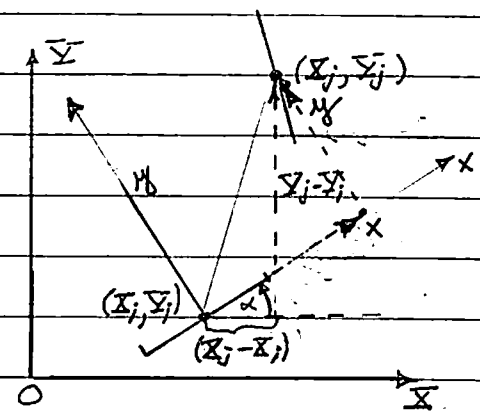
LECTURE 31 THE INDIRECT BOUNDARY INTEGRAL METHOD

LAST TIME WE CONSIDERED THE BOUNDARY INTEGRAL EQUATION

$$-u(\underset{\sim}{x}) + \int_{\partial\Omega} u(\underset{\sim}{x}') \frac{\partial V}{\partial n}(\underset{\sim}{x}', \underset{\sim}{x})\, ds(x') = \int_{\partial\Omega} q(x')\, V(\underset{\sim}{x}', \underset{\sim}{x})\, ds(x')$$

THAT IS OBTAINED WHEN WE LET $x \to \partial\Omega$.



CONSIDERING A GLOBAL COORDINATE SYSTEM OXY IN WHICH THE SENDING ELEMENT HAS COORDINATES (X_i, Y_i) FOR ITS MIDPOINT AND ORIENTATION α, THE LOCAL COORDINATES (x, y) OF THE MIDPOINT (X_j, Y_j) OF THE jTH RECEIVING ELEMENT ARE GIVEN BY

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} X_j - X_i \\ Y_j - Y_i \end{bmatrix}$$

THEN DEFINING $V_{ji} = \bar{V}_i(x_j, y_j) = \frac{1}{2\pi}\int_{-a}^{a} \ln\left[(x_j - s)^2 + y_j^2\right]^{1/2} ds = \int_{\Omega_i} V(\underset{\sim}{x}', \underset{\sim}{x}_j)\, ds(\underset{\sim}{x}')$

AND

$$V_{Nji} = -\frac{\partial \bar{V}_i}{\partial y}(x_j, y_j) = \frac{1}{2\pi}\left\{\mathrm{Tan}^{-1}\left(\frac{y_j}{x_j - a}\right) - \mathrm{Tan}^{-1}\left(\frac{y_j}{x_j + a}\right)\right\} \xrightarrow{y_j \to 0} \begin{cases} 0 & |x| > a \\ \frac{1}{2} & |x| < a \quad y_j \to 0+ \\ -\frac{1}{2} & |x| < a \quad y_j \to 0- \end{cases}$$

THEN $\sum_{i=1}^{N} G_{ji}\, q_i = -u_j + \sum_{i=1}^{N} V_{Nji}\, u_i = \sum_{i=1}^{N} H_{ji}\, u_i$

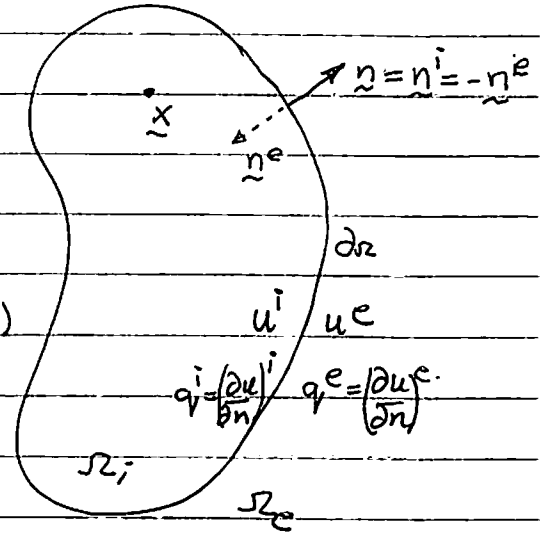
WHERE $G_{ji} = V_{ji}$ AND $H_{ji} = -\delta_{ji} + V_{Nji}$ SO THAT $H_{ii} = -\frac{1}{2}$.

THUS $\quad G q = H u$

$$q = G^{-1} H u = G^{-1} H g \quad \text{FOR THE DIRICHLET PROBLEM}$$

THE INDIRECT BOUNDARY ELEMENT METHOD - SINGLE AND DOUBLE LAYER POTENTIALS

CONSIDER A CLOSED CURVE $\partial\Omega$ THAT DIVIDES $\mathbb{R}^2$ INTO AN INTERIOR REGION Ω_i AND AN EXTERIOR REGION Ω_e. IF $\underset{\sim}{x} \in \Omega_i$ THEN



$$u^i(\underset{\sim}{x}) = \int_{\partial\Omega} u^i(\underset{\sim}{x}') \frac{\partial V(\underset{\sim}{x}', \underset{\sim}{x})}{\partial n} ds(\underset{\sim}{x}') - \int_{\partial\Omega} q^i(\underset{\sim}{x}') V(\underset{\sim}{x}', \underset{\sim}{x}) ds(\underset{\sim}{x}')$$

$$0 = -\int_{\partial\Omega} u^e(\underset{\sim}{x}') \frac{\partial V(\underset{\sim}{x}', \underset{\sim}{x})}{\partial n} ds(x') + \int_{\partial\Omega} q^e(\underset{\sim}{x}') V(\underset{\sim}{x}', \underset{\sim}{x}) ds(x')$$

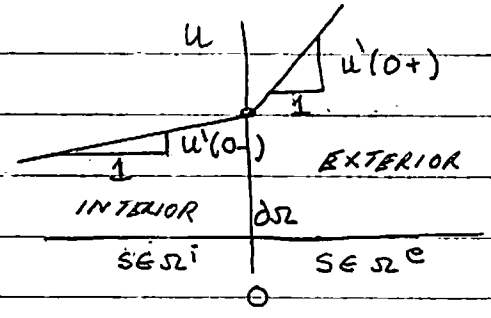
ADDING WE OBTAIN

$$u^i(\underset{\sim}{x}) = \int_{\partial\Omega} \{u^i(\underset{\sim}{x}') - u^e(\underset{\sim}{x}')\} \frac{\partial V(\underset{\sim}{x}', \underset{\sim}{x})}{\partial n} ds(\underset{\sim}{x}')$$

$$- \int_{\partial\Omega} \{q^i(\underset{\sim}{x}') - q^e(\underset{\sim}{x}')\} V(\underset{\sim}{x}', \underset{\sim}{x}) ds(\underset{\sim}{x}') \qquad (*)$$

UP TILL NOW u^i AND u^e AND q^i AND q^e ARE UNRELATED.

SINGLE LAYER POTENTIAL: REQUIRE $[u] = \{u^e - u^i\} = 0$ AND ALLOW THE FLUX JUMP TO BE NON-ZERO $F(x) = [q] = \{q^e - q^i\}$



NOTE: THE FLUX CAN JUMP WHILE THE POTENTIAL IS CONTINUOUS

IN THIS CASE THE INTEGRAL EQUATION (*) REDUCES TO

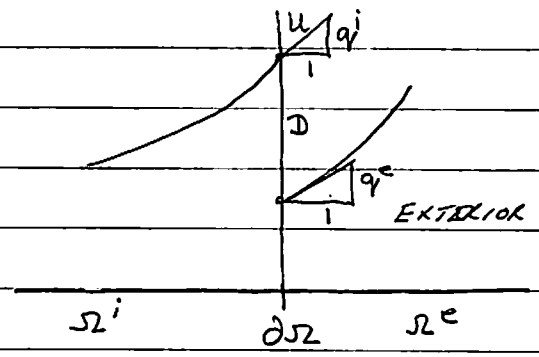
$$\boxed{u^i(x) = \int_{\partial\Omega} F(\underset{\sim}{x}') V(\underset{\sim}{x}', \underset{\sim}{x}) ds(x')}$$

SINGLE LAYER POTENTIAL

- THIS METHOD HAS THE ADVANTAGE THAT ONLY ONE INTEGRAL IS INVOLVED ALONG THE BOUNDARY. FOR EXAMPLE FOR THE DIRICHLET PROBLEM $u = g$ ON $\partial\Omega$ WE HAVE $g_j = u(x_j) = \sum_{i=1}^{N} F_i \int_{\partial\Omega_i} V(\underset{\sim}{x}', \underset{\sim}{x}_j) ds(\underset{\sim}{x}') = GF \Rightarrow F = G^{-1} g$
- $F(\underset{\sim}{x}')$ IS A FICTITIOUS QUANTITY THAT REPRESENTS THE CHARGE DISTRIBUTION ALONG $\partial\Omega$ THAT GENERATES THE SPECIFIED POTENTIAL $g(x)$ ON $\partial\Omega$.

DOUBLE LAYER POTENTIAL: REQUIRE $[q] = \{q^e - q^i\} = 0$ (CONTINUOUS) FLUXES AND ALLOW THE POTENTIAL TO JUMP ACROSS THE BOUNDARY

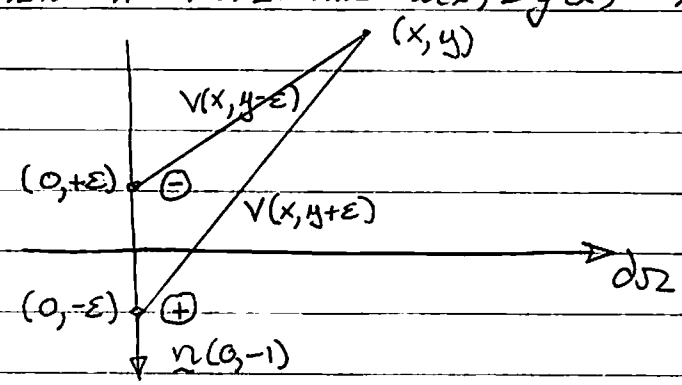
i.e. LET $D(\underset{\sim}{x}') = u^e(\underset{\sim}{x}') - u^i(\underset{\sim}{x}') = [u]$



IN THIS CASE

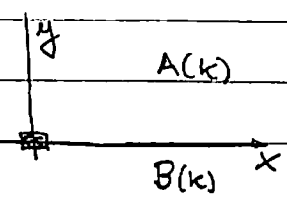
$$\boxed{u^i(\underset{\sim}{x}) = -\int_{\partial\Omega} D(x')\,\frac{\partial V}{\partial n}(x', x)\,ds(x')}$$

NOTE THAT $D(x')$ CAN BE INTERPRETED AS A DIPOLE DISTRIBUTION ALONG $\partial\Omega$ TO ACHIEVE A POTENTIAL $u(x) = g(x)$ ALONG $\partial\Omega$



$$\lim_{\varepsilon \to 0} \frac{V(x, y+\varepsilon) - V(x, y-\varepsilon)}{2\varepsilon} = \frac{\partial V}{\partial y} = -\frac{\partial V}{\partial n}$$

GREEN'S FUNCTION FOR A POINT JUMP IN u:



$$u_{xx} + u_{yy} = 0 \quad \xrightarrow{FT_x} \quad \hat{u}_{yy} - k^2\hat{u} = 0 \qquad k = \sqrt{k^2}$$

$$[u_y] = 0 \qquad [u] = \delta(x)$$

$$\therefore \quad \hat{u}(k,y) = \begin{cases} A(k)\,e^{-ky} & y<0 \\ B(k)\,e^{ky} & y>0 \end{cases} \qquad \hat{u}_y = \begin{cases} -A(k)\,e^{-ky}k & y>0 \\ +B(k)\,e^{ky}k & y<0 \end{cases}$$

$$1 = [\hat{u}(k,0)] = A(k) - B(k) \qquad 0 = [\hat{u}_y(k,0)] = -A(k)k - B(k)k \Rightarrow B = -A \;\&\; A(k) = 1/2$$

$$\therefore \quad \hat{u}(k,y) = \frac{1}{2}\,\mathrm{sgn}(y)\,e^{-k|y|}$$

$$\therefore \quad u(x,y) = \frac{\mathrm{sgn}(y)}{4\pi}\int_{-\infty}^{\infty} e^{-k|y|}e^{-ikx}\,dk = \frac{\mathrm{sgn}(y)}{2\pi}\int_0^{\infty} e^{-k|y|}\cos(kx)\,dx = \frac{1}{2\pi}\frac{y}{x^2+y^2}$$

NOTE IF $V(x,y) = \frac{1}{2\pi}\ln[x^2+y^2]^{1/2}$ $\quad \frac{\partial V}{\partial y} = \frac{1}{2\pi}\frac{y}{x^2+y^2}$ = POTENTIAL DUE TO A POINT JUMP IN u.