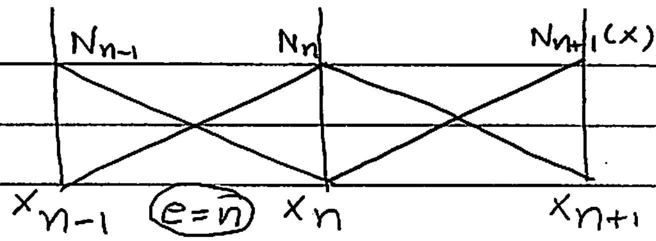
$$\therefore \sum_{m=1}^{N} v_m \left\{ G \int_a^b N_0' N_m' dx + \sum_{n=1}^{N} u_n \int_a^b N_n' N_m' dx - N_m(b) H - \int_a^b f N_m(x) dx \right\} = 0$$

for $m = 1, \ldots, N$

$$\therefore \sum_{n=1}^{N} A_{mn} u_n = N_m(b) H + \int_a^b f N_m dx - G \int_a^b N_0' N_m' dx = b_m$$

WHERE $A_{mn} = \int_a^b N_n'(x) N_m'(x) dx$

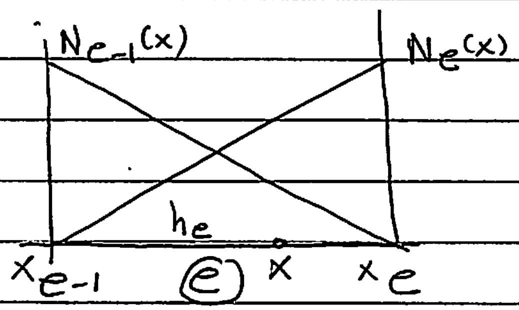
THIS IS A LINEAR SYSTEM $A \underset{\sim}{u} = \underset{\sim}{b}$ FOR THE NODAL VALUES u_n $n = 1, \ldots, N$.

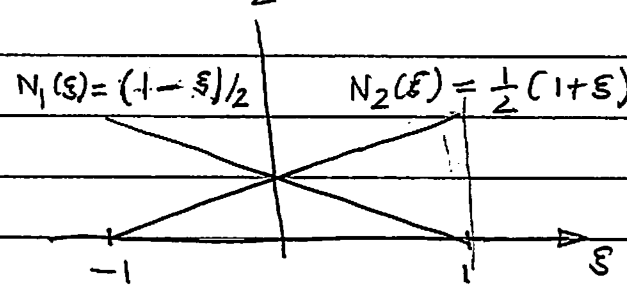


$$A_{mn} = \int_a^b N'_m(x)\, N'_n(x)\, dx$$

$$= \sum_{e=1}^{N} \int_{x_{e-1}}^{x_e} N'_m(x)\, N'_e(x)\, dx$$

$$N_a = \frac{1}{2}(1+\xi_a \xi)$$





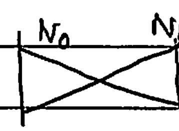
$$x(\xi) = x_{e-1} N_1(\xi) + x_e N_2(\xi) = \left(\frac{x_{e-1}+x_e}{2}\right) + \left(\frac{x_e - x_{e-1}}{2}\right)\xi$$

$$\frac{dx}{d\xi} = \frac{h_e}{2} \qquad \frac{d\xi}{dx} = \frac{2}{h_e}$$

DEFINE THE ELEMENT STIFFNESS MATRIX $A^e_{pq} = \int_{x_{e-1}}^{x_e} N'_p(x)\, N'_q(x)\, dx$

$$A^e_{pq} = \int_{x_{e-1}}^{x_e} N'_p(x)\, N'_q(x)\, dx = \int_{-1}^{1} \frac{dN_a(\xi)}{d\xi}\frac{d\xi}{dx}\frac{dN_b}{d\xi}\cancel{\frac{d\xi}{dx}}\cancel{\frac{dx}{d\xi}}\, d\xi$$

$$= \frac{2}{h_e}\int_{-1}^{1} \frac{\xi_a}{2}\cdot\frac{\xi_b}{2}\, d\xi = \frac{\xi_a \xi_b}{h_e}$$



$$b_1 = \cancelto{0}{N_1(b)}\, H + \int_{x_0}^{x_1} f(x)\, N_1(x)\, dx - G \int_{x_0}^{x_1} N_0(x) N_1(x)\, dx$$

$$b_m = \cancelto{0}{N_m(b)}\, H + \int_{x_{m-1}}^{x_m} f(x)\, N_m(x)\, dx - G \int_{x_{m-1}}^{x_m} \cancelto{0}{N_0(x) N_m(x)}\, dx$$

$$b_N = N_N(b)\, H + \int_{x_{m-1}}^{x_m} f(x)\, N_N(x)\, dx$$

$$A^e_{pq} = \frac{1}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$ ASSUME $h_e = h$ ALL h.

$$\frac{1}{h}\;\begin{array}{ccccccc} u_0 & u_1 & u_2 & u_3 & & u_{N-1} & u_N \\ -1 & 1+1 & -1 & & & & \\ & -1 & 1+1 & -1 & & & \\ & & -1 & 1 & & & \\ & & & & & 1+1 & -1 \\ & & & & & -1 & 1 \end{array} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} b_1 = -G\left(-\frac{1}{h}\right) + \int_{x_1}^{x_2} f N_1\,dx \\ b_2 = \\ \vdots \\ \\ H + \int_{x_{N-1}}^{x_N} f N_N(x)\,dx \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & & \\ & & & 2 & -1 \\ & & & -2 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ \\ \vdots \\ \\ u_N \end{bmatrix} = h \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \\ b_N \end{bmatrix}$$

$u'' + f = 0 \quad f = x$

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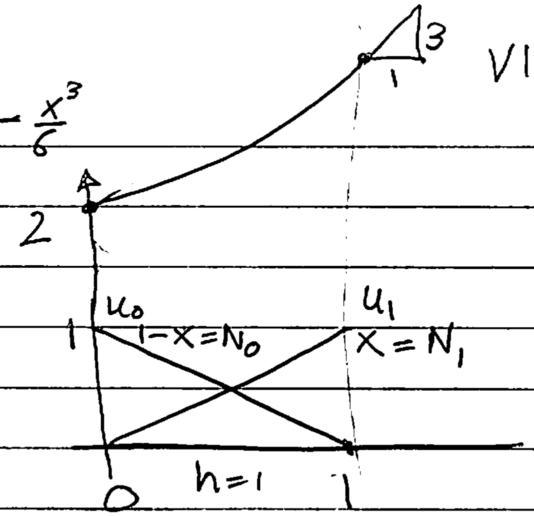
EXAMPLE: $u'' = -x$ $\quad u_{EX} = 2 + \frac{7}{2}x - \frac{x^3}{6}$

$u(0) = G = 2 \quad u'(1) = 3 = H$



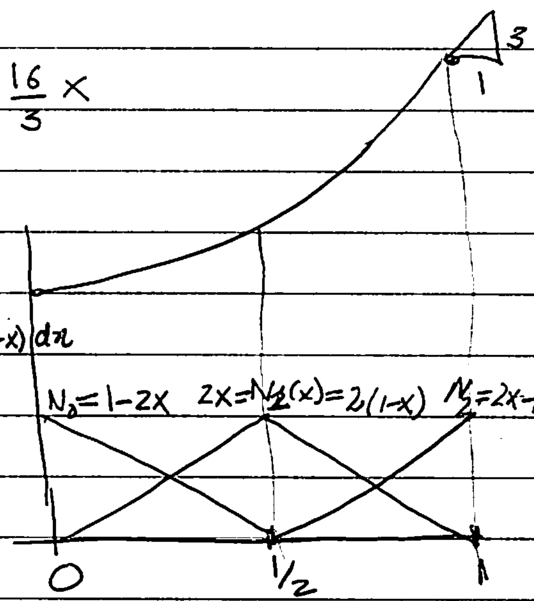
N=1: $\quad A_{01} u_0 + A_{11} u_1 = 3 + \int_0^1 x \overset{x}{\overset{\|}{N_1}}\, dx$

$$-\frac{1}{h}\cdot 2 + \frac{1}{h}\cdot u_1 = 3 + \frac{x^3}{3}\Big|_0^1$$

$h=1 \Rightarrow \quad u_1 = 5 + 1/3 = 16/3$

$$\therefore \bar{u}^h(x) = 2N_1(x) + \frac{16}{3}N_2(x) = 2(1-x) + \frac{16}{3}x$$

N=2: $\quad h = 1/2$



$$\frac{1}{(1/2)}\begin{array}{c c} u_1 & u_2 \\ \end{array}\begin{bmatrix}1+1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} = \begin{bmatrix} 2/(1/2) + \int_0^{1/2} x(2x)dx + \int_{1/2}^1 x2(1-x)\,dx \\ 3.1 + \int_{1/2}^1 x(2x-1)\,dx\end{bmatrix}$$

$$2\begin{bmatrix}2&-1\\-1&1\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} = \begin{bmatrix}4 + \frac{2x^3}{3}\Big|_0^{1/2} + \left[x^2 - \frac{2x^3}{3}\right]_{1/2}^1 \\ 3 + \left[\frac{2x^3}{3} - \frac{x^2}{2}\right]_{1/2}^1\end{bmatrix}$$

$$\begin{bmatrix}4&-2\\-2&2\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} = \begin{bmatrix}4+\frac{1}{12} + (1-2/3) - (1/4 - 1/12)\\ 3 + (\frac{2}{3} - 1/2) - (1/12 - 1/8)\end{bmatrix} = \begin{bmatrix}4 + 1/4\\3+5/24\end{bmatrix} = \begin{bmatrix}17/4\\77/24\end{bmatrix}$$

①+② $\quad 2u_1 = 7 + \frac{5+6}{24} = 7 + 11/24 \Rightarrow u_1 = \frac{7}{2} + \frac{11}{48}$

①+2② $\Rightarrow 2u_2 = 10 + \frac{1}{4} + \frac{10}{24} \Rightarrow u_2 = 5 + \frac{1}{8} + \frac{5}{24}$