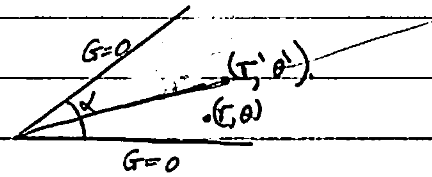
USING EIGENFUNCTION EXPANSIONS – SEPARATION OF VARIABLES

WHEN THE GEOMETRY IS NOT SUFFICIENTLY SIMPLE IT IS NOT CLEAR HOW TO USE THE METHOD OF IMAGES. INSTEAD WE CAN FIND G BY EIGENFUNCTION EXPANSION – SEPARATION OF VARIABLES – TRANSFORM METHODS.

EG: FIND THE GREEN'S FUNCTION FOR THE POISSON EQ ON A WEDGE.

$$\Delta G = \frac{\delta(r-r')\,\delta(\theta-\theta')}{r} \qquad 0\le\theta\le\alpha,\quad 0\le r\le\infty$$



$$G(r,0) = 0 = G(r,\alpha)$$

G BOUNDED AT $r=0$ & $r\to\infty$.

SOLUTION: $$G_{rr} + \frac{1}{r}G_r + \frac{1}{r^2}G_{\theta\theta} = \frac{\delta(r-r')\,\delta(\theta-\theta')}{r} \qquad (*)$$

ASIDE: EIGENFUNCTIONS: $G = R(r)\,\Theta(\theta) \Rightarrow \Theta\left(R'' + \frac{1}{r}R'\right) + \frac{1}{r^2}R\Theta'' = 0.$

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda^2$$

$$r^2R'' + rR' - \lambda^2 R = 0 \qquad \Theta'' + \lambda^2\Theta = 0$$

$R = r^s \Rightarrow s(s-1) + s - \lambda^2 = 0 \qquad \Theta = C\sin\lambda\theta + D\cos\lambda\theta$

$s = \pm\lambda \qquad \Theta(0) = D = 0 \quad \Theta(\alpha) = C\sin\lambda\alpha \qquad \lambda\alpha = n\pi \quad n=1,\dots$

$R = Ar^{\lambda} + Br^{-\lambda}. \qquad \Theta = C\sin\left(\frac{n\pi\theta}{\alpha}\right).$

STARTING POINT: LET $G(r,\theta) = \sum_{n=1}^{\infty}\hat{G}_n(r)\sin\left(\frac{n\pi\theta}{\alpha}\right)$ WHERE

$$\hat{G}_n(r) = \frac{2}{\alpha}\int_0^\alpha G(r,\theta)\sin\left(\frac{n\pi\theta}{\alpha}\right)d\theta.$$

ESSENTIALLY A FINITE FOURIER TRANSFORM PAIR.

$$\widehat{\frac{\partial^2 G}{\partial\theta^2}} = \frac{2}{\alpha}\int_0^\alpha G_{\theta\theta}\sin\left(\frac{n\pi\theta}{\alpha}\right)d\theta = \frac{2}{\alpha}\left\{\cancel{G_\theta\sin\left(\frac{n\pi\theta}{\alpha}\right)\Big|_0^\alpha} - \left[\cancel{\left(\frac{n\pi}{\alpha}\right)G\cos\left(\frac{n\pi\theta}{\alpha}\right)\Big|_0^\alpha} + \left(\frac{n\pi}{\alpha}\right)^2\int_0^\alpha G(r,\theta)\sin\left(\frac{n\pi\theta}{\alpha}\right)d\theta\right]\right\}$$

$$= -\left(\frac{n\pi}{\alpha}\right)^2\hat{G}_n(r).$$

$\therefore$ THE FINITE SINE TRANSFORM OF $(*)$ IS:

$$\hat{G}_n''(r) + \frac{1}{r}\hat{G}_n'(r) - \left(\frac{n\pi}{\alpha}\right)^2\frac{\hat{G}_n(r)}{r^2} = \frac{2}{\alpha}\sin\left(\frac{n\pi\theta'}{\alpha}\right)\frac{\delta(r-r')}{r}$$

SUBJECT TO G_n BOUNDED AT 0 & ∞.

HOMOGENEOUS EQ $\hat{G}_n(r) = r^s \Rightarrow s(s-1) + s - \left(\frac{n\pi}{\alpha}\right)^2 = s^2 - \left(\frac{n\pi}{\alpha}\right)^2 = 0$

$$\therefore \quad \hat{G}_n(r) = \begin{cases} Ar^{\lambda_n} & 0<r<r' \\ Br^{-\lambda_n} & r'<r<\infty \end{cases} \qquad \lambda_n = \left(\frac{n\pi}{\alpha}\right).$$

CONTINUITY: $\hat{G}_n(r'_+) = B(r')^{-\lambda_n} = A(r')^{+\lambda_n} = \hat{G}(r'_-)$

JUMP CONDITION: $(r\hat{G}_n')' - \lambda_n^2 \frac{1}{r}\hat{G}_n = \frac{2}{\alpha}\sin(\lambda_n\theta')\,\delta(r-r')$

$$r\hat{G}_n'\Big|_{r'-\varepsilon}^{r'+\varepsilon} = \frac{2}{\alpha}\sin(\lambda_n\theta')$$

$$r'\left(\hat{G}_n'\big|_{r'_+} - \hat{G}_n'\big|_{r'_-}\right) = r'\left\{-B\lambda_n(r')^{-\lambda_n-1} - A\lambda_n(r')^{\lambda_n-1}\right\} = \frac{2}{\alpha}\sin(\lambda_n\theta')$$

$$A = (r')^{-2\lambda_n}B$$

$$-\lambda_n B(r')^{-\lambda_n} - B\lambda_n(r')^{-\lambda_n} = \frac{2}{\alpha}\sin(\lambda_n\theta')$$

$$\therefore\ B = -(r')^{\lambda_n}\frac{\sin(\lambda_n\theta')}{\alpha\lambda_n} \Rightarrow A = -(r')^{-\lambda_n}\frac{\sin(\lambda_n\theta')}{\alpha\lambda_n}$$

$$\therefore\ \hat{G}_n = -\frac{\sin\left(\frac{n\pi\theta'}{\alpha}\right)}{n\pi}\begin{cases}\left(\frac{r}{r'}\right)^{\lambda_n} & r<r' \\ \left(\frac{r'}{r}\right)^{\lambda_n} & r>r'\end{cases} = -\left(\frac{r_<}{r_>}\right)^{\lambda_n}\frac{\sin\left(\frac{n\pi\theta'}{\alpha}\right)}{n\pi}\ ;\quad \begin{aligned} r_< &= \min(r,r') \\ r_> &= \max(r,r')\end{aligned}$$

$$\cos(A-B) - \cos(A+B) = 2\sin A\sin B$$

$$\therefore\ G(r,\theta) = \frac{-1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r_<}{r_>}\right)^{(n\pi/\alpha)}\frac{\sin\left(\frac{n\pi\theta'}{\alpha}\right)\sin\left(\frac{n\pi\theta}{\alpha}\right)}{n}$$

$$= -\frac{1}{2\pi}\sum_{n=1}^{\infty}\left(\frac{r_<}{r_>}\right)^{(n\pi/\alpha)}\left\{\cos\left(\frac{n\pi}{\alpha}(\theta-\theta')\right) - \cos\left(\frac{n\pi}{\alpha}(\theta+\theta')\right)\right\}/n$$

SUMMING THE SERIES: RECALL $-\log(1-z) = \sum_{n=1}^{\infty}\frac{z^n}{n}$

LET $z_1 = \left(\frac{r_<}{r_>}\right)^{\pi/\alpha} e^{i\pi/\alpha(\theta-\theta')}$ & $z_2 = \left(\frac{r_<}{r_>}\right)^{\pi/\alpha} e^{i\pi/\alpha(\theta+\theta')}$ & $\frac{r_<}{r_>} < 1 \Rightarrow |z_i| < 1$

THEN $G = -\frac{1}{2\pi}\,\mathrm{Re}\left\{\sum_{n=1}^{\infty}\frac{z_1^n}{n} - \sum_{n=1}^{\infty}\frac{z_2^n}{n}\right\}$

$$= \frac{1}{2\pi}\,\mathrm{Re}\log\left(\frac{1-z_1}{1-z_2}\right)$$

$$= \frac{1}{2\pi}\log\left|\frac{1-z_1}{1-z_2}\right|$$

$$= \frac{1}{4\pi}\left\{\ln|1-z_1|^2 - \ln|1-z_2|^2\right\}$$

$$= \frac{1}{4\pi}\ln\left[\frac{\cosh\left(\frac{\pi}{\alpha}\ln\left(\frac{r_<}{r_>}\right)\right) - \cos\left(\frac{\pi}{\alpha}(\theta-\theta')\right)}{\cosh\left(\frac{\pi}{\alpha}\ln\left(\frac{r_<}{r_>}\right)\right) - \cos\left(\frac{\pi}{\alpha}(\theta+\theta')\right)}\right]$$

$\rho = \left(\frac{r_<}{r_>}\right)$

$$|1-z_1|^2 = \left(1-(\rho e^{i\Delta\theta})^{\pi/\alpha}\right)\left(1-(\rho e^{-i\Delta\theta})\right)^{\pi/\alpha}$$

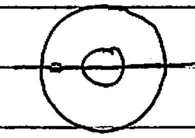
$$= 1 - 2\rho^{\pi/\alpha}\cos(\pi\Delta\theta/\alpha) + \rho^{2\pi/\alpha}$$

$$= \rho^{\pi/\alpha}\left[\rho^{\pi/\alpha} + \rho^{-\pi/\alpha} - 2\cos(\pi\Delta\theta/\alpha)\right]$$

$$= 2\rho^{\pi/\alpha}\left[\cosh\left(\frac{\pi}{\alpha}\ln\rho\right) - \cos\left(\frac{\pi}{\alpha}\Delta\theta\right)\right]$$

IN GENERAL FOR PROBLEMS WITH CIRCULAR SYMMETRY

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$



$$u = R(r)\,\Theta(\theta)$$

$$\frac{r^2R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda^2$$

$$(1)\quad \Theta'' + \lambda^2\Theta = 0 \qquad (2)\quad r^2R'' + rR' - \lambda^2 R = 0$$

(1) $\lambda \neq 0$ $\quad \Theta = A\sin\lambda\theta + B\cos\lambda\theta$; $\quad \lambda = 0 \quad \Theta = A + B\theta \quad B = 0$ FOR PERIODICITY

PERIODICITY: $\Theta(-\pi) = \Theta(\pi) \Rightarrow A\sin\lambda\pi + B\cos\lambda\pi = -A\sin\lambda\pi + B\cos\lambda\pi \Rightarrow A\sin\lambda\pi = 0$

$\Theta'(-\pi) = \Theta'(\pi) \Rightarrow \lambda A\cos\lambda\pi - \lambda B\sin\lambda\pi = A\lambda\cos\lambda\pi + \lambda B\sin\lambda\pi \Rightarrow \lambda B\sin\lambda\pi = 0$

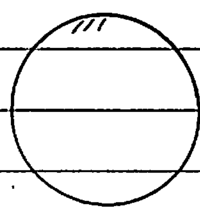
$\therefore\ \lambda = n \quad n = 0, 1, 2, \ldots \quad \lambda = 0 \Rightarrow$

(2) $\underline{\lambda = 0}$: $\quad rR'' + R' = (rR')' = 0 \quad R = C\ln r + D$

$\underline{\lambda = n \neq 0}$: $\quad R = r^\alpha \Rightarrow \alpha(\alpha-1) + \alpha - n^2 = \alpha^2 - n^2 = 0 \quad \alpha = \pm n,$

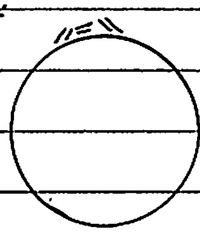
$\quad R = Cr^{-n} + Dr^n$

CIRCLE:



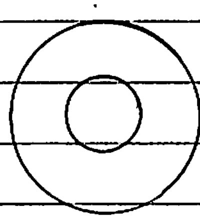
$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n[a_n\cos n\theta + b_n\sin n\theta]$$

CIRCULAR HOLE:



$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^{-n}[a_n\cos n\theta + b_n\sin n\theta]$$

ANNULUS:

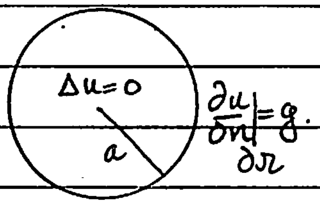


$$u(r,\theta) = A_0 + B_0\ln r + \sum_{n=1}^{\infty} r^n[a_n\cos n\theta + b_n\sin n\theta] + \sum_{n=1}^{\infty} r^{-n}[c_n\cos n\theta + d_n\sin n\theta]$$

A. PEIRCE

EG: THE NEUMANN PROBLEM – AN APPLICATION TO ELECTRICAL IMPEDANCE IMAGING.

$$\Delta u = 0 \,, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = g . \qquad (1)$$



COMPATABILITY CONDITION

$$0 = \int_\Omega \nabla\cdot\nabla u \, dv = \int_{\partial\Omega} \frac{\partial u}{\partial n} ds = \int_{\partial\Omega} g \, ds .$$

SINCE THE HOMOGENEOUS PROBLEM HAS A NONTRIVIAL SOLUTION $\tilde{u} = 1$ THE GREEN'S FUNCTION DOES NOT EXIST SO WE CONSTRUCT A MODIFIED GREENS FUNCTION

$$\left.\begin{aligned} \Delta\tilde{G} &= \delta(\underset{\sim}{x}' - \underset{\sim}{x}) + C.1 \\ \frac{\partial\tilde{G}}{\partial n} &= 0 \quad \text{ON } \partial\Omega . \end{aligned}\right\} \quad (2)$$

C IS CHOSEN SOTHAT $(1, \Delta\tilde{G}) = (\tilde{G}, \Delta 1) = 0 \Rightarrow (1, \delta + C) = 1 + C\,\mathrm{VOL}(\Omega) = 0$

$$\therefore \; C = -1/\mathrm{VOL}(\Omega) = -1/\pi a^2$$

$$\therefore \; \tilde{G}_{rr} + \frac{1}{r}\tilde{G}_r + \frac{1}{r^2}\tilde{G}_{\theta\theta} = \frac{\delta(r - r')\delta(\theta - \theta')}{r} + C \qquad (3)$$

THE APPROPRIATE EXPANSION FOR THIS PROBLEM IS GIVEN BY

$$\tilde{G}(r,\theta) = a_0(r) + \sum_{n=1}^{\infty} \left(a_n(r)\cos n\theta + b_n(r)\sin n\theta\right)$$

WHERE SINCE $\displaystyle\int_{-\pi}^{\pi} \left.\begin{matrix}\cos m\theta \cos n\theta \\ \sin m\theta \sin n\theta \\ \sin m\theta \cos n\theta\end{matrix}\right\} d\theta = \begin{cases}\delta_{mn}\pi \\ \delta_{mn}\pi \\ 0\end{cases}$ AND $\displaystyle\int_{-\pi}^{\pi} 1 \, d\theta = 2\pi$

IT FOLLOWS THAT $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{G}(r,\theta)\,d\theta$ AND $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \tilde{G}(r,\theta)\cos n\theta \, d\theta$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \tilde{G}(r,\theta)\sin n\theta \, d\theta$

$\frac{1}{2\pi}\int_{-\pi}^{\pi}(3)\,d\theta \Rightarrow$ (4) $a_0''(r) + \frac{1}{r}a_0' = \frac{\delta(r - r')}{2\pi r} + C$. SINCE $\tilde{G}_\theta$ IS PERIODIC $\left.\tilde{G}_\theta\right|_{-\pi}^{\pi} = 0$

$\frac{1}{\pi}\int_{-\pi}^{\pi}(3).\cos n\theta \, d\theta \Rightarrow$ (5) $a_n''(r) + \frac{1}{r}a_n' - \frac{n^2}{r^2}a_n = \cos n\theta' \frac{\delta(r - r')}{\pi r}$, SINCE $\frac{1}{\pi}\int_{-\pi}^{\pi} \tilde{G}_{\theta\theta}\cos n\theta \, d\theta = -n^2 a_n$

$\frac{1}{\pi}\int_{-\pi}^{\pi}(3).\sin n\theta \, d\theta \Rightarrow$ (6) $b_n''(r) + \frac{1}{r}b_n' - \frac{n^2}{r^2}b_n = \frac{\delta(r - r')}{\pi r}\sin n\theta'$

THE EQUATIONS SHOULD SATISFY THE B.C $a_0'(a) = 0$, $a_n'(a) = 0 = b_n'(a)$.

SOLVE THE EQ: (4): HOMOG EQ $(r a_0')' = 0 \Rightarrow a_0 = A_0 \ln r + B_0$

PARTICULAR SOLN: $(r a_0')' = rC \Rightarrow r a_0' = \frac{r^2}{2}C \Rightarrow a_0 = \frac{r^2 C}{4}$

$$\therefore \; a_0 = \begin{cases} B_0^- + r^2 C/4 & r < r' \\ A_0^+ \ln r + B_0^+ + r^2 C/4 & r > r' \end{cases}$$

CONTINUITY: $a_0(r'_+) = A_0^+ \ln r' + B_0^+ + \cancel{(r')^2 C/4} = a(r'_-) = B_0^- + \cancel{(r')^2 C/4}$

JUMP $\quad r'\left(a_0'\big|_+ - a_0'\big|_-\right) = \frac{1}{2\pi}$

$$r'\left(A_0^+/r' + \cancel{\frac{r'C}{2}} - \cancel{\frac{r'C}{2}}\right) = \frac{1}{2\pi} \Rightarrow A_0^+ = \frac{1}{2\pi}.$$

BC $\quad \therefore \frac{1}{2\pi}\frac{1}{r} + \frac{r}{2}\left(-\frac{1}{\pi a^2}\right)\Big|_{r=a} = 0. \quad \checkmark$

$$B_0^+ = -\frac{1}{2\pi}\ln r' + B_0^-$$

$$\therefore \boxed{a_0(r) = \begin{cases} B_0^- + Cr^2/4 & r<r' \\ \frac{1}{2\pi}\ln(r/r') + B_0^- + Cr^2/4 & r>r'. \end{cases}}$$

(5): HOMOG EQ $\quad r^2 a_n'' + r a_n' - n^2 a_n = 0$

EQUIDIMENSIONAL $\quad a_n = r^s \quad s(s-1) + s - n^2 = s^2 - n^2 = 0$

$$\therefore a_n(r) = \begin{cases} A_n^- r^n & r<r' \\ A_n^+ r^n + B_n^+ r^{-n} & r'<r. \end{cases}$$

CONTINUITY: $\quad a_n(r'_+) = A_n^+(r')^n + B_n^+(r')^{-n} = a_n(r'_-) = A_n^-(r')^n$

JUMP: $\quad r'(a_n'(r'_+) - a_n'(r'_-)) = \frac{\cos(n\theta')}{\pi}$

$$r'\left[\left(A_n^+ n(r')^{n-1} - B_n^+ n(r')^{-n-1}\right) - A_n^- n(r')^{+n-1}\right] = \frac{\cos(n\theta')}{\pi}$$

$$\therefore n\left[\cancel{A_n^+(r')^n} - B_n^+(r')^{-n} - \cancel{A_n^+(r')^n} - B_n^+(r')^{-n}\right] = \frac{\cos(n\theta')}{\pi}$$

$$B_n^+ = -\frac{(r')^n}{2\pi n}\cos(n\theta')$$

BC: $\quad a_n'(a) = n\left[A_n^+ a^{n-1} - B_n^+ a^{-n-1}\right] = 0 \quad \therefore A_n^+ = B_n^+(a^{-2n})$

$$\therefore A_n^+ = -\left(\frac{r'}{a^2}\right)^n \frac{\cos(n\theta')}{2\pi n}.$$

$$\therefore A_n^- = A_n^+ + B_n^+(r')^{-2n} = -\left[\left(\frac{r'}{a^2}\right)^n - \left(\frac{1}{r'}\right)^n\right]\frac{\cos(n\theta')}{2\pi n}.$$

$$a_n(r) = \begin{cases} -\left[\left(\frac{r'r}{a^2}\right)^n + \left(\frac{r}{r'}\right)^n\right]\frac{\cos(n\theta')}{2\pi n} & r<r' \\ -\left[\left(\frac{r'r}{a^2}\right)^n + \left(\frac{r'}{r}\right)^n\right]\frac{\cos(n\theta')}{2\pi n} & r>r' \end{cases}$$

$$a_n(r) = -\left[\left(\frac{r'r}{a^2}\right)^n + \left(\frac{r_<}{r_>}\right)^n\right]\frac{\cos(n\theta')}{2\pi n} \qquad r_< = \min(r,r') \quad r_> = \max(r,r')$$

SIMILARLY

$$b_n(r) = -\left[\left(\frac{r'r}{a^2}\right)^n + \left(\frac{r_<}{r_>}\right)^n\right]\frac{\sin(n\theta')}{2\pi n}.$$

$$\tilde{G}(r,\theta) = \begin{cases} B_0 - \frac{r^2}{4\pi a^2} - \frac{1}{2\pi}\sum_{n=1}^{\infty}\left[\left(\frac{r'r}{a^2}\right)^n + \left(\frac{r_>}{r_<}\right)^n\right]\frac{(\cos n\theta\cos n\theta' - \sin n\theta \sin n\theta')}{n} & r<r' \\ B_0 + \frac{1}{2\pi}\ln(r/r') - \frac{r^2}{4\pi a^2} - \ \ldots \ \text{SAME} & r>r' \end{cases}$$

$$= B_0 - \frac{r^2}{4\pi a^2} + \frac{1}{2\pi}\ln(r/r')H(r-r') - \frac{1}{2\pi}\sum_{n=1}^{\infty}\left[\left(\frac{r'r}{a^2}\right)^n + \left(\frac{r_>}{r_<}\right)^n\right]\frac{\cos n(\theta-\theta')}{n}$$

$$= B_0 - \frac{r^2}{4\pi a^2} + \frac{1}{2\pi}\ln(r/r')H(r-r') - \frac{1}{2\pi}\mathrm{Re}\left\{\sum_{n=1}^{\infty}\left(\frac{z_1^n}{n} + \frac{z_2^n}{n}\right)\right\}$$

$$-\ln(1-z) = \sum_{n=1}^{\infty}\frac{z^n}{n} \quad \text{WHERE} \quad z_1 = \left(\frac{r'r}{a^2}\right)e^{i(\theta-\theta')} \qquad z_2 = \left(\frac{r_>}{r_<}\right)e^{i(\theta-\theta')}$$

$$\tilde{G} = B_0 - \frac{r^2}{4\pi a^2} + \frac{1}{2\pi}\ln(r/r')H(r-r') + \frac{\mathrm{Re}\ln}{2\pi}(1-z_1)(1-z_2)$$

$$= B_0 - \frac{r^2}{4\pi a^2} + \frac{1}{2\pi}\ln(r/r')H(r-r') + \frac{\ln}{2\pi}|1-z_1||1-z_2|$$

$$|1-z_2|^2 = |1-\rho_2 e^{i(\theta-\theta')}|^2 = (1-\rho_2 e^{i(\theta-\theta')})(1-\rho_2 e^{-i(\theta-\theta')}) = 1 - 2\rho_2\cos(\Delta\theta) + \rho_2^2$$

$$|1-z_1|^2 = 1 - 2\rho_1\cos\Delta\theta + \rho_1^2$$

$$\therefore \tilde{G} = B_0 - \frac{r^2}{4\pi a^2} + \frac{\ln}{4\pi}(r/r')^2 H(r-r') + \frac{1}{4\pi}\ln\left[1 - 2\left(\frac{rr'}{a^2}\right)\cos\Delta\theta + \left(\frac{rr'}{a^2}\right)^2\right]\left[1 - 2\left(\frac{r_<}{r_>}\right)\cos\Delta\theta + \left(\frac{r_<}{r_>}\right)^2\right]$$

$$\boxed{\begin{matrix} r<r' \Rightarrow r_> = r' \\ r>r' \Rightarrow r_> = r \end{matrix}}$$

$$= B_0 - \frac{r^2}{4\pi a^2} + \frac{\ln}{4\pi}(r/r')^2 H(r-r') + \frac{1}{4\pi}\ln\left\{\frac{[a^4 - 2a^2 rr'\cos\Delta\theta + (rr')^2][r_>^2 - 2r_>r_<\cos\Delta\theta + r_<^2]}{r_>^2 a^4}\right\}$$

$$\boxed{\tilde{G} = B_0 - \frac{r^2}{4\pi a^2} + \frac{1}{4\pi}\ln\frac{[a^4 + (rr')^2 - 2a^2 rr'\cos(\theta-\theta')][r^2 + r'^2 - 2rr'\cos(\theta-\theta')]}{(r')^2 a^4}}$$

INTEGRAL REPRESENTATION:

$$\int_\Omega v\Delta u\, dv = \int_{\partial\Omega} v\frac{\partial u}{\partial n}ds - \int_{\partial\Omega} u\frac{\partial v}{\partial n}ds + \int_\Omega u\Delta v\, dv$$

$$v = \tilde{G} \Rightarrow u + c\int_\Omega u\, dv + \int_{\partial\Omega}\tilde{G}g\, ds = 0$$

$$\therefore u = \frac{+1}{4\pi a^2}\int_\Omega u\, dv - \int_{\partial\Omega}\tilde{G}g\, ds$$

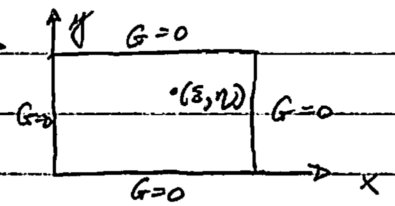
$$u(r',\theta') = A - \int_0^{2\pi} d\theta\, a g(\theta)\left(B_0 - \frac{a^2}{4\pi a^2} + \frac{1}{4\pi}\ln\frac{[a^4 + (r'a)^2 - 2a^3 r'\cos(\theta-\theta')][a^2 + r'^2 - 2ar'\cos(\theta-\theta')]}{(r')^2 a^4}\right)$$

$$= A - \frac{a}{4\pi}\int_0^{2\pi} g(\theta)\ln\frac{[a^4 + (r'a)^2 - 2a^3 r'\cos(\theta-\theta')][a^2 + r'^2 - 2ar'\cos(\theta-\theta')]}{(r')^2 a^4}\, d\theta$$

EG: FIND THE GREEN'S FUNCTION FOR A RECTANGLE DEFINED BY

$$G_{xx}+G_{yy} = \delta(x-\xi)\,\delta(y-\eta) \qquad 0<x<a,\ 0<y<b$$

$$G=0 \qquad x=0,a \ \text{ OR } \ y=0,b$$



TAKE FINITE SINE TRANSFORM

$$\hat{G}_m(y) = \frac{2}{a}\int_0^a G(x,y)\,\sin\left(\frac{m\pi x}{a}\right)dx$$

$$G(x,y) = \sum_{m=0}^{\infty} \hat{G}_m(y)\,\sin\left(\frac{m\pi x}{a}\right)$$

$$\frac{2}{a}\int_0^a \left(G_{xx}+G_{yy}\right)\sin\left(\frac{m\pi x}{a}\right)dx = \frac{2}{a}\int_0^a \delta(x-\xi)\,\delta(y-\eta)\,\sin\left(\frac{m\pi x}{a}\right)dx$$

NOW $\displaystyle\int_0^a G_{xx}\sin\left(\frac{m\pi x}{a}\right)dx = \cancel{G_x \sin\left(\frac{m\pi x}{a}\right)\Big|_0^a} - \left(\frac{m\pi}{a}\right)\left\{\cancel{G\cos\left(\frac{m\pi x}{a}\right)\Big|_0^a} + \left(\frac{m\pi}{a}\right)\int_0^a G\sin\left(\frac{m\pi x}{a}\right)dx\right\}$

$$= -\left(\frac{m\pi}{a}\right)^2 \hat{G}_m\cdot\left(\frac{a}{2}\right)$$

$$\therefore \quad \hat{G}_m''(y) - \left(\frac{m\pi}{a}\right)^2 \hat{G}_m(y) = \left(\frac{2}{a}\right)\sin\left(\frac{m\pi\xi}{a}\right)\delta(y-\eta)$$

$$\hat{G}_m(y) = \begin{cases} A\sinh\left(\frac{m\pi}{a}y\right) & 0<y<\eta \\ B\sinh\left(\frac{m\pi}{a}(y-b)\right) & \eta<y<b. \end{cases}$$

CONTINUITY: $\hat{G}_m(\eta+) = B\sinh\left(\frac{m\pi}{a}(\eta-b)\right) = A\sinh\left(\frac{m\pi}{a}\eta\right) = \hat{G}_m(\eta-)$

$$\therefore \quad A = \frac{B}{\sinh\left(\frac{m\pi}{a}\eta\right)}\sinh\left(\frac{m\pi}{a}(\eta-b)\right) = A'\sinh\left(\frac{m\pi}{a}(\eta-b)\right); \quad B = A'\sinh\left(\frac{m\pi}{a}\eta\right)$$

$$\therefore \quad \hat{G}_m(y) = \begin{cases} A'\sinh\left(\frac{m\pi}{a}y\right)\sinh\left(\frac{m\pi}{a}(\eta-b)\right) & 0<y<\eta \\ A'\sinh\left(\frac{m\pi}{a}\eta\right)\sinh\left(\frac{m\pi}{a}(y-b)\right) & \eta<y<b. \end{cases}$$

LET $y_> = \max[y,\eta]$, $y_< = \min[y,\eta]$

JUMP: $A'\left(\frac{m\pi}{a}\right)\left[\sinh\left(\frac{m\pi\eta}{a}\right)\cosh\left(\frac{m\pi}{a}(\eta-b)\right) - \cosh\left(\frac{m\pi}{a}\eta\right)\sinh\left(\frac{m\pi}{a}(\eta-b)\right)\right] = \frac{2}{a}\sin\left(\frac{m\pi\xi}{a}\right)$

$$A'\left(\frac{m\pi}{a}\right)\sinh\left(\frac{m\pi}{a}b\right) = \frac{2}{a}\sin\left(\frac{m\pi}{a}\xi\right)$$

$$\therefore \quad A' = \frac{2}{m\pi}\sin\left(\frac{m\pi}{a}\xi\right)\Big/\sinh\left(\frac{m\pi b}{a}\right)$$

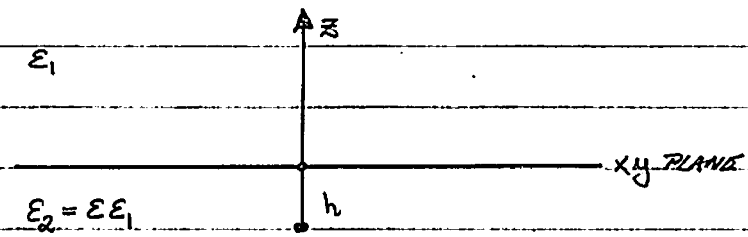
$$\therefore \quad \hat{G}_m(y) = \frac{2}{m\pi}\,\frac{\sin\left(\frac{m\pi}{a}\xi\right)}{\sinh\left(\frac{m\pi b}{a}\right)}\,\sinh\left(\frac{m\pi}{a}y_<\right)\sinh\left(\frac{m\pi}{a}(y_>-b)\right)$$

$$\therefore \quad G(x,y) = \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\,\frac{\sin\left(\frac{m\pi\xi}{a}\right)}{\sinh\left(\frac{m\pi b}{a}\right)}\,\sinh\left(\frac{m\pi}{a}y_<\right)\sinh\left(\frac{m\pi}{a}y_>\right)\sin\left(\frac{m\pi x}{a}\right).$$

EG: TO DETERMINE THE CHARACTERISTIC PARAMETERS OF WAVE PROPAGATION IN MICROSTRIP STRUCTURES IT IS NECESSARY TO DETERMINE THE GREEN'S FUNCTION OF MULTILAYERED DIELECTRIC STRUCTURES. AS A PROTOTYPE DETERMINE THE GREEN'S FUNCTION FOR TWO BONDED HALF-PLANES.

$$\nabla(\varepsilon \nabla G) = \delta(x)\delta(y)\delta(z+h)$$

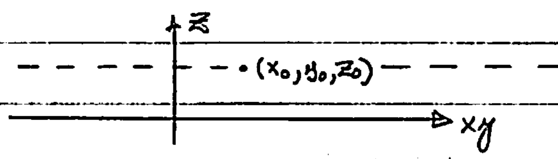
$G \to 0$ AS $r \to \infty$



WARMUP PROBLEM: FREE SPACE GREEN'S FUNCTION USING STITCHING 2 LAYERS

$$\nabla^2 G = \delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0)$$

TAKE A DOUBLE FT IN x & y



$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{+i(k_x x + k_y y)}\,\nabla^2 G\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(k_x x + k_y y)}\,\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\,dx\,dy = e^{i(k_x x_0 + k_y y_0)}\delta(z-z_0)$$

$$\therefore \quad -(k_x^2+k_y^2)\,\hat{G}(k_x,k_y,z) + \hat{G}_{zz}(k_x,k_y,z) = e^{i(k_x x_0 + k_y y_0)}\,\delta(z-z_0) = E\,\delta(z-z_0)$$

$$\therefore \quad \hat{G}'' - \gamma^2\hat{G} = E\,\delta(z-z_0), \qquad E = e^{i(k_x x_0 + k_y y_0)}; \qquad \gamma = \sqrt{k_x^2+k_y^2}$$

HOMOGENEOUS EQ.: $\hat{G} = \begin{cases} A\,e^{\gamma(z-z_0)}, & z < z_0 \\ B\,e^{-\gamma(z-z_0)}, & z > z_0 \end{cases}$

CONTINUITY: $\hat{G}(z_0^+) = B = \hat{G}(z_0^-) = A$

JUMP CONDITION: $\hat{G}_z(z_0^+) - \hat{G}_z(z_0^-) = -\gamma B - \gamma A = E$

$$B = \frac{-E}{2\gamma} = A.$$

$$\therefore \quad \hat{G} = -\frac{E}{2\gamma}\,e^{-\gamma|z-z_0|}$$

$$\therefore \quad G(x,y,z;x_0,y_0,z_0) = -\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-i\{k_x(x-x_0)+k_y(y-y_0)\}}\,e^{-\gamma|z-z_0|}}{2\gamma}\,dk_x\,dk_y$$

$$= \frac{-1}{4\pi r_0}. \qquad \text{WHERE} \quad r_0^2 = (x-x_0)^2+(y-y_0)^2+(z-z_0)^2$$

A RELATED PROBLEM — THE DOUBLE LAYER POTENTIAL

INSTEAD OF HAVING THE FLUX JUMP AT THE POINT (x_0, y_0, z_0) WE REQUIRE A JUMP IN G. IN THIS CASE

$$\hat{G} = \begin{cases} A e^{\gamma(z-z_0)} & z < z_0 \\ B e^{-\gamma(z-z_0)} & z > z_0 \end{cases}$$

$$\left.\begin{aligned} &\text{JUMP} &&: \hat{G}(z_0^+) - \hat{G}(z_0^-) = A - B = \epsilon \quad \therefore \\ &\text{CONTINUITY} &&: \hat{G}_z(z_0^+) = -\gamma B = \hat{G}_z(z_0^-) = \gamma A \quad \Rightarrow A = -B \end{aligned}\right\} A = 1/2$$

$$\hat{G} = \frac{\epsilon}{2} e^{-\gamma|z-z_0|}$$

$$G(x, y, z; x_0, y_0, z_0) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(k_x(x-x_0)+k_y(y-y_0))} \frac{e^{-\gamma|z-z_0|}}{2} dk_x\, dk_y$$

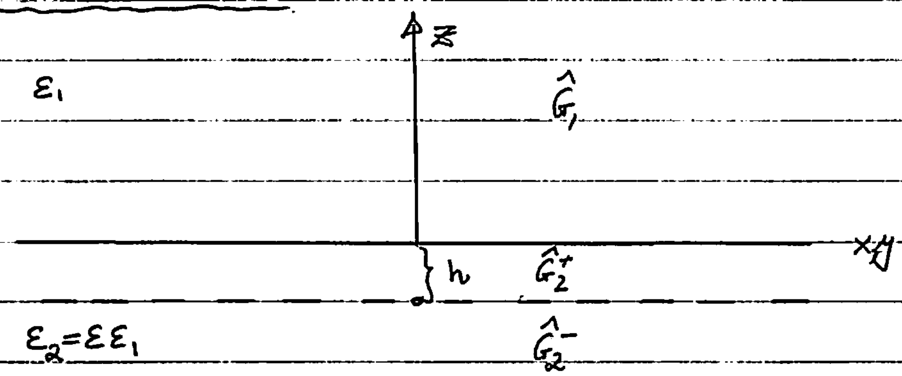
$$= \frac{\partial}{\partial z} \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(k_x(x-x_0)+k_y(y-y_0))} \left(-\frac{e^{-\gamma|z-z_0|}}{2\gamma}\right) dk_x\, dk_y$$

$$= \frac{\partial}{\partial z}\left(-\frac{1}{4\pi r_0}\right). \qquad (*)$$

THUS THE POTENTIAL DUE TO A UNIT DIPOLE IS GIVEN BY $(*)$ SO THAT FOR A DIPOLE DISTRIBUTION $D(x)$:

$$u(x) = \int_{\partial\Omega} D(x') \frac{\partial}{\partial n}\left(-\frac{1}{4\pi|x-x'|}\right) dS(x')$$

FOR BONDED HALF-PLANES



$$\nabla(\varepsilon \nabla G) = \delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0) \qquad z_0 = -h$$

$$G \to 0 \quad \text{AS} \quad r \to \infty$$

$$\varepsilon_K \{ \hat{G}_{zz} - \gamma^2 \hat{G} \} = E\,\delta(z-z_0) \qquad \gamma = \sqrt{k_x^2 + k_y^2}$$

SOLUTION TO HOMOGENEOUS EQ:

$$\hat{G}_1 = A_1 e^{-\gamma z}$$

$$\hat{G}_2^+ = A_2^+ \cosh\gamma(z+h) + B_2^+ \sinh\gamma(z+h)$$

$$\hat{G}_2^- = A_2^- e^{\gamma(z+h)}$$

①→② CONTINUITY: $\hat{G}_1\big|_0 = A_1 = G_2^+\big|_0 = A_2^+ \cosh(\gamma h) + B_2^+ \sinh(\gamma h)$

JUMP: $\varepsilon_1 \hat{G}_{1,z}\big|_0 - \varepsilon_2 \hat{G}_{2,z}^+\big|_0 = \varepsilon_1(-\gamma A_1) - \varepsilon_2(\gamma A_2^+ \sinh\gamma h + \gamma B_2^+ \cosh\gamma h) = 0$

$2^+ \to 2^-$: CONTINUITY: $\hat{G}_2^+\big|_{-h} = \hat{G}_2^-\big|_{-h} \Rightarrow A_2^+ = A_2^-$

JUMP: $\varepsilon_2 (\hat{G}_{2,z}^+ - \hat{G}_{2,z}^-)\big|_{-h} = \varepsilon_2 \{ A_2^+ \gamma \sinh\gamma(z+h) + B_2^+ \gamma \cosh\gamma(z+h) - A_2^+ \gamma e^{\gamma(z+h)} \}\big|_{z=-h} = E$

$$\therefore \quad \varepsilon_2 \gamma (B_2^+ - A_2^+) = E$$

$$\begin{bmatrix} 1 & -\cosh(\gamma h) & -\sinh(\gamma h) \\ \varepsilon_1 & \varepsilon_2 \sinh(\gamma h) & \cosh(\gamma h) \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2^+ \\ B_2^+ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ E/\gamma\varepsilon_2 \end{bmatrix}$$

SOLVE: $A_1 = \dfrac{-E}{\gamma(\varepsilon_1+\varepsilon_2)} e^{-\gamma h}$; $A_2^+ = \dfrac{-E}{\varepsilon_2\gamma} \dfrac{(\varepsilon_1 \sinh(\gamma h) + \varepsilon_2 \cosh(\gamma h))}{(\varepsilon_1+\varepsilon_2)} e^{-\gamma h} = A_2^-$

$$B_2^+ = \frac{E}{\gamma\varepsilon_2} \frac{(\varepsilon_1 \cosh(\gamma h) + \varepsilon_2 \sinh(\gamma h))}{(\varepsilon_1+\varepsilon_2)} e^{-\gamma h}$$

$C(h) = \cosh(\gamma h)$, $S(h) = \sinh(\gamma h)$

$$\hat{G}_1 = \frac{-E}{\gamma(\varepsilon_1+\varepsilon_2)} e^{-\gamma(z+h)}$$

$$\hat{G}_2^+ = -\frac{E e^{-\gamma h}}{\varepsilon_2 \gamma(\varepsilon_1+\varepsilon_2)} \{ [\varepsilon_1 S(h) + \varepsilon_2 C(h)] C(z+h) - [\varepsilon_1 C(h) + \varepsilon_2 S(h)] S(z+h) \}$$

$$= -\frac{E e^{-\gamma h}}{\varepsilon_2 \gamma(\varepsilon_1+\varepsilon_2)} \{ \varepsilon_1 (S(h)C(z+h) - C(h)S(z+h)) + \varepsilon_2 (C(h)C(z+h) - S(h)S(z+h)) \}$$

$$= -\frac{E e^{-\gamma h}}{\varepsilon_2 \gamma(\varepsilon_1+\varepsilon_2)} [\varepsilon_1 \sinh(h-(z+h))\gamma + \varepsilon_2 \cosh((z+h)-h)\gamma]$$

$$= -\frac{E e^{-\gamma h}}{\varepsilon_2 \gamma(\varepsilon_1+\varepsilon_2)} [\varepsilon_2 C(z) - \varepsilon_2 S(z)]$$

$$\hat{G}_2^- = -\frac{E}{\varepsilon_2 \gamma(\varepsilon_1+\varepsilon_2)} \cancel{e^{-\gamma h}} \{\varepsilon_1 S(h) + \varepsilon_2 C(h)\} e^{\gamma(z+\cancel{h})}$$

$$\hat{G}_2 = -\frac{E}{2\varepsilon_2 \gamma(\varepsilon_1+\varepsilon_2)} \begin{cases} [\varepsilon_2(e^{\gamma z}+e^{-\gamma z}) - \varepsilon_1(e^{\gamma z}-e^{-\gamma z})] e^{-\gamma h} & -h<z<0 \\ [\varepsilon_1(e^{\gamma h}-e^{-\gamma h}) + \varepsilon_2(e^{\gamma h}+e^{-\gamma h})] e^{\gamma z} & -\infty<z<-h. \end{cases}$$

$$= -\frac{E}{2\varepsilon_2 \gamma} \begin{cases} \left(\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}\right) e^{\gamma(z-h)} + \left(\frac{\varepsilon_2+\varepsilon_1}{\varepsilon_2+\varepsilon_1}\right) e^{-\gamma(z+h)} & -h<z<0 \\ \left(\frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1+\varepsilon_2}\right) e^{\gamma(z+h)} + \left(\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}\right) e^{\gamma(z-h)} & z<-h \end{cases}$$

$$\hat{G} = \begin{cases} -\frac{E}{2\varepsilon_2} \left[\frac{e^{-\gamma|z+h|}}{\gamma} - K_0 \frac{e^{-\gamma|z-h|}}{\gamma} \right] & z<0. \qquad K_0 = \left(\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}\right) \\ -\frac{E}{\varepsilon_1+\varepsilon_2} \frac{e^{-\gamma(z+h)}}{\gamma} & z>0 \end{cases}$$

$$\therefore \quad G = \begin{cases} -\frac{1}{4\pi\varepsilon_2} \left\{ \frac{1}{r_0^+} - \frac{K_0}{r_0^-} \right\} & z<0 \\ -\frac{1}{4\pi(\varepsilon_1+\varepsilon_2)} \frac{1}{r_0^+} & z>0 \end{cases}$$

$(r_0^+)^2 = (x-x_0)^2 + (y-y_0)^2 + (z+h)^2$

$(r_0^-)^2 = (x-x_0)^2 + (y-y_0)^2 + (z-h)^2$