USING THE FREE SPACE GREEN'S FUNCTION TO CONSTRUCT THE ACTUAL GREEN'S FUNCTION WITH BOUNDARIES — THE METHOD OF IMAGES:

CONSIDER $Lu = \Delta u = f$ ON V (1)

$$u|_S = g$$

$$(G2) \Rightarrow (v, Lu) = (u, Lv) + \int_S v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\, ds$$

IF WE CHOOSE v TO SATISFY $Lv(\underset{\sim}{x}', \underset{\sim}{x}) = \delta(\underset{\sim}{x}' - \underset{\sim}{x})$

THEN

$$u(\underset{\sim}{x}) = \int_V v(\underset{\sim}{x}', \underset{\sim}{x}) f(x')\,dx' + \int_S u(x')\frac{\partial v}{\partial n}(x', x)\,ds - \int_S v(\underset{\sim}{x}', \underset{\sim}{x})\frac{\partial u(x')}{\partial n}\,dx'$$

$$= \int_V v(\underset{\sim}{x}', x) f(x')\,dx' + \int_S g(x')\frac{\partial v}{\partial n}(x', x)\,ds - \int_S v(x', x)\frac{\partial u(x')}{\partial n}\,dx'$$

PROBLEM — WE DON'T KNOW THE FLUX $\frac{\partial u}{\partial n}$!

WE COULD GET RID OF THIS TERM PROVIDED WE REQUIRED THAT v VANISH ON S. IN THIS CASE THE GREENS FUNCTION G IS DEFINED BY THE BOUNDARY VALUE PROBLEM

$$LG(\underset{\sim}{x}', \underset{\sim}{x}) = \delta(\underset{\sim}{x}' - \underset{\sim}{x}) \quad ; \quad G|_S = 0. \qquad (2)$$

EMBED THE BOUNDARY IN ICE

WE NOW USE THE FREE SPACE GREEN'S FUNCTION TO DETERMINE THE GREEN'S FUNCTION THAT SOLVES (2).

LET $G(\underset{\sim}{x}', \underset{\sim}{x}) = V(\underset{\sim}{x}', x) + h$ WHERE $V(\underset{\sim}{x}', \underset{\sim}{x}) = \frac{1}{2\pi}\ln r$

$$LG = LV + Lh = \delta$$

$$G|_S = V + h|_S = 0$$

THUS THE NEW FUNCTION g SATISFIES

$Lh = 0$ — $h(x)$ IS A HARMONIC FUNCTION ON V

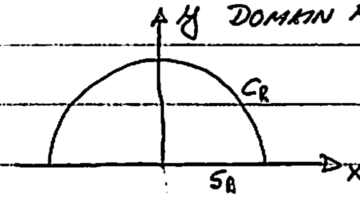
$h|_S = -V|_S$ — THAT TAKES ON THE NEGATIVE VALUES OF V ON S.

EG: THE GREEN'S FUNCTION FOR A HALF-SPACE

$$\left.\begin{aligned} \Delta u &= u_{xx} + u_{yy} = 0 \qquad -\infty < x < \infty,\ y > 0 \\ u(x,0) &= g(x) \qquad y = 0 \end{aligned}\right\} (1)$$

($g(x)$ ← SPECIFIED TEMPERATURE)

DOMAIN FOR u



$\int_S = \int_{S_R} + \int_{C_R}$ ASSUME $u \to 0$ ALONG C_R

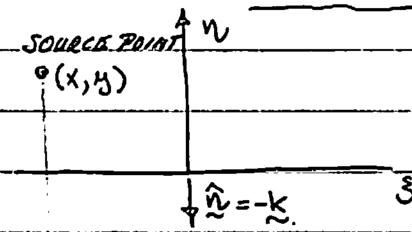
IN THIS CASE

$$u(\underset{\sim}{x}) = \int_S g(\underset{\sim}{x}') \frac{\partial G}{\partial n}(\underset{\sim}{x}', \underset{\sim}{x})\, ds(x')$$

$$\frac{\partial G}{\partial n} = -\frac{\partial G}{\partial \eta}$$

$$= -\int_{-\infty}^{\infty} g(\xi) \frac{\partial G}{\partial \eta}(\xi - x, \eta - y)\Big|_{\eta=0} d\xi \qquad (2)$$

DOMAIN FOR G



NOW: $G(\xi - x, \eta - y) = \frac{1}{2\pi} \ln[(\xi - x)^2 + (\eta - y)^2]^{1/2} + h$

WHERE

$$\left.\begin{aligned} \Delta h &= 0 \\ h(\xi, 0) &= -\frac{1}{2\pi} \ln[(\xi - x)^2 + y^2]^{1/2} \end{aligned}\right\} (*)$$

WE OBSERVE THAT $h = -\frac{1}{2\pi} \ln[(\xi - x)^2 + (\eta + y)^2]^{1/2}$ IS THE SOLUTION TO (*).

$$\therefore\ G(\xi - x, \eta - y) = \frac{1}{2\pi} \ln\left\{\frac{[(\xi - x)^2 + (\eta - y)^2]^{1/2}}{[(\xi - x)^2 + (\eta + y)^2]^{1/2}}\right\}$$

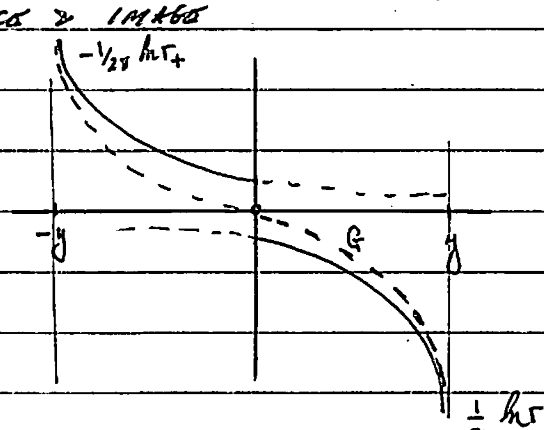
CHECK: $G(\xi - x, 0 - y) = \frac{1}{2\pi} \ln 1 = 0$

TO COMPLETE THE INTEGRAL REPRESENTATION (2) WE NEED $\frac{\partial G}{\partial n}$

$$4\pi \frac{\partial G}{\partial \eta}\Big|_{\eta=0} = [\ ]_-^{-1} \cdot 2(\eta - y) - [\ ]_+^{-1} 2(\eta + y)\Big|_{\eta=0} = \frac{-4y}{[(\xi - x)^2 + y^2]} \qquad \therefore\ \frac{\partial G}{\partial n} = -\frac{\partial G}{\partial \eta} = \frac{y}{[(x - \xi)^2 + y^2]}$$

$$\therefore\ \boxed{u(\underset{\sim}{x}) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(\xi)\, y}{[(\xi - x)^2 + y^2]} d\xi.}$$

PICTURE OF SOURCE & IMAGE



USING THE FT:

$$u_{xx} + u_{yy} = 0$$

$$u(x,0) = g(x)$$

$$\int_{-\infty}^{\infty} u_x e^{ikx}\,dx = e^{ikx} u \Big|_{-\infty}^{\infty} - ik \int_{-\infty}^{\infty} e^{ikx} u\,dx = (-ik)\hat{u}$$

$$\left.\begin{array}{l} \hat{u}_{yy} - k^2 \hat{u} = 0 \\ \hat{u}(k,0) = \hat{g}(k) \end{array}\right\} \quad \hat{u}(k,y) = A e^{-|k|y} + B e^{|k|y} \qquad \hat{u}(k,0) = A = \hat{g}(k)$$

$$k = |k|$$

$$\therefore \quad u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} e^{-|k|y} \hat{g}(k)\,dk$$

$$\int_{-\infty}^{\infty} e^{-ikx} e^{-|k|y}\,dk = 2\mathrm{Re} \int_{0}^{\infty} e^{-k(y+ix)}\,dk$$

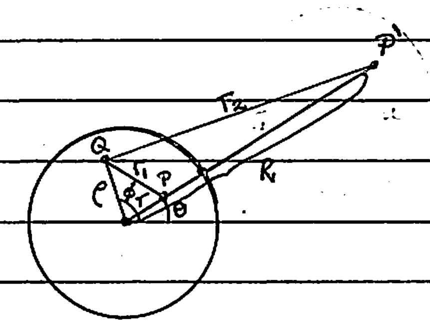
$$= 2\mathrm{Re} \left. \frac{e^{-k(y+ix)}}{-(y+ix)} \right|_{0}^{\infty}$$

$$= 2\mathrm{Re} \frac{1}{(y+ix)} \cdot \frac{(y-ix)}{(y-ix)} = \frac{2y}{x^2+y^2}.$$

USING THE CONVOLUTION THEOREM $\quad \mathcal{F}^{-1}\{\hat{f}\hat{g}\} = \int_{-\infty}^{\infty} f(x-\xi)\, g(\xi)\, d\xi.$

$$\boxed{u(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(\xi)\, y}{(x-\xi)^2 + y^2}\, d\xi.}$$

EG: THE GREEN'S FUNCTION FOR A CIRCLE



$$\Delta u = 0 \qquad 0<r<a \qquad 0<\theta<2\pi$$

$$u(a,\theta) = F(\theta)$$

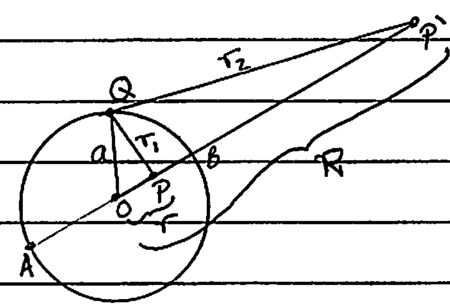
CHOOSE $\Delta G = \delta(x'-x)\delta(y'-y)$

$$G\Big|_{\rho=a} = 0$$

THEN $u(x,y) = \int_S F \dfrac{\partial G}{\partial n}\, ds = \int_0^{2\pi} F(\phi) \dfrac{\partial G}{\partial \rho}\Big|_{\rho=a} a\, d\phi$

IDEA: IF $P=(r,\theta)$ IS A POINT SOURCE WITHIN THE CIRCLE CAN WE LOCATE AN IMAGE SOURCE AT A POINT $P'=(R,\theta)$ SO THAT THE POTENTIAL ON THE BOUNDARY DUE TO THE SOURCE AT P IS CANCELLED BY THAT AT P'.

IN THIS CASE FOR Q ON THE BOUNDARY OF THE CIRCLE, P' MUST BE SO CHOSEN THAT $PQ = K\, P'Q$ FOR ALL Q ON THE CIRCLE.



$$K \hat{=} \frac{PQ}{P'Q} = \frac{a+OP}{a+OP'}\Big|_{Q@A} = \frac{a-OP}{OP'-a}\Big|_{Q@B}.$$

$$\therefore (a+OP)(OP'-a) = (a+OP')(a-OP)$$

$$a\,OP' - a\,OP + OP\,OP' - a^2 = a^2 - a\,OP + a\,OP' - OP\,OP'$$

$$OP' = a^2/OP \qquad (R = a^2/r)$$

$$\therefore K = \frac{a+OP}{a+a^2/OP} = \frac{a+OP}{a(a+OP)/OP} = \frac{OP}{a}$$

$\therefore$ BY CONSTRUCTION FOR Q ON THE CIRCLE

$$1 = \frac{1}{K}\frac{PQ}{P'Q} = \frac{a}{OP}\cdot\frac{PQ}{P'Q} = \frac{a}{r}\frac{r_1}{r_2} \qquad \text{FOR Q ON BDY } \frac{r r_2}{a} = r_1$$

$$0 = \ln 1 = \ln\left(\frac{a}{r}\cdot\frac{r_1}{r_2}\right) \qquad \text{PROVIDED Q IS ON THE BOUNDARY } \frac{r r_2}{a} = r_1$$

NOTE: $OP'\,OP = r.R = a^2 \qquad \sqrt{r.R} = a.$

LET $Q=(\rho,\phi)$ BE THE FIELD POINT AND $P=(r,\theta)$ BE THE SOURCE POINT, THEN

$$G(\rho,\phi;r,\theta) = \frac{1}{2\pi}\ln r_1 + h(\rho,\phi;r,\theta)$$

WHERE h IS HARMONIC ON THE DISK AND SATISFIES

$$\Delta h = 0$$

$$h = -V\Big|_S$$

IF WE TAKE $h = -\frac{1}{2\pi}\ln r_1 = -\frac{1}{2\pi}\ln\left(\frac{r}{a}r_2\right)$ SINCE Q IS ON BDY $r_1 = \frac{r}{a}r_2$

BUT $h = -\frac{1}{2\pi}\ln\left(\frac{r}{a}r_2\right)$ IS HARMONIC (SINCE P' IS OUTSIDE THE CIRCLE)

$$\therefore\ G = \frac{1}{2\pi}\ln\left(\frac{r_1 a}{r r_2}\right) = \frac{1}{2\pi}\left\{\ln r_1 + \ln\left(\frac{a}{r}\right) - \ln r_2\right\} = \frac{1}{4\pi}\left[\ln r_1^2 - \ln r_2^2\right] + \frac{1}{2\pi}\ln\left(\frac{a}{r}\right)$$

CHECK: $G\Big|_{\rho=a} = \frac{1}{2\pi}\ln\left(\frac{a r_1}{r r_2}\right) = \frac{1}{2\pi}\ln(1) = 0.$

BACK TO INTEGRAL REPRESENTATION

$$r_1^2 = r^2 + \rho^2 - 2r\rho\cos(\phi-\theta)$$

$$r_2^2 = R^2 + \rho^2 - 2R\rho\cos(\phi-\theta) \qquad R = OP' = a^2/OP = a^2/r$$

$$= \left(\frac{a^2}{r}\right)^2 + \rho^2 - 2\left(\frac{a^2}{r}\right)\rho\cos(\phi-\theta)$$

$$u = \int_0^{2\pi} F(\phi)\,\frac{\partial G}{\partial \rho}\Big|_{\rho=a}\,d\phi$$

$$\frac{\partial G}{\partial \rho} = \frac{1}{2\pi}\left\{\frac{1}{2r_1^2}\left[2\rho - 2r\cos(\phi-\theta)\right] - \frac{1}{2r_2^2}\left[2\rho - 2\left(\frac{a^2}{r}\right)\cos(\phi-\theta)\right]\right\}$$

$$= \frac{1}{2\pi}\,\frac{r_2^2(\rho - r\cos(\phi-\theta)) - r_1^2(\rho - (a^2/r)\cos(\phi-\theta))}{r_1^2 r_2^2}$$

$\rho = a \Rightarrow r_1 = PQ = K P'Q = K r_2$

$r_2 = r_1/K = \frac{a r_1}{r}$

$$\frac{\partial G}{\partial \rho}\Big|_{\rho=a} = \frac{1}{2\pi}\,\frac{\left[\left(\frac{a r_1}{r}\right)^2\left[a - r\cos(\phi-\theta)\right] - r_1^2\left(a - (a^2/r)\cos(\phi-\theta)\right)\right]}{\left(\frac{a}{r}\right)^2 r_1^4}$$

$$= \frac{1}{2\pi}\,\frac{\left\{\frac{a^2}{r^2}a - \frac{a^2}{r}\cos(\phi-\theta) - a + \frac{a^2}{r}\cos(\phi-\theta)\right\}}{\left(\frac{a}{r}\right)^2 r_1^2}$$

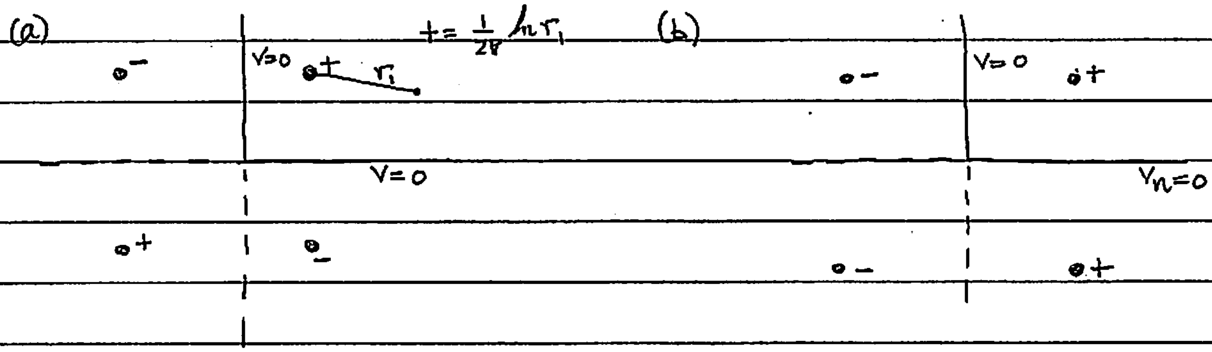
$$a\frac{\partial G}{\partial \rho}\Big|_{\rho=a} = \frac{1}{2\pi}\,\frac{\frac{a^2}{r^2}(a^2 - r^2)}{\frac{a^2}{r^2}\, r_1^2} = \frac{1}{2\pi}\,\frac{(a^2-r^2)}{r^2 + a^2 - 2ar\cos(\phi-\theta)}$$

$$\therefore\ \boxed{u(x) = \frac{1}{2\pi}\int_0^{2\pi}\frac{(a^2-r^2)\quad F(\phi)}{r^2+a^2-2ar(\cos(\phi-\theta))}\,d\phi}$$

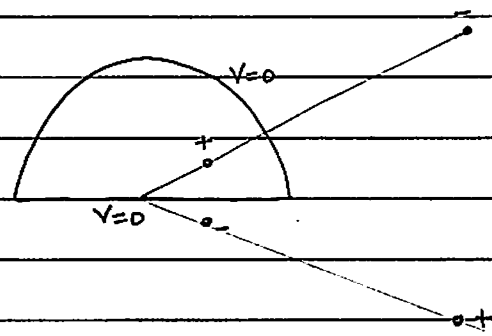
- POISSON INTEGRAL FORMULA
- $\frac{1}{2\pi}\,\frac{(a^2-r^2)}{r^2+a^2-2ar\cos(\phi-\theta)}$ DIRICHLET KERNEL

REMARKS: METHOD OF IMAGES FOR OTHER GEOMETRIES

1. QUARTER PLANE:

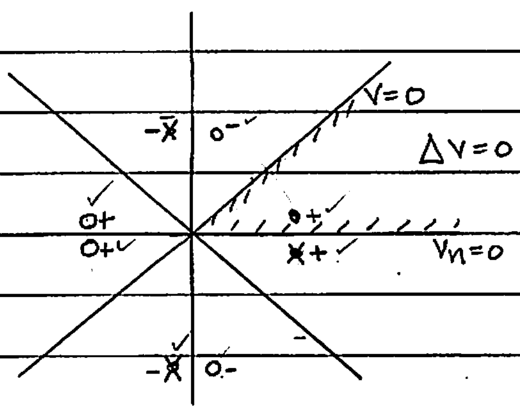


2. SEMICIRCLE

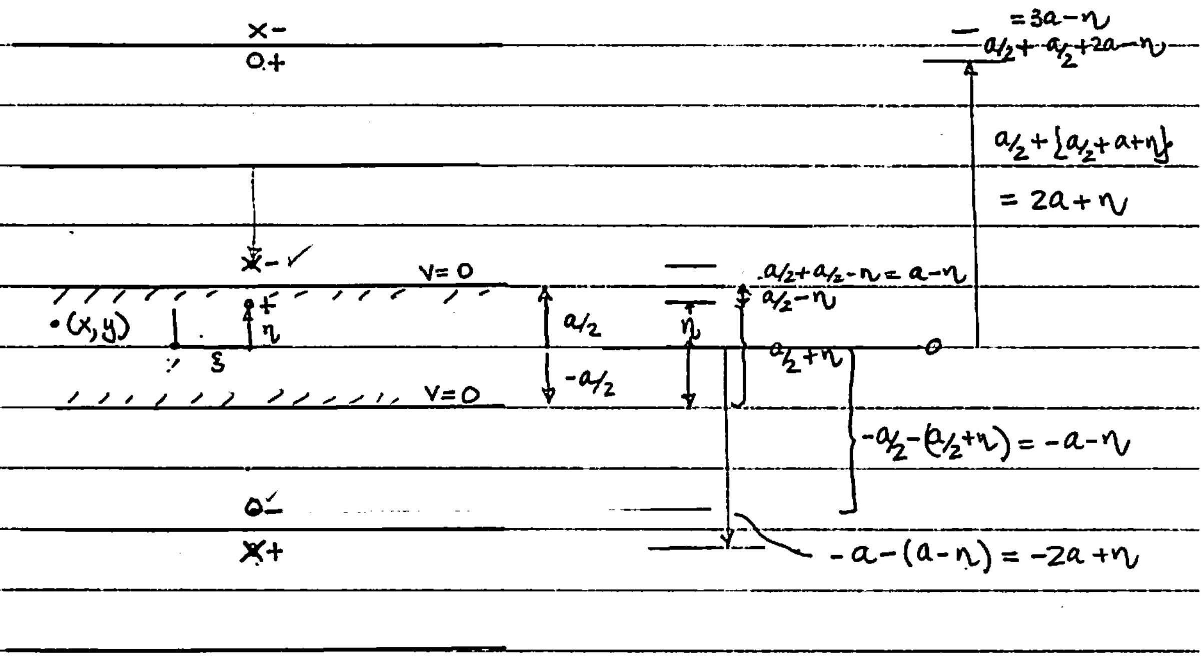


$r \cdot R = a^2 \qquad R = a^2/r$

3. WEDGE:



4) FINITE STRIP:

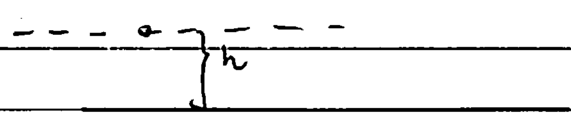


$$G = \frac{1}{2\pi}\ln r_1 - \frac{1}{4\pi}\sum \ln\left[(x-\xi)^2+(y-(2n+1)a+\eta)^2\right]$$

$$+ \frac{1}{4\pi}\sum \ln\left[(x-\xi)^2+(y+(2n)a-\eta)^2\right].$$

5) INHOMOGENEOUS MATERIAL – BONDED HALFSPACES

$$\nabla(\sigma_1 \nabla u) = \delta(\underset{\sim}{x}' - \underset{\sim}{x}).$$



$$\nabla(\sigma_2 \nabla u) = 0$$

$$\sigma_1(u_{xx} + u_{yy}) = \delta(x' - x)$$

$$-k^2\hat{u} + u_{yy} = 0$$

$$\hat{u} = A_1(k)e^{-ky} + A_2(k)e^{ky}$$

$$u^1 = A_1^1 e^{-ky} + \cancel{A_2^1 e^{ky}}$$

$$u^2 = A_1^2 e^{-ky} + A_2^2 e^{ky}$$

$$u^3 = \cancel{A_1^3 e^{-ky}} + A_2^3 e^{ky}$$

@ $y=0$: $u_1^1(k,0) = u^2(k,0)$: $A_1^1 = A_1^2 + A_2^2$

$$\sigma_1 (u_y^+ - u_y^-)\Big|_{y=0} = 1 \Rightarrow \sigma_1 \{kA_1^1 - [-kA_1^2 + kA_2^2]\} = 1.$$

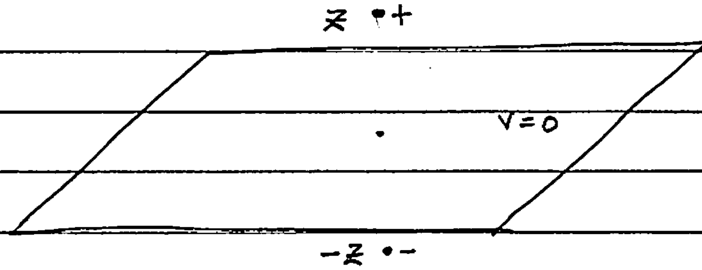
@ $y=-h$: $u^2 = u^3 \Rightarrow A_1^2 e^{+kh} + A_2^2 e^{-kh} = A_2^3 e^{-kh}$

$$\sigma_1 u_y^2 - \sigma_2 u_y^3 = 0 \Rightarrow \sigma_1 \{-kA_1^2 e^{kh} + kA_2^2 e^{-kh}\} = \sigma_2 \{kA_2^3 e^{-kh}\}$$

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ -k\sigma_1 & k\sigma_1 & -k\sigma_1 & 0 \\ 0 & e^{kh} & e^{-kh} & -e^{-kh} \\ 0 & -\sigma_1 k e^{kh} & \sigma_1 k e^{-kh} & -\sigma_2 k e^{-kh} \end{bmatrix} \begin{bmatrix} A_1^1 \\ A_1^2 \\ A_2^2 \\ A_2^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

3D: $\Delta V = \delta(\xi - x)\,\delta(\eta - y)\,\delta(\zeta - z) = \frac{\delta(r)}{4\pi r^2}$ $\quad V = -\frac{1}{4\pi r}$

HALFSPACE: $\Delta V = 0$

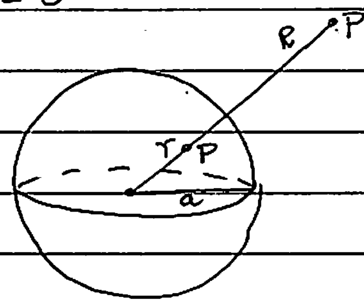


$V = \frac{-1}{4\pi r_-} + \frac{1}{4\pi r_+}$

$r_\mp = [(x-\xi)^2 + (y-\eta)^2 + (z \mp \zeta)^2]^{1/2}$

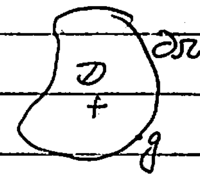
SPHERE: $\Delta G = \delta(\underset{\sim}{x}' - \underset{\sim}{x})$ $\quad \rho^2 = \xi^2 + \eta^2 + \zeta^2 \le a$

$G = 0$ $\quad \rho = a$.



PLACE IMAGE CHARGE FOR $P(r, \theta, \phi)$
AT POINT $P'(a^2/r, \theta, \phi)$

EG: MODIFIED GREEN'S FUNCTION



$$\Delta u = f(x,y) \quad \text{IN } D$$

$$\frac{\partial u}{\partial n} = g \quad \text{ON } \partial D$$

$$\int_D \nabla \cdot \nabla u \, dv = \int_{\partial\Omega} \frac{\partial u}{\partial n} ds = \int_{\partial D} g \, ds \Rightarrow \int_D f \, dv = \int_{\partial D} g \, ds$$ A COMPATIBILITY CONDITION

$$(v, \Delta u) = \int_{\partial D} v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \, ds + (u, \Delta v)$$

NOTICE THAT THE HOMOGENEOUS ADJOINT PROBLEM

$$\Delta v_H = 0 \; ; \; \left.\frac{\partial v_H}{\partial n}\right|_{\partial D} = 0$$ HAS A NONTRIVIAL SOLUTION $v_H = 1$

∴ THE GREEN'S FUNCTION DOES NOT EXIST. HOWEVER WE CONSTRUCT A MODIFIED GREEN'S FUNCTION BY REQUIRING $\tilde{G}$ SATISFY

$$\Delta \tilde{G} = \delta(x' - x) + C \cdot 1 \qquad (*)$$

$$G_n = 0 \quad \text{ON } \partial D$$

WHERE C IS CHOSEN SO THAT THE RHS OF (*) SATISFY THE SOLVABILITY CONDITION

$$(v_H, \Delta \tilde{G}) = (\tilde{G}, \Delta v_H) = 0 \Rightarrow (v_H, \delta + C \cdot 1) = 1 + C \int_D dv = 0$$

$$C = -1/\text{VOL}(D).$$

$$\therefore \quad \Delta \tilde{G} = \delta(\xi - x, \eta - y) - 1/\text{VOL}(D) \quad \text{IN } D$$

$$\tilde{G}_n = 0$$

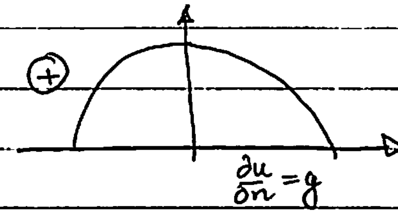
$$\therefore \quad u(x,y) = A + \int_D \tilde{G} f \, dv(\xi,\eta) - \int_{\partial D} \tilde{G} g \, ds$$

WHERE A IS ARBITRARY AND $\tilde{G}$ IS THE MODIFIED GREENS FUNCTION.

WHAT HAPPENS IF WE WANTED TO SOLVE THE PROBLEM IN WHICH THE FLUX IS SPECIFIED ALONG THE BOUNDARY OF THE HALF PLANE.

IN THIS CASE $Lu = u_{xx} + u_{yy} = 0$

$$\frac{\partial u}{\partial n}(x,0) = g(x)$$

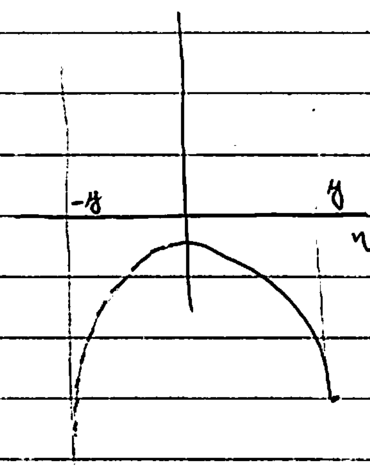


$$u(x) = -\int_S u(x') \frac{\partial v}{\partial n}(x',x)\,ds - \int_S v(x',x)\, g(x')\,ds$$

THE GREEN'S FCN SHOULD SATISFY

$$L\tilde{G} = \delta(x'-x) = 0$$

$$\frac{\partial \tilde{G}}{\partial n} = 0$$



$$G = \frac{1}{2\pi} \ln[(\xi-x)^2 + (\eta-y)^2]^{1/2} + h$$

$$\Delta h = 0 \qquad \frac{\partial h}{\partial n} = -\frac{\partial}{\partial n} \frac{1}{4\pi} \ln[(\xi-x)^2 + (\eta-y)^2]\Big|_{\eta=0}$$

$$= \frac{1}{4\pi}\frac{\partial}{\partial \eta} \ln[\;\;]\Big|_{\eta=0}$$

$$= \frac{1}{4\pi} \cdot \frac{2(\eta-y)}{[(x-\xi)^2+(\eta-y)^2]}\Big|_{\eta=0} = -\frac{2y}{4\pi[(x-\xi)^2+y^2]}$$

$h = \frac{1}{2\pi} \ln[(\xi-x)^2 + (\eta+y)^2]^{1/2}$ DOES THE JOB SINCE $\frac{\partial}{\partial n} \frac{1}{2\pi} \ln[\;]^{1/2}\Big|_{\eta=0} = -\frac{\partial}{\partial \eta} \frac{1}{2\pi} \ln[\;]^{1/2}\Big|_{\eta=0} = -\frac{2y}{4\pi[(x-\xi)^2+y^2]}$

$$u(x) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \ln[(\xi-x)^2 + y^2]\, g(\xi)\,d\xi + A.$$