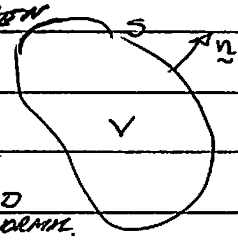
GREEN'S FUNCTIONS FOR ELLIPTIC PDE

1.3.1 GREEN'S SECOND IDENTITY

LET V BE A REGION IN $\mathbb{R}^2$ OR $\mathbb{R}^3$ WITH SURFACE S. THEN



$$\int_V u_1 \Delta u_2 - u_2 \Delta u_1 \, dv = \int_S (u_1 \frac{\partial u_2}{\partial n} - u_2 \frac{\partial u_1}{\partial n} \, ds \qquad (*)$$

WHERE $\Delta u = u_{xx} + u_{yy} + u_{zz}$ IN $\mathbb{R}^3$ & $\frac{\partial u}{\partial n} = \hat{n} \cdot \nabla u$ $\quad \hat{n}$ = OUTWARD UNIT NORMAL.

LET $(f,g) = \int_V fg \, dv$. SO THAT $(*)$ CAN BE WRITTEN IN THE FORM: $(u_1, \Delta u_2) = (u_2, \Delta u_1) + \underbrace{\int_S (u_1 \frac{\partial u_2}{\partial n} - u_2 \frac{\partial u_1}{\partial n}) ds}_{B.T.}$

PROOF: $\nabla \cdot (u_1 \nabla u_2) = \nabla u_1 \cdot \nabla u_2 + u_1 \nabla^2 u_2$

$$(u_1, \Delta u_2) = \int_V u_1 \nabla^2 u_2 \, dv = \int_V \nabla \cdot (u_1 \nabla u_2) - \nabla u_1 \cdot \nabla u_2 \, dv$$

$$= \int_S \hat{n} \cdot (u_1 \nabla u_2) ds - \int_V \nabla u_1 \cdot \nabla u_2 \, dv \quad (1)$$ USING THE DIVERGENCE THEOREM.

SWAPPING u_1 & u_2 IN (1) WE HAVE:

$$(u_2, \Delta u_1) = \int_S \hat{n} \cdot (u_2 \nabla u_1) \, ds - \int_V \nabla u_2 \cdot \nabla u_1 \, dv \qquad (2)$$

SUBTRACTING (2) FROM (1) WE OBTAIN

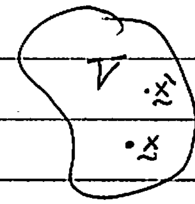
$$(u_1, \Delta u_2) - (u_2, \Delta u_1) = \int_S u_1 \frac{\partial u_2}{\partial n} - u_2 \frac{\partial u_1}{\partial n} \, ds \qquad \square$$

NOTE: 1) $(*)$ IS THE GREEN'S IDENTITY FOR THE LAPLACIAN, BY SUBTRACTING THE TERM $+k^2(u_1, u_2)$ FROM $(*)$ WE OBTAIN THE GREEN'S IDENTITY FOR THE HELMHOLTZ OPERATOR

$$(u_1, \Delta u_2 + k^2 u_2) = (u_2, \Delta u_1 + k^2 u_1) + \int_S u_1 \frac{\partial u_2}{\partial n} - u_2 \frac{\partial u_1}{\partial n} \, ds.$$

EXAMPLE: CONSIDER THE DIRICHLET PROBLEM

$$\left.\begin{aligned} \Delta u &= f(x) \quad \text{IN } V \\ u &= g \quad \text{ON } S. \end{aligned}\right\} (1)$$



ON A TWO OR THREE DIMENSIONAL DOMAIN.

LET $\underset{\sim}{x}' = (x', y', z')$ THEN GREEN'S IDENTITY YIELDS

$$(v, \Delta u) = (u, \Delta v) + \int_S v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\, ds$$

$$\int_V v\,\Delta u\, dV(x') = \int_V u\,\Delta v\, dV(x') + \int_S v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\, dS(x').$$

WE WANT TO CHOOSE v SO THAT WE OBTAIN A FORMULA FOR u AND THEREBY INVERT THE OPERATOR GIVEN IN (1).

LET $\Delta v(\underset{\sim}{x}', \underset{\sim}{x}) = \delta(\underset{\sim}{x}' - \underset{\sim}{x})$

$v = 0$ ON S.

THEN $$u(x) = \int_V v(\underset{\sim}{x}', \underset{\sim}{x}) f(x')\, dV(x') + \int_S g(x') \frac{\partial v}{\partial n}(\underset{\sim}{x}', \underset{\sim}{x})\, dS(x')$$

SIMILARLY FOR THE HELMHOLTZ EQUATION:

$$\Delta u + k^2 u = f(x) \quad \text{IN } V$$
$$u = g \quad \text{ON } S$$

IN THIS CASE THE NOTE ON THE PREVIOUS PAGE IMPLIES

$$u(x) = \int_V G(\underset{\sim}{x}', x) f(x')\, dV(x') + \int_S g(x') \frac{\partial G}{\partial n}(\underset{\sim}{x}', x)\, dS(x')$$

WHERE $\Delta G + k^2 G = \delta(\underset{\sim}{x}' - \underset{\sim}{x})$ ON V.

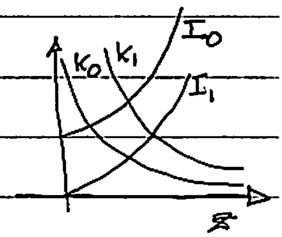
$G = 0$ ON S

OTHER TYPES OF EQUATIONS:

THE MODIFIED HELMHOLTZ EQUATION

EG: ABSORBING MEDIUM – FREE SPACE GREEN'S FUNCTION

2D: $\Delta v - k^2 v = \delta(\underset{\sim}{x}' - \underset{\sim}{x})$ $\qquad v = A K_0(kr)$



$v_{rr} + \frac{1}{r} v_r - k^2 v = 0$

$r^2 v_{rr} + r v_r - (kr)^2 v = 0 \qquad z = kr \qquad \frac{d}{dr} = k\frac{d}{dz}$

$z^2 v''(z) + z v' - z^2 v = 0.$ MODIFIED BESSEL EQUATION

$v = A K_0(kr)$ FOR SOME A

WHERE $K_0(kr)$ IS THE ZEROTH ORDER MODIFIED BESSEL FUNCTION OF THE SECOND KIND.

3D: $\Delta v - k^2 v = \delta(\underset{\sim}{x}' - \underset{\sim}{x})$ $\qquad v = \frac{A e^{-kr}}{r}$ FOR SOME A.

MULTIDIMENSIONAL DELTA FUNCTIONS

2D: $\int_V \delta(\underset{\sim}{x}'-\underset{\sim}{x})\, f(\underset{\sim}{x}')\, dv = f(\underset{\sim}{x}) \qquad \underset{\sim}{x} \in V.$

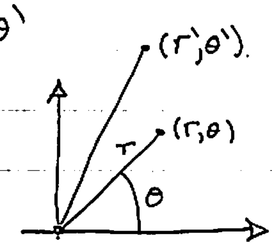
$$\delta(\underset{\sim}{x}'-\underset{\sim}{x}) = \delta(x'-x,\, y'-y)$$

CARTESIANS: $\delta(x'-x,\, y'-y) = \delta(x'-x)\,\delta(y'-y)$ PRODUCT

$$\int_V \delta(x'-x,\, y'-y)\, f(x',y')\, d\underset{\sim}{v} = \int_V \delta(x'-x)\delta(y'-y)\, f(x',y')\, dV = f(x,y).$$

POLARS: $\int_V \delta(x'-x, y'-y)\, f(x',y')\, dv = \int_{V_p} \delta(r'-r,\, \theta'-\theta)\, r'\, \tilde{f}(r',\theta')\, dr'\, d\theta' = \tilde{f}(r,\theta).$

$$= \int_{V_p} \delta(r'-r)\,\delta(\theta'-\theta)\,\tilde{f}(r',\theta')\, dr'\, d\theta'$$



$$\therefore\ \delta(r'-r,\, \theta'-\theta) = \frac{\delta(r'-r)\,\delta(\theta'-\theta)}{r'}$$

SINGULAR POINTS: CONSIDER

$$\int_{V_p} \frac{\delta(r'-r)\,\delta(\theta'-\theta)}{r'}\, f(r',\theta')\, r'\, dr'\, d\theta' = f(r,\theta)$$

WHAT HAPPENS IF $r=0$ THEN SINCE $r=0$ DEFINES A SINGLE POINT IN THE DOMAIN $f(r,\theta)$ CANNOT DEPEND ON θ IF f IS SINGLE VALUED.

IN THIS CASE THE $\delta(\theta'-\theta)$ IN THE ABOVE REPRESENTATION BECOMES REDUNDANT. IN ORDER TO REMOVE THIS UNNECESSARY REDUNDANCY (WHICH LEADS TO MORE COMPLICATED RHS.) WE PROCEED AS FOLLOWS. LET $F(0) = f(0,\theta)$

$$\int_{-\pi}^{\pi}\int_0^{\infty} \frac{\delta(r')}{r'}\, f(r',\theta')\, r'\, dr'\, d\theta' = \int_{-\pi}^{\pi} F(0)\, d\theta' = 2\pi F(0)$$

$$\therefore\ \int_{-\pi}^{\pi}\int_0^{\infty} \frac{\delta(r')}{2\pi r'}\, f(r',\theta')\, r'\, dr'\, d\theta' = F(0)$$

$$\therefore\ \delta(\underset{\sim}{x}'-\underset{\sim}{x}) = \delta(x'-x)\,\delta(y'-y) = \frac{\delta(r')}{2\pi r'}$$

FOR A COORDINATE SYSTEM (r',θ') CENTERED AT $\underset{\sim}{x}' = \underset{\sim}{x}$

3D: CARTESIANS: $\delta(\underset{\sim}{x}' - \underset{\sim}{x}) = \delta(x'-x)\delta(y'-y)\delta(z'-z)$

CYLINDRICALS: $\delta(\underset{\sim}{x}' - \underset{\sim}{x}) = \dfrac{\delta(r'-r)\,\delta(\theta'-\theta)\,\delta(z'-z)}{r}$

SINGULAR POINT $\delta(x')\,\delta(y')\,\delta(z'-z) = \dfrac{\delta(r')\,\delta(z'-z)}{2\pi r'}$

SPHERICALS: $\delta(\underset{\sim}{x}' - \underset{\sim}{x}) = \dfrac{\delta(r'-r)\,\delta(\theta'-\theta)\,\delta(\phi'-\phi)}{r'^2 \sin\phi'}$

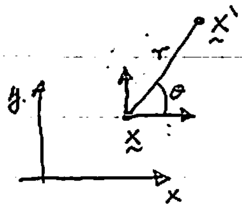
SINGULAR POINT: $\delta(x')\,\delta(y')\,\delta(z') = \dfrac{\delta(r')}{4\pi r'^2}$

CHECK $\displaystyle\int_{-\pi}^{\pi}\int_0^{\pi}\int_0^{\infty} \frac{\delta(r')}{4\pi r'^2}\cdot r'^2 \sin\phi'\,dr'\,d\phi'\,d\theta = \frac{1}{4\pi}\int_{-\pi}^{\pi} d\theta \cdot \int_0^{\pi} \sin\phi'\,d\phi' = 1.$

THE FREE SPACE GREEN'S FUNCTION

BECAUSE OF LARGER VARIETY OF POSSIBLE BOUNDARIES IN 2D & 3D IT MAY BE DIFFICULT TO CONSTRUCT GREEN'S FUNCTIONS FOR GENERAL DOMAINS. IT IS USEFUL, THEREFORE, TO CONSTRUCT WHAT IS KNOWN AS THE FREE SPACE GREEN'S FUNCTION WHICH CAPTURES THE SINGULARITY OF THE DELTA FUNCTION BUT IGNORES THE BOUNDARIES. THE FREE SPACE GREEN'S FUNCTION CAN THEN BE USED TO OBTAIN THE SOLUTION INCLUDING BOUNDARIES USING SUPERPOSITION.

1. FREE SPACE GREEN'S FUNCTION FOR THE LAPLACIAN IN $\mathbb{R}^2$:



$$\Delta V = \delta(\underset{\sim}{x}' - \underset{\sim}{x}) \qquad x \in \mathbb{R}^2$$

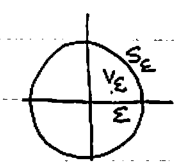
CONVERT TO POLAR COORDINATES AND LET $r = |\underset{\sim}{x}' - \underset{\sim}{x}|$

$$V_{rr} + \frac{1}{r} V_r + \frac{1}{r^2} V_{\theta\theta} = \frac{\delta(r)}{2\pi r}$$

AND LOOK FOR A SOLUTION THAT IS INDEPENDENT OF θ.

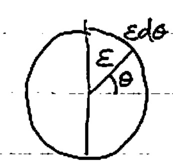
$$V_{rr} + \frac{1}{r} V_r = \frac{\delta(r)}{2\pi r} \qquad (*)$$

HOMOGENEOUS EQ.: $V_{rr} + \frac{1}{r} V_r = 0 \quad r > 0 \Rightarrow V(r) = A \ln r + B$.



CHOOSE $B = 0$ SO THAT $V(r) = A \ln r$. TO MATCH A TO THE STRENGTH OF THE FORCING TERM, ENCLOSE THE SOURCE AT THE ORIGIN IN A CIRCLE OF RADIUS ε. AND INTEGRATE $(*)$ OVER THIS VOLUME

$$\int_{V_\varepsilon} V_{rr} + \frac{1}{r} V_r \, dv = \int_{V_\varepsilon} \underset{\sim}{\nabla} \cdot \underset{\sim}{\nabla} v \, dv = \int_{-\pi}^{\pi} \int_0^{\infty} \frac{\delta(r)}{2\pi r} \cdot r \, dr \, d\theta = 1$$



$$\therefore \quad \int_{S_\varepsilon} \underset{\sim}{n} \cdot \underset{\sim}{\nabla} v \, dS = 1 \qquad\qquad \underset{\sim}{n} \cdot \underset{\sim}{\nabla} v \Big|_{r=\varepsilon} = \frac{\partial v}{\partial r}\Big|_{r=\varepsilon} = \frac{A}{r}\Big|_{r=\varepsilon} = \frac{A}{\varepsilon}$$

$$\therefore \quad \int_{-\pi}^{\pi} \frac{A}{\varepsilon} \cdot \varepsilon \, d\theta = 1 \qquad\qquad ds = \varepsilon \, d\theta$$

$$\therefore \quad A = 1/2\pi$$

$$\therefore \quad V(r) = \frac{\ln r}{2\pi}$$

ANOTHER WAY TO CONSTRUCT THE FREE SPACE GREEN'S FCN USING THE FT

METHOD: $v_{xx} + v_{yy} = \delta(x)\,\delta(y)$

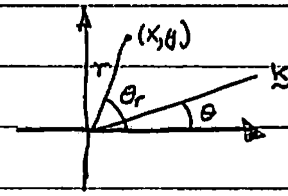
$$\hat{v}(k_1, k_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(k_1 x + k_2 y)}\, v(x,y)\,dx\,dy$$

$$\hat{v}_x = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\underline{k}\cdot\underline{x}}\, v_x\,dx\,dy = \int_{-\infty}^{\infty} \underbrace{\left. v e^{i\underline{k}\cdot\underline{x}}\right|_{-\infty}^{\infty}}_{\to 0} dy - ik_1 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\underline{k}\cdot\underline{x}}\, v\,dx\,dy$$

$$= -ik_1\,\hat{v}$$

$$\therefore \left\{(-ik_1)^2 + (-ik_2)^2\right\}\hat{v} = -(k_1^2 + k_2^2)\,\hat{v} = 1$$

$$\therefore \hat{v} = -\frac{1}{(k_1^2 + k_2^2)}$$



$$v(x,y) = \frac{-1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-i\underline{k}\cdot\underline{x}}}{k^2}\,dx\,dy$$

$$= -\frac{1}{(2\pi)^2}\int_0^{\infty}\int_0^{2\pi} \frac{e^{-ikr\cos(\theta-\theta_r)}}{k^2}\,k\,dk\,d\theta$$

$\theta = \bar{\theta} + \frac{\pi}{2}$ $\quad d\theta = d\bar{\theta}$

$$= -\frac{1}{(2\pi)^2}\int_0^{\infty}\frac{1}{k}\int_{-3\pi/2-\theta_r}^{\pi/2-\theta_r} e^{ikr\sin t}\,dt\,dk$$

$\theta - \theta_r = \frac{\pi}{2} + (\bar{\theta} - \theta_r)$

$\cos(\theta-\theta_r) = -\sin(\bar{\theta}-\theta_r)$

$$= -\frac{1}{2\pi}\int_0^{\infty}\frac{1}{k}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikr\sin t}\,dt\right\}dk$$

$t = \bar{\theta} - \theta_r$ $\quad d\bar{\theta} = dt$

$$= -\frac{1}{2\pi}\int_0^{\infty}\frac{J_0(kr)}{k}\,dk$$

NOW $\displaystyle\int_0^{\infty}\frac{J_0(kr)}{k}\,ds = -\ln r$

$F(s) = \int_0^{\infty} e^{-ks} J_0(kr)/k\,dk$

$F'(s) = -\int_0^{\infty} e^{-ks} J_0(kr)\,dk = \frac{-1}{\sqrt{s^2+r^2}}$

$\therefore v(x,y) = \frac{1}{2\pi}\ln(x^2+y^2)^{1/2}$

$F(s) = -\ln(s + \sqrt{s^2+r^2}) + c$

$F(0) = -\ln r$

2. FREE SPACE GREEN'S FUNCTION IN 3D

$$\Delta V = \delta(\underset{\sim}{x}' - \underset{\sim}{x})$$

METHOD 1: STITCHING:

FOR $r = |\underset{\sim}{x}' - \underset{\sim}{x}|$ AND A SOLUTION DEPENDING ONLY ON r:

$$V_{rr} + \frac{2}{r}V_r = \frac{\delta(r)}{4\pi r^2}$$

HOMOGENEOUS EQ: $V(r) = V_{rr} + \frac{2}{r}V = 0$ $\quad (r^2 V_r)_r = 0$, $V_r = \frac{-A}{r^2}$, $V = \frac{A}{r} + B$

CHOOSE $B = 0$ $\quad V = \frac{A}{r}$

TO ADJUST STRENGTH, CONSTRUCT A SPHERE V_ε ABOUT SOURCE OF RADIUS ε AND INTEGRATE

$$\int_{V_\varepsilon} \underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla} V\, dv = \int_{V_S} \underset{\sim}{n}\cdot\underset{\sim}{\nabla} V\, dv = \int_0^{\pi}\int_{-\pi}^{\pi}\int_0^{\infty} \frac{\delta(r)}{4\pi \cancel{r^2}}\cancel{r^2}\sin\phi\, dr\, d\phi\, d\theta = 1$$

$$\therefore 1 = \int_{S_\varepsilon} \frac{\partial V}{\partial r}\, ds = \int_0^{\pi}\int_{-\pi}^{\pi} -\frac{A}{\varepsilon^2}\cdot \varepsilon^2 \sin\phi\, d\phi\, d\theta = -A\,4\pi$$

$\therefore \quad V(r) = -\frac{1}{4\pi r}$ IS THE FREE SPACE GREEN'S FUNCTION

METHOD 2: USING FOURIER TRANSFORMS.

$$\nabla^2 V = V_{xx} + V_{yy} + V_{zz} = \delta(x,y,z)$$

DEFINE THE FOURIER TRANSFORM $\hat{V}(k_1,k_2,k_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(k_1x + k_2y + k_3z)} V(x,y,z)\,dx\,dy\,dz$

$$\hat{V}_x = \iiint_{-\infty}^{\infty} e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} V_x\, dx^3 = \iint_{-\infty}^{\infty} \cancel{\left[e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} V\right]_{-\infty}^{\infty}} dy\,dz - ik_1 \iint_{-\infty}^{\infty} e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} V\, dy\,dz = -ik_1\hat{V}$$

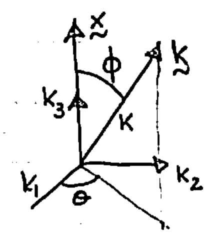
INVERSE TRANSFORM $V(x,y,z) = \frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty} e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}}\hat{V}(\underset{\sim}{k})\, dk^3$

$$\{(-ik_1)^2 + (-ik_2)^2 + (-ik_3)^2\}\hat{V}(\underset{\sim}{k}) = -k^2\hat{V} = 1 \qquad k^2 = k_1^2 + k_2^2 + k_3^2$$

$$\therefore \hat{V}(k) = -1/k^2$$

$$\therefore V(x,y,z) = \frac{-1}{(2\pi)^3}\iiint_{-\infty}^{\infty} \frac{e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}}}{k^2}\, dk^3$$

CHECK SIGN



$$= \frac{-1}{(2\pi)^3}\int_{-\pi}^{\pi}\int_0^{\pi}\int_0^{\infty} \frac{e^{-ikr\cos\phi}}{\cancel{k^2}}\cancel{k^2}\sin\phi\, dk\, d\phi\, d\theta$$

$$= \frac{-1}{(2\pi)^2}\int_0^{\infty}\left[\frac{e^{-ikr\cos\phi}}{+ikr}\right]_0^{\pi} dk \qquad x = |\underset{\sim}{x}| = r$$

$$= \frac{-1}{(2\pi)^2}\int_0^{\infty} \frac{e^{ikr} - e^{-ikr}}{+ikr}\, dk$$

$$= -\frac{2}{(2\pi)^2}\int_0^{\infty} \frac{\sin(kr)}{kr}\, dk \qquad \mathcal{K} = kx \qquad dk = \frac{d\mathcal{K}}{r}$$

$$= -\frac{1}{2\pi^2 r}\int_0^{\infty} \frac{\sin\mathcal{K}}{\mathcal{K}}\, d\mathcal{K}$$

$$= -\frac{1}{2\pi^2 r}\cdot\frac{\pi}{2} = -\frac{1}{4\pi r}$$