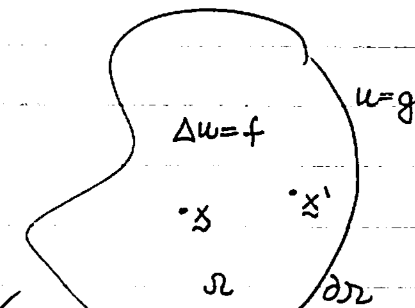
AN INTEGRAL REPRESENTATION OF THE SOLUTION USING THE FREE SPACE GREEN'S FUNCTION — BOUNDARY INTEGRAL METHODS.

$$(V, Lu) = (u, LV) + \int_{\partial\Omega} V\frac{\partial u}{\partial n} - u\frac{\partial V}{\partial n}\, ds.$$

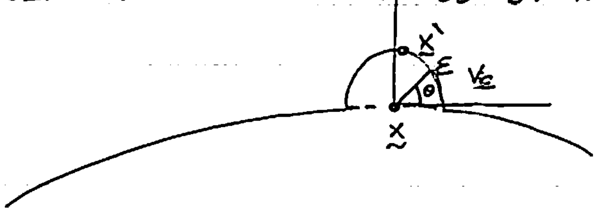


IF WE CHOOSE V TO BE THE FREE SPACE GREEN'S FUNCTION $V = \frac{1}{2\pi}\ln|x'-x|$ THEN

$$u(\underset{\sim}{x}) = \int_{\Omega} V(\underset{\sim}{x}'-\underset{\sim}{x}) f(x')\, dv(x') - \int_{\partial\Omega} V(\underset{\sim}{x}'-\underset{\sim}{x})\frac{\partial u}{\partial n} - u(\underset{\sim}{x}')\frac{\partial V}{\partial n}(\underset{\sim}{x}'-\underset{\sim}{x})\, ds(x') \qquad (*)$$

HOW CAN WE DETERMINE THE UNKNOWN FLUXES $p = \frac{\partial u}{\partial n}$? FIRSTLY WE ALLOW THE SOURCE POINT TO APPROACH THE BOUNDARY AND WRITE AN INTEGRAL EQUATION FOR $u(\underset{\sim}{x})$ FOR $\underset{\sim}{x} \in \partial\Omega$.

TO SEE HOW THE INTEGRAL EQUATION CHANGES WHEN $x \to \partial\Omega$ CONSIDER $\underset{\sim}{x}$ TO BE LOCATED AT SOME POINT ON THE BOUNDARY OF $\partial\Omega$. AND DEFORM THE BOUNDARY Ω BY A SMALL CIRCLE CENTRED AT $\underset{\sim}{x}$.



$$\underset{\sim}{x}' = \underset{\sim}{x} + \varepsilon(\cos\theta, \sin\theta) = \underset{\sim}{x} + \varepsilon\,\hat{\underset{\sim}{n}}$$

$$\frac{\partial V}{\partial n} = \frac{\partial}{\partial r}\left(\frac{1}{2\pi}\ln r\right) = \frac{1}{2\pi r}$$

$$\int_{\partial\Omega_\varepsilon} u(\underset{\sim}{x}')\frac{\partial V}{\partial n}(\underset{\sim}{x}'-\underset{\sim}{x})\, ds(x') = \int_0^{\pi} \frac{u(\underset{\sim}{x}+\varepsilon\hat{\underset{\sim}{n}}(\theta))}{2\pi\cancel{\varepsilon}}\,\cancel{\varepsilon}\, d\theta \underset{\varepsilon\to 0}{\approx} \frac{u(\underset{\sim}{x})}{2}$$

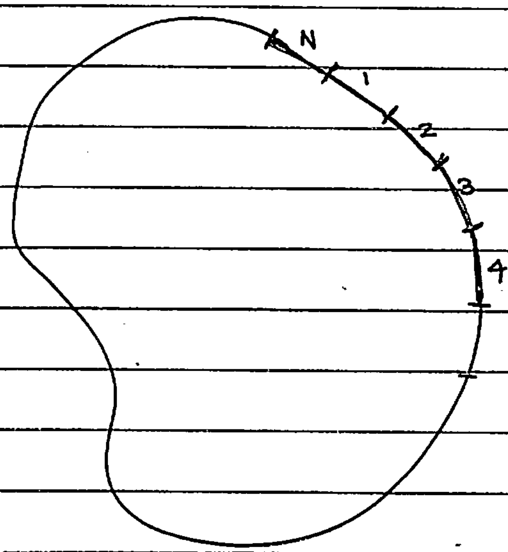
$$\int_{\Omega_\varepsilon} p(x')\, V(\underset{\sim}{x}'-\underset{\sim}{x})\, ds(x') = \int_0^{\pi} p(\underset{\sim}{x}+\varepsilon\hat{\underset{\sim}{n}}(\theta))\ln\varepsilon\,.\,\varepsilon\, d\theta$$

$$\approx p(x)\,\varepsilon\ln\varepsilon\,.\,\pi \to 0 \quad \text{AS } \varepsilon \to 0.$$

$$\therefore \quad \boxed{\frac{u(x)}{2} = \int_{\Omega} V(\underset{\sim}{x}'-\underset{\sim}{x}) f(x')\, dv(x') + \fint_{\partial\Omega} u(x')\frac{\partial V}{\partial n}(x'-x)\, ds(x') - \int_{\partial\Omega} V(x'-x)\, p(x')\, ds(x')}$$

BOUNDARY INTEGRAL EQUATION FOR u

DISCRETIZATION PROCEDURE



- DISCRETIZE BOUNDARY INFO
  N ELEMENTS $i=1,\ldots,N$.
- ASSUME u & p ARE PIECEWISE CONSTANT ALONG EACH ELEMENT

INFLUENCE FUNCTIONS

ASSUME $f=0$

$$\int_{\partial\Omega} v(x'-x)\,p(x')\,ds(x') = -\tfrac{1}{2}u(x) + \fint_{\partial\Omega} u(x')\frac{\partial v}{\partial n}(x'-x)\,ds(x')$$

$$\sum_{i} p_i \int_{\partial\Omega_i} v(x'-x_j)\,ds(x) = -\tfrac{1}{2}u(x_j) + \sum_{k=1}^{N} u_k \fint_{\partial\Omega_k} \frac{\partial v}{\partial n}(x'-x_j)\,ds$$

$$\sum_i G_{ji}\,p_i = \sum H_{jk}\,u_k$$

$$\underset{\approx}{G}\,\underset{\sim}{p} = \underset{\approx}{H}\,\underset{\sim}{u} = \underset{\sim}{b}$$

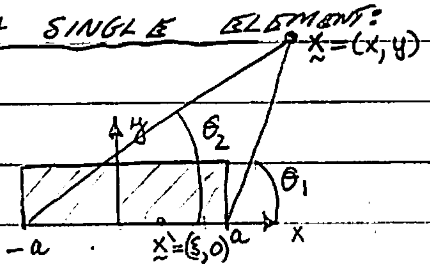
~~BENCHMARKS~~

HOW DO WE DETERMINE $u(x)$ WITHIN Ω ONCE THE p_i HAVE BEEN FOUND?

IN THIS CASE

$$u(\underset{\sim}{x}) = \sum_{i=1}^{N} u_i \int_{\Omega_i} \frac{\partial v}{\partial n}(\underset{\sim}{x}'-\underset{\sim}{x})\,d\underset{\sim}{x}' - \sum_{i=1}^{N} p_i \int_{\Omega_i} v(\underset{\sim}{x}'-\underset{\sim}{x})\,d\underset{\sim}{x}'$$

INFLUENCE OF A SINGLE ELEMENT:



$$V(x,y) = \frac{1}{2\pi}\int_{-a}^{a} \ln[(x-\xi)^2+y^2]^{1/2}\, d\xi$$

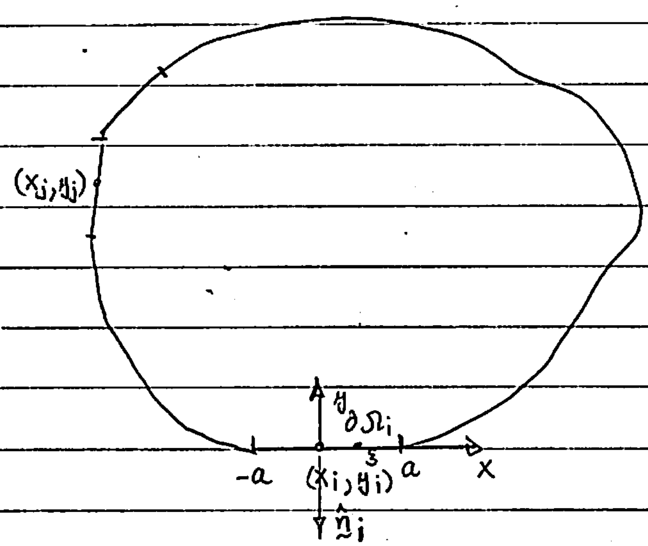
$$= \frac{1}{2\pi}\left\{ y\left(\mathrm{Tan}^{-1}\left(\frac{y}{x-a}\right) - \mathrm{Tan}^{-1}\left(\frac{y}{x+a}\right)\right) - (x-a)\ln[(x-a)^2+y^2]^{1/2} + (x+a)\ln[(x+a)^2+y^2]^{1/2}\right\}$$

$$V_{,x} = \frac{-1}{2\pi}\left[\ln[(x-a)^2+y^2]^{1/2} - \ln[(x+a)^2+y^2]^{1/2}\right]$$

$$V_{,y} = \frac{-1}{2\pi}\left[\mathrm{Tan}^{-1}\left(\frac{y}{x-a}\right) - \mathrm{Tan}^{-1}\left(\frac{y}{x+a}\right)\right]$$

NOTE: $$\lim_{y\to 0}\left[\mathrm{Tan}^{-1}\left(\frac{y}{x-a}\right) - \mathrm{Tan}^{-1}\left(\frac{y}{x+a}\right)\right] = \begin{cases} 0 & |x|>a \\ \pi & |x|<a \quad y\to 0+ \\ -\pi & |x|<a \quad y \to 0- \end{cases}$$

CALCULATING THE ELEMENTS OF $\underset{\sim}{H}$ AND $\underset{\sim}{G}$:



$$\begin{bmatrix} G_{1i} \\ G_{2i} \\ G_{ii} \\ \vdots \\ G_{Ni} \end{bmatrix} \begin{bmatrix} \\ \\ \cdot i \\ \\ \\ \end{bmatrix}$$

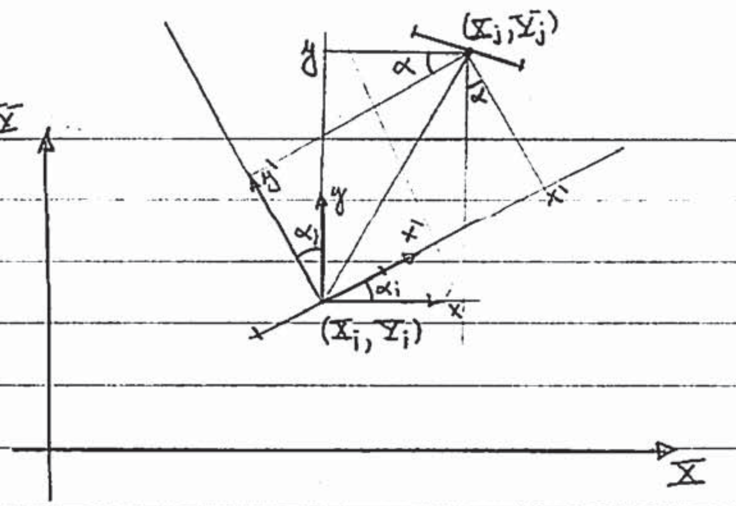
$$G_{ji} = \int_{\partial\Omega_i} V(\underset{\sim}{x}' - \underset{\sim}{x}_j)\, ds(x') = \frac{1}{2\pi}\int_{-a}^{a} \ln[(x_j-\xi)^2+y_j^2]^{1/2}\, d\xi$$

$$= \frac{1}{2\pi}\left\{ y_j\left(\mathrm{Tan}^{-1}\left(\frac{y_j}{x_j-a}\right) - \mathrm{Tan}^{-1}\left(\frac{y_j}{x_j+a}\right)\right) - (x_j-a)\ln[(x_j-a)^2+y_j^2]^{1/2} + (x_j+a)\ln[(x_j+a)^2+y_j^2]^{1/2}\right\}$$

$$H_{ji} = \int_{\partial\Omega_i} \frac{-\partial V(\underset{\sim}{x}' - \underset{\sim}{x}_j)}{\partial n}\, ds(x') = \frac{1}{2\pi}\int_{-a}^{a} -\frac{\partial}{\partial y}\ln[(x-\xi)^2+y_i^2]^{1/2}\Big|_{(x_j,y_j)} d\xi$$

$$= \frac{1}{2\pi}\left[\mathrm{Tan}^{-1}\left(\frac{y_j}{x_j-a}\right) - \mathrm{Tan}^{-1}\left(\frac{y_j}{x_j+a}\right)\right]$$

USE LOCAL COORDINATE SYSTEM TO ASSEMBLE INFLUENCE MATRIX:



$$x = \bar{X} - \bar{X}_i \qquad y = \bar{Y} - \bar{Y}_i$$

$$x' = x\cos\alpha + y\sin\alpha$$

$$y' = -x\sin\alpha + y\cos\alpha$$

$$u(x) = \int_{\partial\Omega} u(\underset{\sim}{x}) \frac{\partial v}{\partial n}(\underset{\sim}{x}' - \underset{\sim}{x})\, ds(x') - \int_{\partial\Omega} p(x')\, V(\underset{\sim}{x}' - \underset{\sim}{x})\, ds(x')$$

$$\sum_{i=1}^{N} p_i \int_{\Omega_i} v(\underset{\sim}{x}' - \underset{\sim}{x})\, ds = -u(x) + \sum_{i=1}^{N} u_i \int_{\Omega_i} \frac{\partial v}{\partial n}(\underset{\sim}{x}' - \underset{\sim}{x})\, ds$$

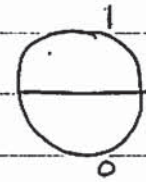
$$\sum_{i=1}^{N} G_{ji} p_i = -u(x_j) + \sum_{i=1}^{N} u_i H_{ji} = \sum_i (-\delta_{ji} + H_{ji}) u_i$$

$$\underset{\approx}{G}\, \overset{?}{\underset{\sim}{p}} = \underset{\approx}{H}\, \overset{\checkmark}{\underset{\sim}{u}}$$

SPECIAL SOLUTION: – USEFUL TEST PROBLEM FOR BEM CODE

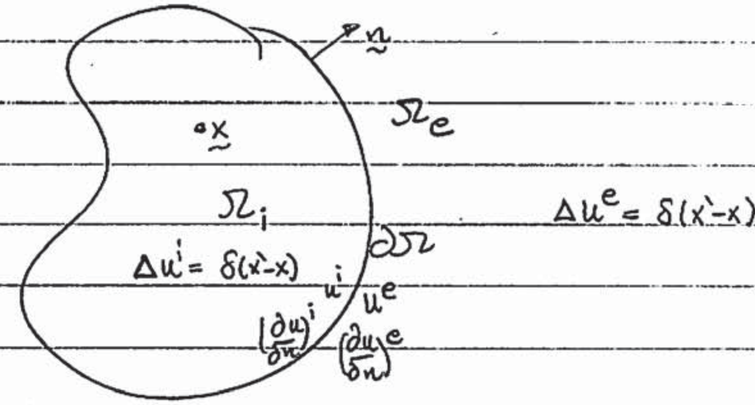
$$\nabla^2 V = 0$$

$$V(a,\theta) = \begin{cases} 1 & 0<\theta<\pi \\ 0 & -\pi<\theta<0 \end{cases}$$



$$V(r,\theta) = \frac{1}{2} + \frac{1}{\pi} \mathrm{Tan}^{-1}\left(\frac{2ar\sin\theta}{a^2 - r^2}\right).$$

THE INDIRECT BOUNDARY INTEGRAL EQUATION METHOD – SINGLE & DOUBLE LAYER POTENTIAL



$$u^i(x) = \int_{\partial\Omega} u^i(x') \frac{\partial v}{\partial n}(x',x)\, ds(x') - \int_{\partial\Omega} p^i(x')\, v(x',x)\, ds(x')$$

$$0 = -\int_{\partial\Omega} u^e(x') \frac{\partial v}{\partial n}(x',x)\, ds(x') + \int_{\partial\Omega} p^e(x')\, v(x',x)\, ds(x')$$ (USING THE SAME NORMAL).

$$u^i(x) = \int_{\partial\Omega} \{u^i(x') - u^{(e)}(x')\} \frac{\partial v}{\partial n}(x',x)\, ds(x') - \int_{\partial\Omega} \{p^i(x') - p^e(x')\}\, v(x',x)\, ds(x') \qquad (1)$$

FORMULATION 1: SINGLE LAYER POTENTIAL –

UNTIL NOW u^e & u^i HAVE DEFINE TWO SOLUTIONS (ONE FOR AN EXTERIOR PROBLEM AND ONE FOR AN INTERIOR PROBLEM) THAT ARE UNRELATED. NOW IMPOSE THE CONSTRAINT THAT THE u VALUES ACROSS THE BOUNDARY ARE CONTINUOUS. IN THIS CASE (1) REDUCES TO

$$u^i(x) = \int_{\partial\Omega} F(x')\, v(x',x)\, ds(x') \qquad (2)$$

WHERE $F(x') = p^e(x') - p^i(x')$ IS THE FLUX JUMP ACROSS THE BOUNDARY.

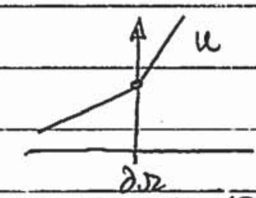
~~PROS & CONS: & NOTES.~~

1) – $F(x')$ IS A FICTICIOUS QUANTITY THAT IS NOT PART OF THE ORIGINAL PROBLEM BUT CAN BE INTERPRETED AS THE CHARGE DISTRIBUTION ALONG $\partial\Omega$ TO ACHIEVE POTENTIAL $u(x)$.

3) + TO SOLVE A BVP ALL WE NEED TO DO IS TO DISCRETIZE (2), DETERMINE THE APPROXIMATE VALUES OF F ALONG $\partial\Omega$ AND TO USE (2) TO DETERMINE u WITHIN Ω_i.

$$\check{\underset{\sim}{u}} = \underset{\approx}{V}\,\underset{\sim}{F} \Rightarrow \underset{\sim}{F} = \underset{\approx}{V}^{-1}\check{\underset{\sim}{u}} \qquad u(x) = \sum_{i=1}^{N} F_i \int_{\partial\Omega_i} v(x',x)\, dx'$$

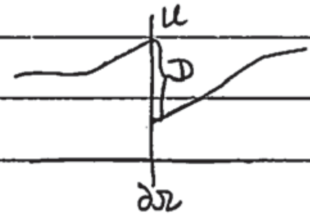
3)



4) (2) CAN BE INTERPRETED AS THE APPROPRIATE POINT CHARGE DISTRIBUTION ALONG $\partial\Omega$ TO MODEL A GIVEN POTENTIAL FIELD $u(x)$ SPECIFIED ALONG $\partial\Omega$.

FORMULATION 2: DOUBLE LAYER POTENTIAL

IMPOSE THE CONSTRAINT THAT THE FLUXES ACROSS THE BOUNDARY ARE CONTINUOUS BUT ALLOW A JUMP IN THE SOLUTION VALUES.

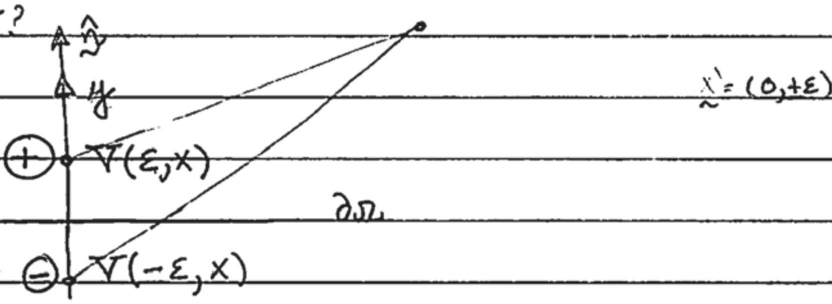


IN THIS CASE (1) REDUCES TO:

$$u^i(x) = \int_{\partial\Omega} D(x') \frac{\partial V}{\partial n}(x', x)\, ds(x') \quad (3) \quad \text{WHERE} \quad D(x') = u^i(x') - u^e(x')$$

NOTES: 1) (3) CAN BE INTERPRETED AS REQUIRING THE APPROPRIATE DIPOLE DISTRIBUTION $D(x')$ ALONG $\partial\Omega$ TO MODEL A GIVEN POTENTIAL FIELD $u(x)$ SPECIFIED ALONG $\partial\Omega$.

WHY A DIPOLE?



$$\lim_{\varepsilon \to 0} \frac{V(\varepsilon, x) - V(-\varepsilon, x)}{2\varepsilon} = \frac{\partial V}{\partial n}$$

2) ALTHOUGH IT IS MORE SINGULAR, THE KERNEL IN (3) DECAYS MORE RAPIDLY $\quad V(x', x) = O(|x' - x|^{-1})$

THAN THE KERNEL IN (2) AND IS THEREFORE BETTER CONDITIONED.