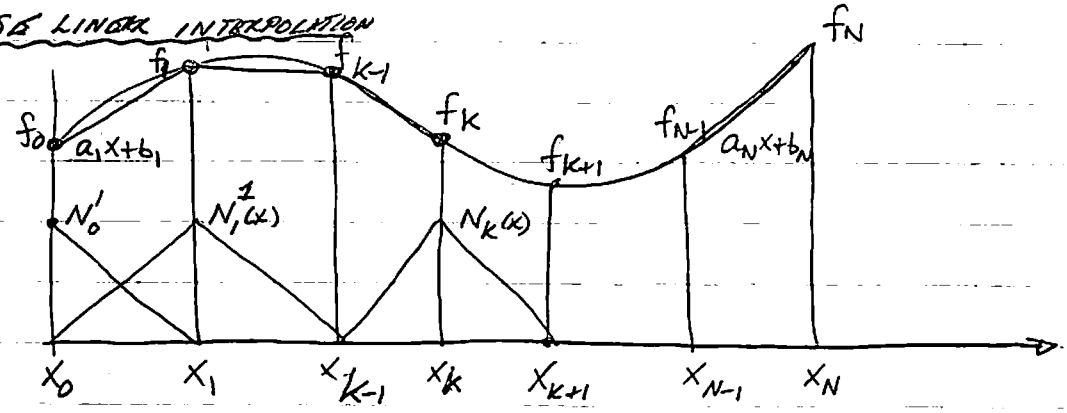
LECTURE 5

PIECEWISE POLYNOMIAL APPROXIMATION – FINITE ELEMENT BASIS FUNCTIONS

IDEA: LIMIT OSCILLATIONS OF HIGH DEGREE BY RESTRICTING OURSELVES TO LOW DEGREE POLYNOMIALS OVER SUBINTERVALS

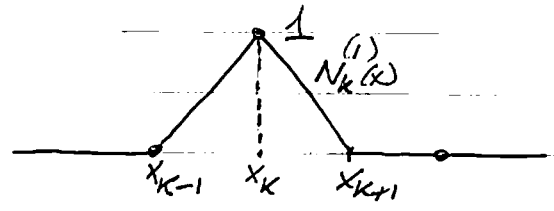
PIECEWISE LINEAR INTERPOLATION



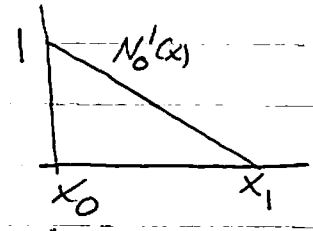
DOF: 2N UNKNOWNS – (N−1) CONSTRAINTS C^0 CONTINUITY = N+1 DOF GIVEN BY $f_0, \dots, f_N$

PIECEWISE LINEAR LAGRANGE BASIS FUNCTIONS – HAT FUNCTIONS

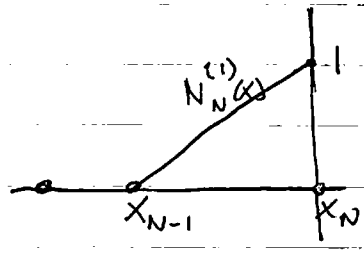
$$N_K^{(1)}(x) = \begin{cases} \left(\dfrac{x-x_{K-1}}{x_K-x_{K-1}}\right) & x \in [x_{K-1}, x_K] \\ \left(\dfrac{x_{K+1}-x}{x_{K+1}-x_K}\right) & x \in [x_K, x_{K+1}] \\ 0 & \text{OTHERWISE} \end{cases}$$



$$N_0^{(1)}(x) = \begin{cases} \left(\dfrac{x_1-x}{x_1-x_0}\right) & x \in [x_0, x_1] \\ 0 & \text{OTHERWISE} \end{cases}$$



$$N_N^{(1)}(x) = \begin{cases} 0 & \text{OTHERWISE} \\ \left(\dfrac{x-x_{N-1}}{x_N-x_{N-1}}\right) & x \in [x_{N-1}, x_N] \end{cases}$$



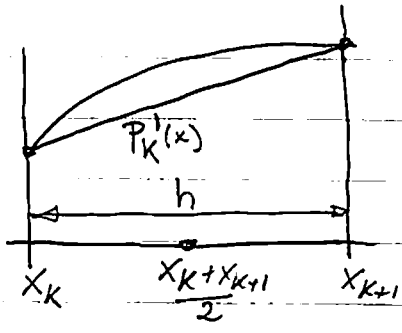
$$p_N^1(x) = \sum_{K=0}^{N} f_K N_K^{(1)}(x)$$

ERROR: $e(x) = f(x) - p_K^{(1)}(x) = (x-x_K)(x-x_{K+1})\dfrac{f(\xi)}{2}$

NOW $(x-x_K)(x-x_{K+1}) = \left(\dfrac{x_{K+1}-x_K}{2}\right)\left(\dfrac{x_K-x_{K+1}}{2}\right) = -\dfrac{h^2}{4}$

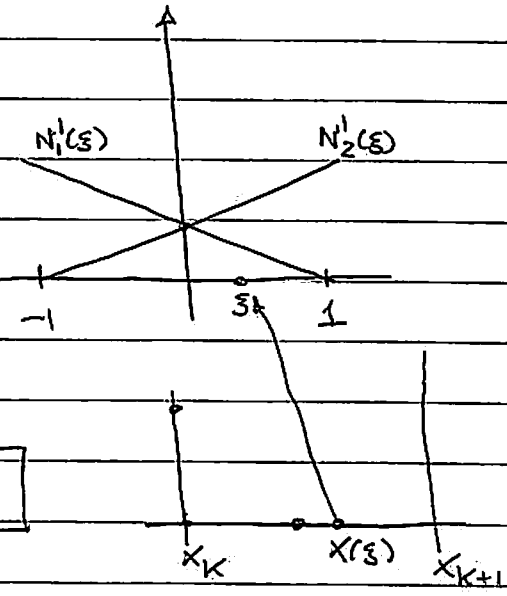
IS A MAX/MIN AT THE MIDPOINT $\rightarrow x = \dfrac{x_K+x_{K+1}}{2}$

$$|e(x)| \le \frac{h^2}{8}\|f\|_\infty$$



REPRESENTATION ON A CANONICAL INTERVAL

IT IS OFTEN USEFUL TO PERFORM CALCULATIONS (eg INTEGRATION) USING THESE BASIS FUNTIONS ON A SPECIAL (SO-CALLED CANONICAL INTERVAL) $[-1,1]$



$$N_1^1(\xi) = \frac{1}{2}(1-\xi) \qquad N_2(\xi) = \frac{1}{2}(1+\xi)$$

$$\boxed{N_a^1(\xi) = \frac{1}{2}(1+\xi_a\xi) \qquad \xi_1=-1,\ \xi_2=1}$$

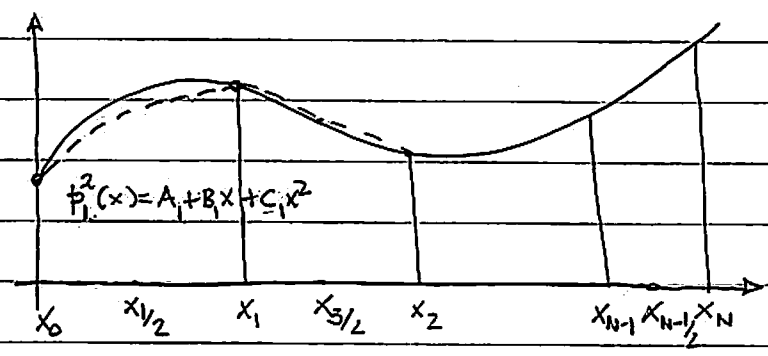
- TYPICALLY THE INTERVAL $[x_K, x_{K+1}]$ IS MAPPED ONTO THE INTERVAL $[-1,1]$ BY THE LINEAR TRANSFORMATION

$$x(\xi) = x_K N_1(\xi) + x_{K+1} N_2(\xi) = \frac{1}{2}(1-\xi)x_K + \frac{1}{2}(1+\xi)x_{K+1} = \left(\frac{x_K+x_{K+1}}{2}\right) + \left(\frac{x_{K+1}-x_K}{2}\right)\xi$$

$$p_1(\xi) = f_K N_1(\xi) + f_{K+1} N_2(\xi)$$

IS CALLED AN ISOPARAMETRIC REPRESENTATION OF THE FUNCTION f.

PIECEWISE QUADRATIC INTERPOLATION



DOF: $3N$ UNKNOWNS $A_K, B_K, C_K \quad K=1,\dots,N$.

- ENFORCE CONTINUITY AT INTERIOR NODES $\rightarrow$ $(N-1)$ CONSTRAINTS

$$3N-(N-1) = \underbrace{N+1}_{f_0,\dots f_N} + \underbrace{N}_{\text{AN EXTRA SAMPLE OF } f \text{ WITHIN EACH SUBINTERVAL } f_{1/2},f_{3/2},\dots f_{K-1/2}}$$