GREEN'S FUNCTIONS

LINEAR ALGEBRA

$$Au = f$$

$$u^T A^T v = v^T A u = v^T f$$

IF v_k SOLVES $A^T v_k = e_k = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow k$

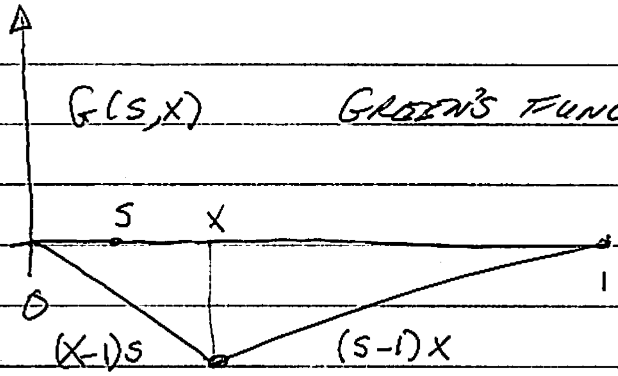
THEN THE kTH COMPONENT OF u IS GIVEN BY

$$u_k = v_k^T f$$

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \\ \vdots \\ u_N \end{bmatrix} = \begin{bmatrix} v_1^T f \\ v_2^T f \\ \vdots \\ v_k^T f \\ \vdots \\ v_N^T f \end{bmatrix} = \begin{bmatrix} \text{---} v_1 \text{---} \\ \text{---------} \\ \text{---} v_k \text{---} \\ \\ \text{---} v_N \text{---} \end{bmatrix} f = \bar{V} f$$

$$\therefore \; u = \bar{V} f$$

WHERE THE ROWS OF $\bar{V}$ SATISFY $A^T v_k = e_k$



GREEN'S FUNCTION.

$G(S,X) = G(X,S)$

METHOD II ADJOINT OPERATOR & STITCHING

$Lu = u'' = f. \qquad u(0) = 0 = u(1)$

$$0 = \int_0^1 V L u \, dx = \int_0^1 V u'' dx$$

$$= V u' \Big|_0^1 - \int_0^1 u' V' dx$$

$$= \left[V u' - u V' \right]_0^1 + \int_0^1 u V'' dx$$

$$= V(1) u'(1) - V(0) u'(0) - \cancel{u(1)} V'(1) + \cancel{u(0)} V(0) + \int_0^1 u V'' dx$$

NOW CHOOSE $V(0) = 0 = V(1)$ AND $V_{SS} = \delta(S-X)$

$$V_{SS} = \delta(S-X)$$

$$V_S = H(S-X) + A$$

$$V(S,X) = (S-X) H(S-X) + AS + B$$

$$V(0,X) = -X \cancel{H(0-X)} + A0 + B = 0$$

$$V(1,X) = (1-X) H(1-X) + A \Rightarrow A = (X-1)$$

$$\therefore V(S,X) = (S-X) H(S-X) + S(X-1)$$

$$= \begin{cases} S(X-1) & S < X \\ S - X + S(X-1) & S > X \end{cases}$$

$$= \begin{cases} S(X-1) & S < X \\ X(S-1) & S > X \end{cases}$$

EG: BVP $Lu = u''(x) = f(x) \qquad u(0) = 0 = u(1)$

$$u = Bf = L^{-1}f \quad ?$$

① VARIATION OF PARAMETERS

H) $Lu = 0 \Rightarrow u = C_1 x + C_2 1$

P) LET $u = xV_1 + 1V_2$

$u' = V_1 + xV_1' + V_2' = V_1 + \{xV_1' + V_2'\}$

$u'' = V_1' + \{xV_1' + V_2'\}'$

REQUIRE $\{xV_1' + V_2'\} = 0$

$\therefore Lu = u'' = V_1' = f$

$$\begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} V_1' \\ V_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

$V_1' = -f/(-1) = f(x) \Rightarrow V_1(x) = \int_0^x f(s)\,ds$

$V_2' = xf/-1 \qquad V_2(x) = -\int_0^x s f(s)\,ds$

$$\therefore u(x) = x\int_0^x f(s)\,ds - 1\int_0^x s f(s)\,ds$$

$$u(x) = c_1 x + c_2 + x\int_0^x f(s)\,ds - \int_0^x s f(s)\,ds$$

$0 = u(0) = c_2 + 0 - 0.$

$0 = u(1) = c_1 + 1\int_0^1 f(s)\,ds - \int_0^1 s f(s)\,ds$

$$\therefore c_1 = \int_0^1 (s-1) f(s)\,ds$$

$$\therefore u(x) = x\left(\int_0^x (s-\cancel{1}) f(s)\,ds + \int_x^1 (s-1) f(s)\,ds\right) + x\int_0^x \cancel{1 f(s)\,ds} - \int_0^x s f(s)\,ds$$

$$\therefore = \int_0^x (x-1)s f(s)\,ds + \int_x^1 (s-1)x f(s)\,ds$$

$$G(s,x) = \begin{cases} (x-1)s & 0 < s < x \\ (s-1)x & x < s < 1 \end{cases}$$

$$\therefore u(x) = \int_0^1 G(s,x) f(s)\,ds.$$

SOLUTION BY STITCHING

SOLVE $\qquad L\,V(s,x) = \delta(s-x) \qquad V(0,x) = 0 = V(1,x)$

HOMOG $\qquad V(s,x) = A \;\; + B$

$$V(s,x) = \begin{cases} A_- s & 0 < s < x \\ A_+(s-1) & x < s < 1 \end{cases}$$

CONTINUITY

$$V(s,x)\Big|_{s=x_+} = V(s,x)\Big|_{s=x_-}$$

$$A_+(x-1) = A_- x$$

$$V_{ss} = \delta(s-x)$$

$$\int_{x-\varepsilon}^{x+\varepsilon} V_{ss}\,ds = V_s = 1$$

$$\therefore \quad \frac{\partial V}{\partial s}(s,x)\Big|_{s=x_+} - \frac{\partial V}{\partial s}(s,x)\Big|_{s=x_-} = 1$$

$$A_+ - A_- = 1 \qquad\qquad A_+ = A_- + 1$$

$$\therefore \quad (A_- + 1)(x-1) = A_- x$$

$$\cancel{A_- x} - A_- + (x-1) = \cancel{A_- x}$$

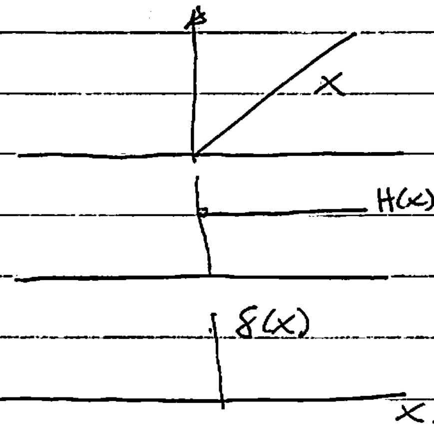
$$A_- = (x-1) \qquad\qquad A_+ = (x-1) + 1 = x$$

$$\therefore \quad V(s,x) = \begin{cases} s(x-1) & 0 < s < x \\ x(s-1) & x < s < 1. \end{cases}$$

GENERALIZED FUNCTIONS – DISTRIBUTIONS

IDEA: TRY TO GIVE SOME RIGOUR TO THE δ FUNCTION AND ITS DERIVATIVES.

EG: $F(x) = \begin{cases} 0 & x < 0 \\ x & x \geq 0 \end{cases}$



$F'(x) = H(x)$

$F''(x) = \delta(x)$

INTERPRET $F''(x)$ THROUGH ITS ACTION AGAINST AN INFINITELY SMOOTH FUNCTION $\phi(x)$

$$\int_a^b F''(x)\,\phi(x)\,dx = F'\phi + F\phi' \Big|_a^b + \int_a^b F(x)\,\phi''(x)\,dx.$$

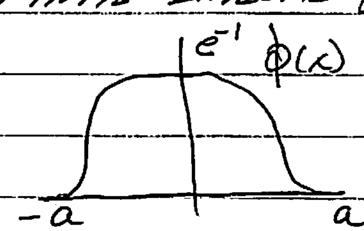
IF WE CHOOSE a & b TO BE $\pm\infty$ AND ϕ TO VANISH OUTSIDE SOME FINITE INTERVAL AND BE INFINITELY DIFFERENTIABLE

$$\int_a^b F''(x)\phi(x)\,dx = \int_a^b F(x)\,\phi''(x)\,dx$$

ALLOWS US TO INTERPRET $F''(x)$.

DEF: $\phi(x)$ IS A TEST FUNCTION IF IT IS INFINITELY DIFFERENTIABLE AND $\phi(x)$ VANISHES OUTSIDE SOME FINITE INTERVAL $[-a, a]$

eg: $\phi(x) = \begin{cases} e^{-\frac{a^2}{a^2-x^2}} & |x| < a \\ 0 & |x| \geq a. \end{cases}$



DEF: THE GENERALIZED FUNCTION $f(x)$ IS DEFINED WITH RESPECT TO THE SET OF TEST FUNCTIONS $\phi(x)$ TO BE

$$T_f(\phi) = \int_{-\infty}^{\infty} f(x)\,\phi(x)\,dx = (f, \phi)$$

1) $T_f(\cdot)$ IS A LINEAR OPERATOR SINCE

$$T_f(\alpha\phi_1+\beta\phi_2) = \alpha T_f(\phi_1)+\beta T_f(\phi_2)$$

2) IF f IS AN ORDINARY FUNCTION (eg A PW CONTINUOUS FCN) THEN f IS ALSO A GENERALIZED FCN.

$$T_f(\phi) = \int f(x)\,\phi(x)\,dx.$$

3) DEFINITION OF THE DIRAC DELTA FUNCTION

$$(\delta(x),\phi) = \phi(0).$$

4) PROPERTIES OF THE δ FUNCTION:

a) SCALING

$$(\delta(Cx),\phi) = \int_{-\infty}^{\infty} \delta(Cx)\,\phi(x)\,dx \qquad C\geq 0$$

$$= \int_{-\infty}^{\infty} \delta(\chi)\,\phi(\chi/c)\,\frac{d\chi}{c} \qquad \chi = Cx, \quad dx = \frac{d\chi}{c}$$

$$= \frac{1}{c}\phi(0).$$

IF $c<0$ $\quad C=-|C|$

LET $\chi = Cx = -|C|x \qquad d\chi = -|C|\,dx \qquad dx = -\frac{d\chi}{|C|}$

$$\therefore\ (\delta(cx),\phi) = \int_{\infty}^{-\infty} \delta(\chi)\,\phi(-\chi/|c|)\,\frac{-d\chi}{|C|} = \frac{1}{|C|}\phi(0)$$

(b) TRANSLATION

$$(\delta(x-y),\phi) = \int_{-\infty}^{\infty} \delta(x-y)\,\phi(x)\,dx \qquad \chi = x-y$$

$$= \int_{-\infty}^{\infty} \delta(\chi)\,\phi(\chi+y)\,d\chi \qquad x = \chi+y$$

$$= \phi(y)$$

(c) $(x\,\delta(x),\phi) = \int_{-\infty}^{\infty} \delta(x)\{x\,\phi(x)\}\,dx = 0 \implies x\,\delta(x)=0.$

(d) $(\delta'(x),\phi) = \int_{-\infty}^{\infty} \delta'(x)\,\phi(x)\,dx = \delta(x)\phi\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\delta(x)\,\phi'(x)\,dx$

$$= -\phi'(0)$$

(e) DERIVATIVE OF THE HEAVISIDE FUNCTION

$$(H', \phi) = \int_{-\infty}^{\infty} H' \phi \, dx = -\int_{-\infty}^{\infty} H \phi'(x) \, dx$$

$$= -\int_{0}^{\infty} \phi'(x) \, dx =$$

$$= -\{\phi(\infty) - \phi(0)\} = \phi(0)$$

$\therefore \; H'(x) = \delta(x)$

(f) $(D^n f, \phi) = (-1)^n (f, \phi^{(n)})$

(g) COMPOSITION OF A GENERALIZED FUNCTION WITH AN INVERTIBLE FUNCTION.

EG. LET $g : \mathbb{R} \to \mathbb{R}$ BE 1-1 & ONTO DIFFERENTIABLE FUNCTION WITH $g'(x) > 0$.

$f \circ g(x) = f(g(x))$

$$(f(g(x)), \phi) = \int_{-\infty}^{\infty} f(g(x)) \, \phi(x) \, dx$$

$$= \int_{-\infty}^{\infty} f(y) \, \phi(g^{-1}(y)) \, \frac{dy}{g'(g^{-1}(y))}$$

LET $y = g(x)$

$x = g^{-1}(y)$

dx.

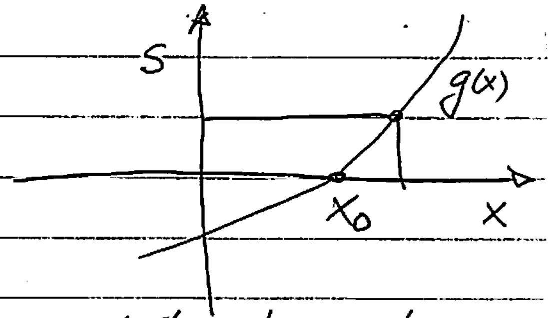
$g(g^{-1}(y)) = 1$

$g' \frac{d}{dy} g^{-1}(y) = 1$

$\frac{d}{dy} g^{-1}(y) = \frac{1}{g'}$

$dx = \frac{dg^{-1}}{dy} dy = \frac{1}{g'(g^{-1}(y))}$

(h) LET $g(x)$ BE A SMOOTH FUNCTION THAT VANISHES AT x_0 WITH $g'(x_0) > 0\ \forall x$ IE. g IS 1-1



$$(\delta(g(x)), \phi(x)) = \int_{-\infty}^{\infty} \delta(g(x))\, \phi(x)\, dx$$

LET $s = g(x)$ $\quad x = g^{-1}(s)$ $\quad dx = \frac{dg^{-1}}{ds} ds$

$x = g^{-1}(g(x))$ $\quad 1 = \frac{dg^{-1}}{ds}\frac{ds}{dx} \Rightarrow \frac{dg^{-1}}{ds} = \frac{1}{g'(x)} = \frac{1}{g'(g^{-1}(s))}$

$$\therefore (\delta(g(x)), \phi(x)) = \int_{-\infty}^{\infty} \delta(s)\, \phi(g^{-1}(s))\, \frac{ds}{g'(g^{-1}(s))}$$

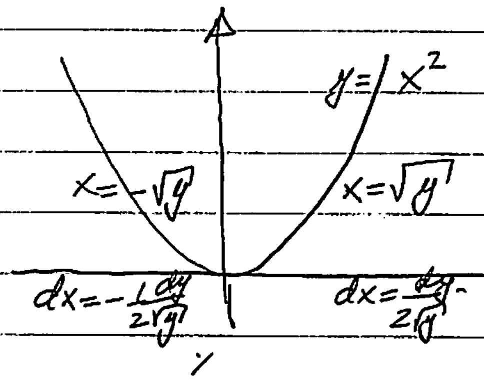
$$dx = \frac{dg^{-1}}{ds} ds = \frac{1}{g'(g^{-1}(s))}$$

$$= \frac{\phi(x_0)}{g'(x_0)} = \frac{\delta(x - x_0)}{g'(x_0)}$$

IF $g(x)$ IS MONOTONIC

$$(\delta(g(x)), \phi(x)) = \frac{\phi(x_0)}{|g'(x_0)|} = \frac{\delta(x - x_0)}{|g'(x_0)|}$$

EG: COMPOSITION AT A REPEATED ROOT



$$(\delta(x^2), \phi(x)) = \int_{-\infty}^{\infty} \delta(x^2)\, \phi(x)\, dx$$

$$= \int_{+\infty}^{0} \delta(y)\, \phi(-\sqrt{y})\, \frac{dy}{-2\sqrt{y}}$$

$$+ \int_{0}^{\infty} \delta(y)\, \phi(\sqrt{y})\, \frac{dy}{2\sqrt{y}}$$

$$= \frac{1}{2}\int_{0}^{\infty} \delta(y) \left\{ \phi(-\sqrt{y}) + \phi(\sqrt{y}) \right\} \frac{dy}{\sqrt{y}}$$

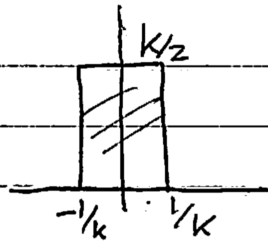
$$= \frac{1}{4}\int_{-\infty}^{\infty} \delta(y) \left\{ \phi(-\sqrt{|y|}) + \phi(\sqrt{|y|}) \right\} \frac{dy}{\sqrt{|y|}}$$

$$= \frac{1}{4} \lim_{x \to 0} \frac{\phi(-x) + \phi(x)}{|x|}$$

$$|x|\, \delta(x^2) = \frac{1}{4}\left\{ \delta(x-) + \delta(x+) \right\}$$

δ-SEQUENCES:

$$f_K(x) = \begin{cases} K/2 & |x| < 1/K \\ 0 & \end{cases}$$



$$\int_{-\infty}^{\infty} f_K(x)\,dx = 1$$

$$(f_K, \phi) = \int_{-\infty}^{\infty} f_K \phi\, dx = \frac{K}{2}\int_{-1/K}^{1/K} \phi(x)\,dx = \frac{K\phi(\bar{x})}{2}\int_{-1/K}^{1/K} dx = \phi(\bar{x}) = \phi(0)$$

$$\bar{x} \in (-1/K, 1/K)$$

$$\lim_{K\to\infty} (f_K, \phi) = \phi(0) = \delta(x) \quad \text{WEAK CONVERGENCE}$$

NOTE $\displaystyle \lim_{K\to\infty} (f_K, \phi) = \delta(x)$

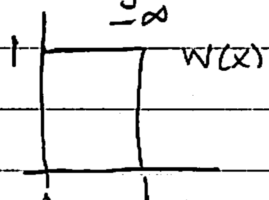
$$\neq$$

$$\left(\lim_{K\to\infty} f_K, \phi\right) = (0, \phi)$$

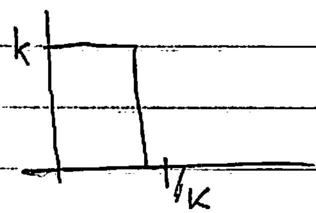
CONSTRUCTING δ SEQUENCES

IF $\displaystyle\int_{-\infty}^{\infty} W(x)\,dx = 1$ THEN $W_K(x) = K W(KX)$ IS A δ SEQUENCE

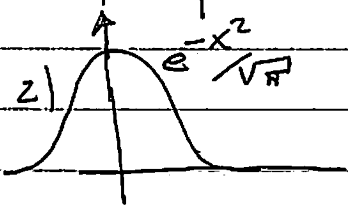
EG: 1)



$W_K = K W(KX)$



2)



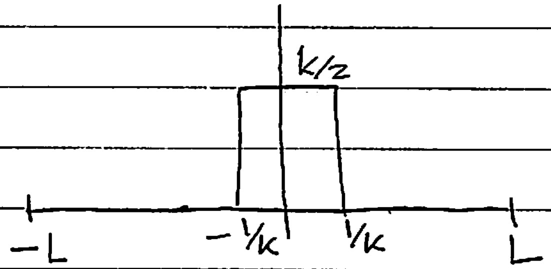
$$W_K = \frac{K e^{-K^2 x^2}}{\sqrt{\pi}}$$

$$u(x,t) = \frac{e^{-x^2/4t}}{2\sqrt{\pi t}} \qquad K = \frac{1}{2\sqrt{t}}$$

IS A δ SEQUENCE

δ FUNCTION AND FOURIER SERIES

$$\delta_K(x) = \begin{cases} K/2 & |x| < 1/K \\ 0 & \end{cases}$$



EXPAND $\delta_K(x)$ AS A FS:

$$\delta_K(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_n = \frac{2}{L}\int_0^{1/K} \frac{K}{2}\cos\left(\frac{n\pi x}{L}\right)dx \qquad a_0 = \frac{\cancel{2}}{L}\int_0^{1/K} \frac{K}{\cancel{2}}\,dK = \frac{1}{L}$$

$$= \frac{K}{L}\,\frac{\sin\left(\frac{n\pi x}{L}\right)}{(n\pi/L)}\Bigg|_{\cancel{0}}^{1/K}$$

$$= \frac{K}{n\pi}\sin\left(\frac{n\pi}{KL}\right)$$

$$\delta_K(x) = \frac{1}{2L} + \sum_{n=1}^{\infty}\frac{K}{n\pi}\sin\left(\frac{n\pi}{KL}\right)\cos\left(\frac{n\pi x}{L}\right)$$

$$\xrightarrow{K\to\infty} \frac{1}{2L} + \frac{1}{L}\sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{L}\right)$$

$$\lim_{K\to\infty}\frac{K}{n\pi}\sin\left(\frac{n\pi}{KL}\right)$$

$$\lim_{S\to\infty} S\sin\left(\frac{1}{LS}\right)$$

$$\lim_{\lambda\to 0}\frac{\sin(\lambda/L)}{\lambda} = \frac{\frac{1}{L}\cos(\lambda}{1}$$

FORMALLY:

$$a_n = \frac{1}{L}\int_{-L}^{L}\delta(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\cdot 1$$

$$b_n = 0$$

$$\delta(x) = \frac{1}{2L} + \frac{1}{L}\sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{L}\right)$$

GREEN'S FUNCTIONS FOR ODE'S

$$Lu = a_0 u'' + a_1 u' + a_2 u \qquad (u,v) = \int_{x_0}^{x_1} u\, v\, dx$$

$$u(x_0) = u_0 \qquad u(x_1) = u_1$$

$$(v, Lu) = \int_{x_0}^{x_1} v\{a_0 u'' + a_1 u' + a_2 u\}\, dx$$

$$= v a_0 u' + v a_1 u \Big]_{x_0}^{x_1} - \int_{x_0}^{x_1} u'(a_0 v)' + u(a_1 v)' - u a_2 v\, dx$$

$$= \left[v a_0 u' - u(a_0 v)' + v a_1 u\right]_{x_0}^{x_1} + \int_{x_0}^{x_1} u\{(a_0 v)'' - (a_1 v)' + a_2 v\}\, dx$$

$$= BT + \int_{x_0}^{x_1} u\, L^* v\, dx.$$

WHERE $\quad L^* v = (a_0 v)'' - (a_1 v)' + a_2 v = 0.$

AND $\quad BT = \left[v a_0 u' - u a_0' v - u a_0 v' + a_1 v u\right]_{x_0}^{x_1}$

IF WE CHOOSE $v(x_0) = 0$ & $v(x_1) = 0$ THEN

$$BT = -u(x_1) a_0(x_1) v'(x_1) + u(x_0) a(x_0) v'(x_0)$$

IF $\quad L_s^* v(s,x) = \dfrac{d^2}{ds^2}(a_0(s) v(s,x)) - \dfrac{d}{ds}(a_1 v(s,x)) + a_2 v(s,x) = \delta(s-x)$

IMPOSE THE BC $\quad v(x_0, x) = 0 = v(x_1, x)$

THEN $\quad u(x) = \displaystyle\int_{x_0}^{x_1} v(s,x) f(s)\, ds + u(x_1) a_0(x_1) v'(x_1, x) - u(x_0) a(x_1) v'(x_0, x)$

EG 1 $Lu = x^2u'' - 2xu' - 4u = f(x) \quad 0<x<1 \quad u(0)<\infty \quad u(1)=1$ (1)

$(v,Lu) = \int_0^1 v\,Lu\,dx = \int_0^1 v\{x^2u'' - 2xu' - 4u\}\,dx$

$= \left[v x^2 u' - v2xu\right]_0^1 - \int_0^1 u'(x^2v)' - u(2xv)' + 4uv\,dx$

$= \left[vx^2u' - u(x^2v)' - v2xu\right]_0^1 + \int_0^1 u\{(x^2v)'' + (2xv)' - 4v\}\,dx$

$= vx^2u'\Big]_0^1 - u2xv\Big]_0^1 - ux^2v'\Big|_0^1 - v2xu\Big]_0^1 + \int_0^1 uL^*v\,dx$

TO MAKE THIS TERM INVOLVING UNKNOWNS $u'(0)$ & $u'(1)$ VANISH CHOOSE

$v(0)<\infty \quad v(1)=0$

IN THIS CASE ALL THE BOUNDARY TERMS VANISH EXCEP $ux^2v'\Big|^1$

$(v,Lu) = -u(1).1^2v'(1) + \int_0^1 uL^*v\,dx$

THE ADJOINT PROBLEM WE NEED TO SOLVE IS

$$\left.\begin{aligned} L_s^*v &= (s^2v)'' + (2sv)' - 4v = \delta(s-x) \\ v(0)&<\infty \quad v(1)=0 \end{aligned}\right\} \quad (2)$$

IN WHICH CASE $\quad u(x) = \int_0^1 v(s,x)f(s)\,ds + v'(1)$

THE GREEN'S FUNCTION FOR THE ADJOINT PROBLEM (2).

$L_s^*v = (s^2v'' + 4sv' + 2v) + (2sv' + 2v) - 4v$

$= s^2v'' + 6sv' = 0$

HOMOGENEOUS EQ $\quad v = s^r \Rightarrow r(r-1) + 6r = r^2 + 5r = r(r+5) = 0 \quad r = 0, 5$

$\therefore \quad v(s,x) = \begin{cases} A_- + B_-s^{-5} & 0<s<x \\ A_+ + B_+s^{-5} & x<s<1 \end{cases}$

$\left.\begin{aligned} v(0,x)<\infty &\Rightarrow B=0 \\ v(1,x) = A_+ + B_+ = 0 &\quad B_+ = -A_+ \end{aligned}\right\} \quad v(s,x) = \begin{cases} A_- \\ A_+(1-s^{-5}) \end{cases}$

<u>CONTINUITY:</u> $\quad v(x_-,x) = A_- = A_+(1-x^{-5}) = v(x_+,x)$

<u>JUMP CONDITION:</u> $\quad \int_{x-\varepsilon}^{x+\varepsilon} L_x v\,ds = \int_{x-\varepsilon}^{x+\varepsilon} [(s^2v)'' + (2sv)' - 4v]\,ds = 1$

$\therefore \quad (s^2v)'\Big|_{x-\varepsilon}^{x+\varepsilon} + \overset{\text{CONT}}{\cancel{[2sv]_{x-\varepsilon}^{x+\varepsilon}}} - 4\cancel{\int_{x-\varepsilon}^{x+\varepsilon} v\,ds} = 1$

$s^2v'\Big|_{x-\varepsilon}^{x+\varepsilon} + \cancel{2sv\Big|_{x-\varepsilon}^{x+\varepsilon}}$ CONT $= 1$

$\therefore \quad s^2v'\Big|_{x-}^{x+} = 1$

$v_s = \begin{cases} 0 \\ A_+5s^{-6} \end{cases} \quad \therefore \quad x^2[A_+5x^{-6} - 0] = 1 \quad A_+ = \frac{1}{5}x^4$

$\therefore \quad A_- = \frac{x^4}{5}(1-x^{-5}) = \frac{1}{5}(x^4 - x^{-1})$

$$\therefore \quad V(s,x) = \begin{cases} \frac{1}{5}(x^4 - x^{-1}) & 0 < s < x \\ \frac{x^4}{5}(1 - s^{-5}) & x < s < 1 \end{cases}$$

$$V_s = \frac{x^4}{5} \cdot 5 s^{-6} = x^4 s^{-6} \qquad V_s(1,x) = x^4$$

$$\therefore \quad u(x) = x^4 + \int_0^1 V(s,x) f(s)\, ds$$

SELF-ADJOINT OPERATORS

DEFINITION: FORMALLY SELF ADJOINT: AN OPERATOR L IS SAID TO BE FORMALLY SELF-ADJOINT IF $L=L^*$ (FSA)

WHEN IS $L=L^*$?

$$LV = a_0 V'' + a_1 V' + a_2 V$$

$$L^*V = (a_0 V)'' - (a_1 V)' + a_2 V$$

$$= a_0 V'' + (2a_0' - a_1)V' + (a_0'' - a_1' + a_2)V$$

$2a_0' - a_1 = a_1$ PROVIDED $\boxed{a_0' = a_1}$

$a_0'' - a_1' + a_2 = a_2$ ALSO PROVIDED $a_0' = a_1$

IN THIS CASE

$$LV = a_0 V'' + a_0' V' + a_2 V = (a_0 V')' + a_2 V$$

IF L IS NOT FSA CAN WE MAKE IT SELF ADJOINT BY MULTIPLYING BY A SUITABLE FACTOR?

$$LV = a_0 V'' + a_1 V' + a_2 V$$

$$\mathcal{L}V = (FL)V = Fa_0 V'' + Fa_1 V' + Fa_2 V$$

$$(Fa_0)' = Fa_1$$

$$F' + \left(\frac{a_0'}{a_0} - \frac{a_1}{a_0}\right)F = 0$$

$$\left[e^{\int \frac{a_0'}{a_0} - \frac{a_1}{a_0} dx} F\right]' = 0$$

$$\therefore \quad F = e^{-\ln a_0 + \int \frac{a_1}{a_0} dx}$$

$$\therefore \quad \boxed{F = \frac{e^{\int \frac{a_1}{a_0} dx}}{a_0}} \qquad \text{ABEL'S FORMULA}$$

EG 1 REVISITED BY TURNING THE PROBLEM TO A SELF-ADJOINT ONE

$$Lu = x^2u'' - 2xu' - 4u = f(x) \quad 0<x<1 \quad u(0)<\infty \quad u(1)=1$$

$a_0 = x^2 \quad a_1 = -2x$

BY ABEL'S FORMULA $\quad F(x) = \dfrac{e^{\int \frac{a_1}{a_0}dx}}{a_0} = \dfrac{e^{-\int \frac{2x}{x^2}dx}}{x^2} = \dfrac{e^{\ln x^{-2}}}{x^2} = x^{-4}$

$$\therefore \mathcal{L}u = (FL)u = x^{-2}u'' - 2x^{-3}u' - \frac{4}{x^4}u = \frac{f(x)}{x^4}$$

$$\mathcal{L}u = (x^{-2}u')' - \frac{4}{x^2}u = \frac{f(x)}{x^4} \qquad \mathcal{L} \text{ IS FORMALLY SELF ADJOINT}$$

$$\int_0^1 v\mathcal{L}u\,dx = \int_0^1 v\left\{(x^{-2}u')' - \frac{4}{x^2}u\right\}dx$$

$$= vx^{-2}u'\Big|_0^1 - v'x^{-2}u\Big|_0^1 + \int_0^1 u\mathcal{L}v\,dx.$$

$v(0)<\infty \quad v(1)=0$

AND SOLVE THE BVP $\quad \mathcal{L}_s v = (s^{-2}v')' - \frac{4}{s^2}v = \delta(s-x) \quad v(0)<\infty, \; v(1)=0$

HOMOG. EQ: $\quad s^2v'' - 2sv' - 4v = 0$

$$v = s^r \Rightarrow r(r-1) - 2r - 4 = r^2 - 3r - 4 = (r+1)(r-4) = 0 \quad r=-1,4$$

$$v(s,x) = \begin{cases} A_-s^4 + B_-s^{-1} \\ A_+s^4 + B_+s^{-1} \end{cases}$$

$v(0,x) < \infty \Rightarrow B_- = 0$

$0 = v(1,x) = A_+ + B_+ \Rightarrow B_+ = -A_+.$

$$\therefore v(s,x) = \begin{cases} A_-s^4 & 0<s<x \\ A_+(s^4 - s^{-1}) & x<s<1. \end{cases} \qquad v_s(s,x) = \begin{cases} 4A_-s^3 \\ A_+(4s^3 + s^{-2}) \end{cases}$$

$$v(x_-,x) = A_-x^4 = A_+(x^4 - x^{-1}) = v(x_+,x) \text{ CONT} \quad -(3)$$

$$\int_{x-\varepsilon}^{x+\varepsilon} (s^{-2}v')' - \frac{4}{s^2}v\,ds = s^{-2}v'\Big|_{x-\varepsilon}^{x} - 4\int_{x-\varepsilon}^{x+\varepsilon} \frac{v}{s^2}\,ds = 1$$

$$\therefore x^{-2}A_+(4x^3 + x^{-2}) - x^{-2}A_-(4x^3) = 1$$

$$A_+(4x + x^{-4}) - A_-(4x) = 1$$

$(3)\; 4x^{-3} \Rightarrow \quad A_+(4x - 4x^{-4}) - A_-(4x) = 0$

$$\therefore A_+5x^{-4} = 1 \quad \text{OR} \quad A_+ = \frac{x^4}{5}$$

$$A_-x^4 = \frac{x^4(x^4 - x^{-1})}{5}$$

$$\therefore A_- = \frac{1}{5}(x^4 - x^{-1})$$

$$\therefore \quad v(s,x) = \begin{cases} \frac{1}{5} s^4 (x^4 - x^{-1}) & 0 < s < x \\ \frac{1}{5} x^4 (s^4 - s^{-1}) & x < s < 1 \end{cases}$$

$$v_s(s,x) = \frac{x^4}{5}(4s^3 + s^{-2}) \qquad v_s(1,x) = x^4$$

$$\therefore \quad u(x) = \int_0^1 \begin{Bmatrix} \frac{1}{5} s^4 (x^4 - x^{-1}) \\ \frac{1}{5} x^4 (s^4 - s^{-1}) \end{Bmatrix} \frac{f(s)}{s^4}\, ds + v_s(1,x)$$

$$= \int_0^1 \begin{Bmatrix} \frac{1}{5}(x^4 - x^{-1}) \\ \frac{x^4}{5}(1 - s^{-5}) \end{Bmatrix} f(s)\, ds + x^4 \qquad \text{AS BEFORE}$$

USING THE WRONSKIAN FORMULA:

$$v(s,x) = \begin{cases} A_- w_0 \\ A_+ w_1 \end{cases} \qquad w_0 = s^4 \quad w_1 = s^4 - s^{-1}$$

$$w_0' = 4s^3 \quad w_1' = 4s^3 + s^{-2}$$

$$\overline{W}(x) = \begin{vmatrix} w_0 & -w_1 \\ -w_0' & w_1' \end{vmatrix} = w_0 w_1' - w_0' w_1 = x^4(\cancel{4x^3} + x^{-2}) - 4x^3(\cancel{x^4} - x^{-1})$$

$$= 5x^2$$

$$\therefore \quad v(s,x) = \frac{1}{p(x)\overline{W}(x)} \begin{cases} w_0(s)\, w_1(x) & 0 < s < x \\ w_0(x)\, w_1(s) & x < s < 1 \end{cases}$$

$$= \frac{1}{5} \begin{cases} s^4 (x^4 - x^{-1}) & 0 < s < x \\ x^4 (s^4 - s^{-1}) & x < s < 1 \end{cases}$$

GREEN'S FUNCTION FOR A SELF ADJOINT PROBLEM

$$Lu = (pu')' + qu = f(x) \qquad 0<x<1$$

$$B_0 u = u'(0) + \alpha u(0) = 0 \qquad B_1 = u'(1) + \beta u(1) = 0.$$

$$\int_0^1 v\, Lu\, dx = \int_0^1 v\{(pu')' + q u\} dx$$

$$= v\, p u' \Big|_0^1 - u(pv')\Big|_0^1 + \int_0^1 u\{(pv')' + qv\} dx$$

$$\int_0^1 vLu\,dx - \int_0^1 uLv\,dx = p(1)\{v(1)\, u'(1) - v'(1)\, u(1)\}$$

$$- p(0)\{v(0)\, u'(0) - v'(0)\, u(0)\}$$

$$= p(1)\{v(1)[u'(1) + \beta u(1)] - u(1)[v'(1) + \beta v(1)]$$

$$- p(0)\{v(0)[u'(0) + \alpha u(0)] - u(0)[v'(0) + \alpha v(0)]\}$$

IF WE CHOOSE v: $B_1 v = v'(1) + \beta v(1) = 0$ & $B_0 v = v'(0) + \alpha v(0) = 0$ THEN ALL THE BOUNDARY TERMS VANISH AND

$$\int_0^1 vLu\,dx = \int_0^1 uLv\,dx$$

IF $v(s,x)$ SATISFIES $L_s v(s,x) = (pv')' + qv = \delta(s-x)$

$$B_1 v = v'(1) + \beta v(1) = 0 \qquad B_0 v = v'(0) + \alpha v(0) = 0$$

THEN $\quad u(x) = \int_0^1 v(s,x) f(s)\, ds.$

SOLUTION BY STITCHING:

LET $W_0(s)$ SOLVE THE HOMOG. EQ $L_s W_0(s) = 0$ SUCH THAT $B_0 W_0(s) = 0.$

AND $W_1(s)$ SOLVE THE HOMOG EQ $L_s W_1(s) = 0$ SUCH THAT $B_1 W_1(s) = 0$

THEN $\quad v(s,x) = \begin{cases} A_- W_0(s) & 0<s<x \\ A_+ W_1(s) & x<s<1 \end{cases}$

CONTINUITY $\qquad A_- W_0(x) = A_+ W_1(x)$

JUMP: $\qquad \int_{x-\varepsilon}^{x+\varepsilon} (pv') + qv\, ds = pv'\Big|_{x-\varepsilon}^{x+\varepsilon} + \int_{x-\varepsilon}^{x+\varepsilon} q \cancel{v}\, ds = 1$ CONT

$$\therefore\ p(x)[v'(x_+) - v'(x_-)] = p(x)[A_- W_0' - A_+ W_1'] = 1$$

$$\therefore \begin{bmatrix} W_0 & -W_1 \\ -W_0' & W_1' \end{bmatrix} \begin{bmatrix} A_- \\ A_+ \end{bmatrix} = \begin{bmatrix} 0 \\ 1/p \end{bmatrix}$$

$$\therefore A_- = W_1/p\, /\, (W_0 W_1' - W_0' W_1) \qquad \mathcal{W}(x) = W_0 W_1' - W_0' W_1$$

$$A_+ = W_0/p/\mathcal{W}(x)$$

$$\therefore\ v(s,x) = \frac{1}{p(x)\mathcal{W}(x)} \begin{cases} W_0(s) W_1(x) & 0<s<x \\ W_0(x) W_1(s) & x<s<1 \end{cases}$$

GREEN'S FUNCTION EXISTS PROVIDED $\mathcal{W}(x) \neq 0.$

DEFINITION: ESSENTIALLY SELF ADJOINT

$$Lu = a_0 u'' + a_1 u' + a_2 u$$

$$B_0 u(0) = \alpha_0 \qquad B_1 u(1) = \alpha_1$$

THE OPERATOR (L, B_0, B_1) IS ESSENTIALLY SELF ADJOINT IF

1) $L^* = L \qquad (a_0' = a_1)$

2) THE BOUNDARY CONDITIONS ON V REQUIRED TO MAKE THE UNKNOWN BOUNDARY TERMS VANISH ARE HOMOGENEOUS VERSIONS OF THOSE ON u, i.e.

$$B_0 V(0) = 0 = B_1 V(1)$$

NOTE: 1) FOR A SELF-ADJOINT PROBLEM

$$G(s,x) = G(x,s) \qquad \text{MAXWELL RECIPROCITY}$$

TO SEE THIS CHECK THE WRONSKIAN EXPRESSION FOR $G(s,x)$

2) HOW CAN WE BE SURE $p(x)\bar{W}(x)$ DOES NOT VANISH?

$$0 = w_1 L w_0 - w_0 L w_1 = w_1 (p w_0')' - w_0 (p w_1')'$$

$$\int^x w_1 (p w_0')' - w_0 (p w_1')' \, dx = \int^x 0 \, dx + c$$

$$\therefore \; w_1 p w_0' - w_0 p w_1' - \int w_1' p w_0' - w_0' p w_1' \, dx = c$$

$$\therefore \; p(x)(w_1 w_0' - w_0 w_1') = c$$

$$\therefore \; p(x)\bar{W}(x) = p(x)(w_0 w_1' - w_1 w_0') = \bar{c}$$

3) WHAT HAPPENS IF $\bar{W}(x) = w_0 w_1' - w_1 w_0' = 0$?

$$w_1 = \mu w_0$$

THEN $L w_0 = 0$ AND $B_0 w_0 = 0$
AND $L w_1 = 0$ AND $B_1 w_1 = 0$ } DEF OF w_0 & w_1

BUT $L w_1 = \mu L w_0 = 0$

$$0 = B_1 w_1 = \mu B_1 w_0$$

THUS w_0 SATISFIES $L w_0 = 0$

AND $B_0 w_0 = 0$ AND $B_1 w_0 = 0$

THUS w_0 IS AN EIGENFUNCTION OF THE BVP WITH EIGENVALUE 0.

RELATIONSHIP BETWEEN G AND THE SPECTRAL THEORY IF L IS ESSENTIALLY S.A.

CONSIDER THE STURM-LIOUVILLE OPERATOR WITH A FORCING TERM $f(x)$.

$$\left.\begin{aligned} Lu &= -(pu')' + qu = \mu r(x) u + f(x) \\ B_0 u &= u'(0) + \alpha u(0) = 0 \quad B_1 u = u'(1) + \beta u(1) = 0 \end{aligned}\right\} (1)$$

THEN THE EIGENVALUE PROBLEM $Lu = \lambda r u \quad B_0 u = 0 = B_1 u$

HAS EIGENVALUES $\lambda_1 < \lambda_2 \ldots < \lambda_n < \ldots$ AND ORTHONORMAL

EIGENFUNCTIONS $\{\phi_m(x)\}$

SUCH THAT $\int_0 \phi_m(x)\phi_n(x) r(x)\,dx = \delta_{mn}$

WE ARE LOOKING FOR A GREEN'S FUNCTION THAT SATISFIES

$$L_s G(s,x) - r(s)\mu G(s,x) = \delta(s-x)$$

$$B_0 G(0,x) = 0 = B_1 G(1,x)$$

NOW ASSUME AN EIGENFUNCTION EXPANSION FOR $G(s,x)$

$$G(s,x) = \sum_{n=1}^{\infty} g_n(x)\phi_n(s)$$

THUS $L_s G = \sum_{n=1}^{\infty} g_n(x) L_s\phi_n(s) = \sum_{n=1}^{\infty} g_n(x)\{\lambda_n r(s)\phi_n(s) - \mu r(s)\phi_n(s)\} = \delta(s-x)$

$$\therefore \sum_{n=1}^{\infty} g_n(x) r(s)\phi_n(s)(\lambda_n - \mu) = \delta(s-x)$$

$\int_0^1 \phi_m(s)\,ds. \Rightarrow \quad \sum_{n=1}^{\infty} g_n(x)\int_0^1 r(s)\phi_m(s)\phi_n(s)\,ds\,(\lambda_n - \mu) = \phi_m(x)$

$$\therefore \quad g_m(x) = \phi_m(x)/(\lambda_m - \mu)$$

$$\therefore \quad G(s,x) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(s)}{(\lambda_n - \mu)} \qquad (2)$$

NOTE: 1) ONCE THE EIGENFUNCTIONS OF L ARE KNOW $G(s,x)$ IS GIVEN BY (2)

2) $u(x) = \int_0^1 f(s)G(s,x)\,ds = \sum_{n=1}^{\infty} \phi_n(x) \dfrac{\int_0^1 f(s)\phi_n(s)\,ds}{\lambda_n - \mu}$.

3) IF $\mu = \lambda_m$ FOR SOME m THEN THE GREEN'S FUNCTION DOES NOT EXIST SINCE THERE IS A NONTRIVIAL $\phi_m \not\equiv 0$ THAT SOLVES THE HOMOGENEOUS PROBLEM. IN THIS CASE FOR THERE TO BE A SOLUTION WE REQUIRE $\int f(s)\phi_m(s)\,ds = 0$ AND $u(x) = C\phi_m(x) + \sum_{n=1}^{\infty} \phi_n(x)\,(\phi_n(s), f(s))/(\lambda_n - \mu) \quad m \neq n$.

SPECIAL CASE $\mu = 0 = \lambda_1$

IF THERE EXISTS A NONTRIVIAL SOLUTION $V(x)$ TO THE HOMOGENEOUS PROBLEM

$$LV = 0 \qquad (0)$$
$$B_0 V = 0 = B_1 V$$

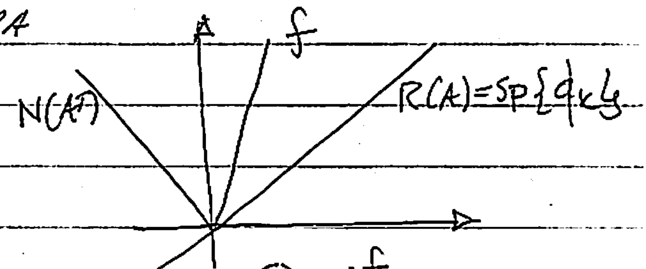
THEN CONSIDER SOLVING

$$\left.\begin{array}{l} Lu = f \\ B_0 u = 0 = B_1 u \end{array}\right\} (1)$$

$$(V, f) = (V, Lu) \doteq (u, LV) = 0$$

THUS IF (1) IS TO HAVE A SOLUTION f MUST SATISFY THE SOLVABILITY CONDITION.

NOTE: 1) ANALOGUS WITH LINEAR ALGEBRA



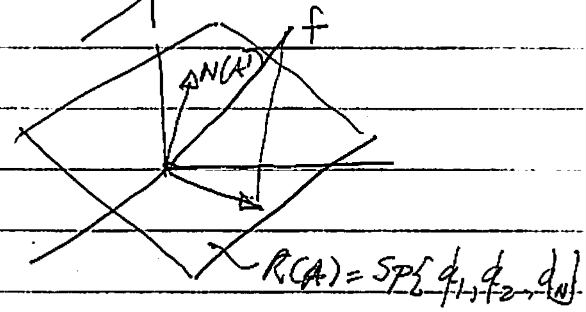
$$Au = f$$

R(A)

$$u_1 q_1 + u_2 q_2 + \cdots + u_n q_n = f$$

N(A^T) THE VECTORS V SUCH THAT

$$A^T V = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} V = 0$$



IF $Au = f$ IS TO HAVE A SOLUTION

$$V^T f = V^T A u = u A^T V = 0$$

$\therefore$ $V^T f = 0$ — SOLVABILITY CONDITION.

IF $Au = f$ IS TO HAVE A SOLUTION f HAS TO LIE IN THE RANGE SPACE OF A.

FREDHOLM ALTERNATIVE:

CONSIDER THE SELF ADJOINT PROBLEM

$$\left.\begin{aligned} Lu &= (p(x)u')' + q\,u = f(x) \quad (a) \\ B_0 u &= 0 \qquad B_1 u = 0 \quad (b) \end{aligned}\right\} \quad (SA)$$

THEN EITHER

(I) $Lu = f$ HAS EXACTLY ONE SOLUTION.

OR (II) THE EIGENVALUE PROBLEM $Lu = \lambda u$ + BC SA (b)
HAS AN EIGENVALUE $\lambda = 0$ WITH EIGENFUNCTION $\phi_0(x)$
IF (II) HOLDS THEN (SA) HAS A SOLUTION
IF AND ONLY IF f SATISFIES THE SOLVABILITY
CONDITION: $\int_0 f(x)\,\phi_0(x)\,dx = 0$.

AND IT IS ONLY DETERMINED UP TO AN
ARBITRARY MULTIPLE OF $\phi_0(x)$.

THE MODIFIED GREEN'S FUNCTION:

CONSIDER THE ESSENTIALLY SELF ADJOINT PROBLEM

$$Lu = (pu')' + qu = f(x) \qquad (a) \Big\} \; (1)$$
$$B_0 u = u'(0) + \alpha u(0) = 0 \quad B_1 u = u'(1) + \beta u(1) = 0 \quad (b)$$

SUPPOSE THAT THERE EXISTS A NONTRIVIAL SOLUTION $\tilde{u}$ TO THE HOMOGENEOUS FORM $(f(x)=0)$ OF THE BVP (1) i.e.

$$L\tilde{u} = 0$$

AND $\quad B_0 \tilde{u} = 0 \quad B_1 \tilde{u} = 0$

MULTIPLYING (1) BY $\tilde{u}$ AND INTEGRATING WE OBTAIN

$$(\tilde{u}, f) = (\tilde{u}, Lu) = (u, L\tilde{u}) = 0$$

WHICH IS JUST THE SOLVABILITY CONDITION

$$\int_0^1 \tilde{u} f \, dx = 0$$

REQUIRED BY THE FREDHOLM ALTERNATIVE.

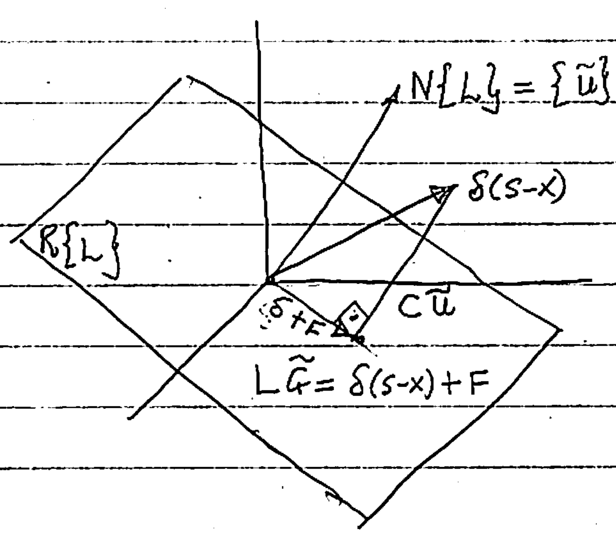
HOW CAN WE CONSTRUCT A GREEN'S FUNCTION?

- BY THE FREDHOLM ALTERNATIVE THE GREEN'S FUNCTION DOES NOT EXIST UNLESS $0 = \int_0^1 G(s,x)\,\delta(s-x)\,ds = G(x,x)$

- HOWEVER IT IS POSSIBLE TO CONSTRUCT A "MODIFIED GREEN'S FUNCTION" $\tilde{G}(s,x)$ THAT IS CONSTRUCTED TO SATISFY

$$L_s G(s,x) = \delta(s-x) + F(s,x)$$
$$B_0 G = 0 \qquad B_1 G = 0$$

WHERE $F(s,x)$ IS CHOSEN SO THAT $\delta(s-x) + F(s,x)$ IS ORTHOGONAL TO $\tilde{u}$.



THUS LET $\underset{s}{L}\tilde{G}(s,x) = \delta(s-x) + F(s,x)$ $(\tilde{2})$

$$B_0\tilde{G} = 0 = B_1\tilde{G}$$

FOR A SOLUTION TO $(\tilde{2})$ TO EXIST THE FREDHOLM ALTERNATIVE DICTATES THAT WE MUST CONSTRUCT F SUCH THAT THE FORCING FUNCTION $\delta + F$ SHOULD SATISFY THE SOLVABILITY CONDITION

$$\int_0^1 \tilde{u}(\delta + F)\, ds = 0 \qquad (3)$$

NOW CHOOSE $\delta(s-x) + F(s,x) = \delta(s-x) + C\,\tilde{u}(s)$ (SEE DIAGRAM FOR MOTIVATION)

I.E $F(s,x) = C\,\tilde{u}(s)$

IN ORDER THAT (3) IS SATISFIED

$$0 = \int_0^1 \tilde{u}(s)\left[\delta(s-x) + C\tilde{u}(s)\right] ds = \tilde{u}(x) + C\int_0^1 [\tilde{u}(s)]^2 ds$$

$$\therefore \quad C = -\tilde{u}(x)/(\tilde{u},\tilde{u})$$

THUS $F(s,x) = -\dfrac{\tilde{u}(x)\,\tilde{u}(s)}{(\tilde{u},\tilde{u})}$ (4)

IF $\tilde{G}(s,x)$ SATISFIES $(\tilde{2})$ WITH F GIVEN BY (4) WE OBTAIN THE FOLLOWING REPRESENTATION FOR $u(x)$:

$$(\tilde{G}, f) = (\tilde{G}, Lu)$$

$$= (u, L\tilde{G}) \quad (L \text{ + BC ARE ESSENTIALLY SELF-ADJOINT})$$

$$= \left(u(s),\ \delta(s-x) - \frac{\tilde{u}(x)\tilde{u}(s)}{(\tilde{u},\tilde{u})}\right)$$

$$= u(x) - \frac{(u,\tilde{u})}{(\tilde{u},\tilde{u})}\tilde{u}(x)$$

$$\therefore \quad u(x) = A\,\tilde{u}(x) + \int_0^1 \tilde{G}(s,x) f(s)\, ds.$$

EXAMPLE 1: $Lu = u'' = f \qquad u'(0) = 0 = u'(1)$.

$\tilde{u} = 1$ IS AN EIGENFUNCTION WITH EIGENVALUE 0.

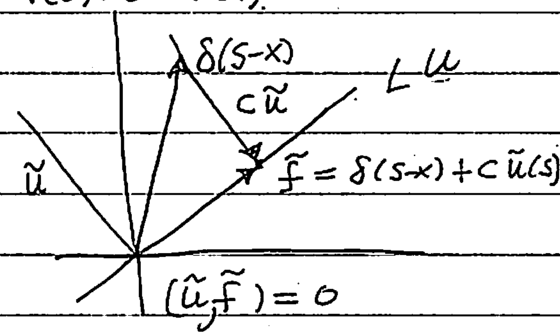
FIND THE MODIFIED GREEN'S FUNCTION.

$(v, Lu) = \int_0^1 v u'' \, dx = [v u' - v' u]_0^1 + \int_0^1 u v'' \, dx \Rightarrow v'' = 0 \quad v'(0) = 0 = v'(1)$.

WE LOOK FOR A MODIFIED GREEN'S FUNCTION

$$L_s \tilde{G}(s,x) = \delta(s-x) + c\,\tilde{u}(s)$$

$$G_s(0,X) = 0 \;\&\; G_s(1,0) = 0$$



WHERE C IS CHOSEN SUCH THAT $\delta + c\tilde{u}$ SATISFIES THE SOLVABILITY CONDITION

$$0 = \int_0^1 \tilde{u}(s)\{\delta(s-x) + c\,\tilde{u}(s)\}\,ds = \tilde{u}(x) + c(\tilde{u},\tilde{u}) \Rightarrow c = -\frac{\tilde{u}(x)}{(\tilde{u},\tilde{u})}$$

$$\therefore L_s \tilde{G}(s,x) = \delta(s-x) - \tilde{u}(x)\tilde{u}(s)/(\tilde{u},\tilde{u})$$

$$\tilde{u} = 1 \Rightarrow (\tilde{u},\tilde{u}) = \int_0^1 1\,dx = 1$$

$$\therefore L_s \tilde{G}(s,x) = -\delta(s-x) - 1$$

$$\tilde{G}_s(0,x) = 0 = \tilde{G}_s(1,x)$$

EQUATION WITHOUT THE δ FUNCTION

$$V_{ss} = -1 \Rightarrow V_s = -s + A \qquad V = -\frac{s^2}{2} + As + B$$

$\therefore$ LET $\tilde{G}(s,x) = \begin{cases} -s^2/2 + A_- s + B_- \\ -s^2/2 + A_+ s + B_+ \end{cases} \qquad G_s(s,x) = \begin{cases} -s + A_- \\ -s + A_+ \end{cases}$

$$0 = \tilde{G}_s(0,x) = A_- \Rightarrow A_- = 0$$

$$0 = \tilde{G}_s(1,x) = -1 + A_+ \Rightarrow A_+ = 1$$

$$\therefore \tilde{G}(s,x) = \begin{cases} -s^2/2 + B_- \\ -s^2/2 + s + B_+ \end{cases}$$

CONTINUITY: $\tilde{G}(x_-,x) = -\cancel{x^2/2} + B_- = \tilde{G}(x_+,x) = -\cancel{x^2/2} + x + B_+ \Rightarrow B_- = (x + B_+)$

JUMP: $\int_{x-\varepsilon}^{x+\varepsilon} \tilde{G}_{ss}\,ds = \tilde{G}_s(x+,x) - \tilde{G}_s(x-,x) = \int_{x-\varepsilon}^{x+\varepsilon} \{\delta(s-x) - 1\}\,ds = 1$

$1 \doteq \tilde{G}_s(x_+,x) - \tilde{G}_s(x_-,x) = (-\cancel{x} + 1) - (-\cancel{x}) = 1$ SATISFIED AUTOMATICALLY

$$\therefore \tilde{G}(s,x) = \begin{cases} -s^2/2 + x + B_+ & 0 < s < x \\ -s^2/2 + s + B_+ & x < s < 1 \end{cases}$$

$\therefore \tilde{G}(s,x) = -\frac{s^2}{2} + B_+ + \begin{cases} x & 0 < s < x \\ s & x < s < 1 \end{cases}$ KNOWN UP TO AN ARBITRARY CONSTANT B_+

NOW $(\tilde{G}, Lu) = (u, L\tilde{G})$

$\therefore \quad \int_0^1 \tilde{G}(s,x) f(s)\, ds = \int_0^1 u(s)\{\delta(s-x) - 1\}\, ds = u(x) - \int_0^1 u(s)\, ds$

$= u(x) - K.$

$\therefore \quad u(x) = K + \int_0^1 \left\{\left(-\frac{s^2}{2} + B_1\right) + \begin{Bmatrix} x & 0<s<x \\ s & x<s<1 \end{Bmatrix}\right\} f(s)\, ds$

$= K + \int_0^1 \left(-\frac{s^2}{2} + B_1\right) f(s)\, ds + x\int_0^x f(s)\, ds + \int_x^1 s f(s)\, ds$

$= C\tilde{u} - \int_0^1 \frac{s^2}{2} f(s)\, ds + x \int_0^x f(s)\, ds + \int_x^1 s f(s)\, ds.$

EXAMPLE: $Lu = u'' + \pi^2 u = f \quad 0<x<1$

$u(0)=0 \quad u(1)=0$

DETERMINE A SOLVABILITY CONDITION ON $f(x)$. ASSUMING THIS CONDITION IS SATISFIED DETERMINE THE MODIFIED GREEN'S FUNCTION AND AN INTEGRAL REPRESENTATION OF THE SOLUTION.

• THIS PROBLEM IS ESSENTIALLY SELF ADJOINT.

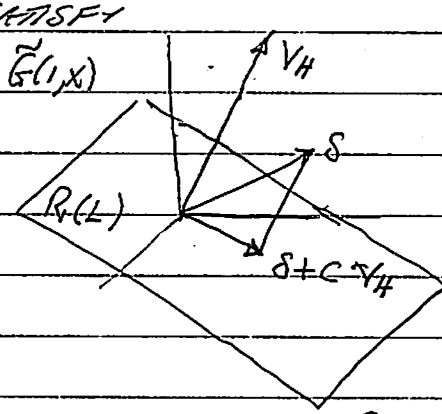
$V_H(x) = \sin\pi x$ IS A NONTRIVIAL FUNCTION THAT SATISFIES $LV_H=0$ AND THE HOMOGENEOUS BC

• $(V_H, Lu) = (u, LV_H) = 0$

THUS $\int_0^1 f(x)\sin(\pi x)\,dx = 0$ IS THE SOLVABILITY CONDITION.

• THE MODIFIED GREEN'S FUNCTION SHOULD SATISFY

$$L_s\tilde{G}(s,x) = \delta(s-x) + c\sin(\pi s) \qquad \tilde{G}(0,x)=0=\tilde{G}(1,x)$$



WHERE C IS CHOSEN SO THAT

$$0 = \int_0^1 \{\delta(s-x)+c\sin(\pi s)\}\sin(\pi s)\,ds = \sin\pi x + c\int_0^1 \sin^2(\pi s)\,ds$$
$$= \sin(\pi x) + c/2$$

∴ $c = -2\sin(\pi x)$

∴ $L_s\tilde{G}(s,x) = \delta(s-x) - 2\sin(\pi x)\sin(\pi s)$

$\tilde{G}(0,x) = 0 = \tilde{G}(1,x)$ PARTICULAR SOL'N TO $L_s V = \sin(\pi s)$ IS $-\frac{s}{2\pi}\cos(\pi s)$

$$\tilde{G}(s,x) = -2\sin(\pi x)\left\{-\frac{s}{2\pi}\cos(\pi s)\right\} + \begin{cases} A_-\sin(\pi s) & 0<s<x \\ A_+\sin(\pi s) + B_+\cos(\pi s) & x<s<1 \end{cases}$$

NOW $\tilde{G}(0,x)=0$

$$\tilde{G}(1,x) = \frac{1}{\pi}\sin(\pi x)\cos(\pi) + A_+\cancel{\sin\pi} + B_+\cos\pi \Rightarrow B_+ = -\frac{\sin\pi x}{\pi}$$

CONTINUITY:

$$\tilde{G}(x_-,x) = \frac{x}{\pi}\cancel{\sin(\pi x)\cos\pi x} + A_-\sin\pi x = \frac{x}{\pi}\cancel{\sin(\pi x)\cos(\pi x)} + A_+\sin\pi x + B_+\cos\pi x$$

$$A_-\sin(\pi x) - A_+\sin(\pi x) = B_+\cos\pi x$$

JUMP: $1 = \tilde{G}(x_+,x) - \tilde{G}(x_-,x) = A_+\pi\cos\pi x - B_+\pi\sin(\pi x) - A_+\pi\cos(\pi x)$

$$Au = \begin{bmatrix} -\pi\cos\pi x & \pi\cos\pi x \\ \sin(\pi x) & -\sin(\pi x) \end{bmatrix}\begin{bmatrix} A_- \\ A_+ \end{bmatrix} = \begin{bmatrix} 1 + B_+\pi\sin(\pi x) \\ B_+\cos\pi x \end{bmatrix} = b$$

$$DET(A) = DET\begin{bmatrix} -\pi\cos\pi x & \pi\cos\pi x \\ \sin\pi x & -\sin\pi x \end{bmatrix} = \pi\cos\pi x\sin\pi x - \pi\cos\pi x\sin\pi x = 0$$

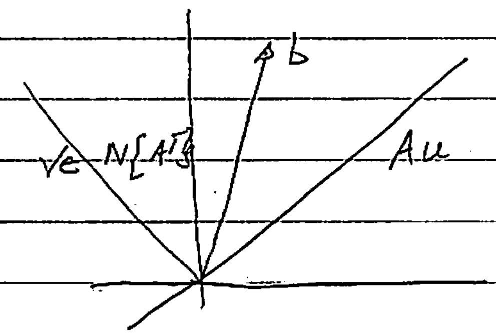
THUS THE MATRIX IS SINGULAR AND HAS A NONTRIVIAL NULLSPACE.

FOR A SOLUTION TO EXIST WE MUST ENSURE THAT THE RIGHT HAND SIDE VECTOR SHOULD BE ORTHOGONAL TO THE NULL SPACE OF A^T:

i.e., $Au = b$

IF $v \in N\{A^T\}$ THEN

$0 = uA^Tv = v^TAu = v^Tb$



NOTE THAT $v = [\sin\pi x \quad \pi\cos\pi x]$

SINCE

$$A^Tv = \begin{bmatrix} -\pi\cos\pi x & \sin\pi x \\ \pi\cos\pi x & -\sin\pi x \end{bmatrix}\begin{bmatrix} \sin\pi x \\ \pi\cos\pi x \end{bmatrix} = \begin{bmatrix} -\pi\cos\pi x\sin\pi x + \pi\cos\pi x\sin\pi x \\ \pi\cos\pi x\sin\pi x - \pi\cos\pi x\sin\pi x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

FOR A SOLUTION TO EXIST WE MUST HAVE THAT $v^Tb = 0$

$$v^Tb = [\sin\pi x \quad \pi\cos\pi x]\begin{bmatrix} 1 + B_+\pi\sin\pi x \\ B_+\cos\pi x \end{bmatrix} = \sin\pi x + \left(\frac{-\sin\pi x}{\pi}\right)\pi\sin^2\pi x + \left(\frac{-\sin\pi x}{\pi}\right)\pi\cos^2\pi x$$

$$= \sin\pi x - \sin\pi x\left[\sin^2\pi x + \cos^2\pi x\right] = 0.$$

THE INFORMATION WE REQUIRE CAN COME FROM EITHER EQUATION:

$$\therefore \quad A_- = A_+ + B_+\frac{\cos\pi x}{\sin\pi x} = A_+ - \frac{\sin\pi x}{\pi}\frac{\cos\pi x}{\sin\pi x} = A_+ - \frac{1}{\pi}\cos\pi x.$$

$$\therefore \quad \tilde{G}(s,x) = \frac{1}{\pi}\sin\pi x \cdot s\cos\pi s + \begin{cases} \left(A_+ - \frac{1}{\pi}\cos\pi x\right)\sin\pi s & 0 < s < x \\ A_+\sin\pi s - \frac{1}{\pi}\cos\pi s\,\sin\pi x & x < s < 1 \end{cases}$$

$$= \frac{1}{\pi}s\cos\pi s\,\sin\pi x + A_+\sin\pi s - \frac{1}{\pi}\begin{cases} \cos\pi x\,\sin\pi s & 0 < s < x \\ \cos\pi s\,\sin\pi x & x < s < 1 \end{cases}$$

SINCE

$$(\tilde{G}, Lu) = (u, L\tilde{G}) = (u, \delta(s-x) - 2\sin\pi x\sin\pi s)$$

$$\int_0^1 \tilde{G}(s,x)f(s)\,ds = u(x) - 2(u, \sin\pi s)\cdot\sin\pi x$$

$$\therefore \quad u(x) = 2(u,\sin\pi s)\sin\pi x + \int_0^1\left\{\frac{1}{\pi}s\cos\pi s\,\sin\pi x + \cancel{A_+\sin\pi s}\right\}f(s)\,ds$$
$$- \frac{\cos\pi x}{\pi}\int_0^x \sin\pi s\cdot f(s)\,ds - \frac{\sin\pi x}{\pi}\int_x^1 \cos\pi s\,f(s)\,ds$$

$$\therefore \quad u(x) = C\sin\pi x - \frac{\cos\pi x}{\pi}\int_0^x \sin\pi s\,f(s)\,ds - \frac{\sin\pi x}{\pi}\int_x^1 \cos\pi s\,f(s)\,ds$$