LECTURE 3

BUILDING HIGHER ORDER INTERPOLATION POLYNOMIALS FROM LOWER DEGREE ONES

LET $\Pi_N = \{$ THE SET OF POLYNOMIALS OF DEG $N\}$

LET $P_{N-1}(x) \in \Pi_{N-1}$ INTERPOLATE f AT $x_0, x_1, \ldots, x_{N-1}$ AND $Q_{N-1} \in \Pi_{N-1}$ INTERPOLATE f AT $x_1, \ldots, x_N$ THEN $R_N(x) \in \Pi_N$

$$R_N(x) = \frac{(x-x_0)\,Q_{N-1}(x) - (x-x_N)\,P_{N-1}(x)}{x_N - x_0}$$

INTERPOLATES f AT $x_0, x_1, \ldots, x_N$

PROOF: $R_N(x_0) = \dfrac{-(x_0-x_N)\,P_{N-1}(x_0)}{(x_N-x_0)} = f_0$

$$R_N(x_N) = \frac{(x_N-x_0)\,Q_{N-1}(x_N)}{(x_N-x_0)} = f_N$$

$k = 1, \ldots, N-1$ $\quad R_N(x_k) = \dfrac{(x_k-x_0)\,Q_{N-1}(x_k) - (x_k-x_N)\,P_{N-1}(x_k)}{x_N-x_0}$

$$= \frac{(x_k-x_0)\,f_k - (x_k-x_N)\,f_k}{x_N-x_0} = f_k$$

□

NEWTON'S DIVIDED DIFFERENCE FORMULA

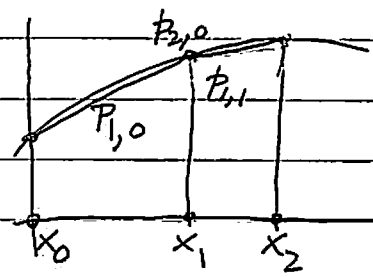
$$P_{1,0}(x) = f_0\frac{(x_1-x)}{(x_1-x_0)} + f_1\frac{(x-x_0)}{(x_1-x_0)}$$

$$= f_0\frac{(x_1-x_0+x_0-x)}{(x_1-x_0)} + f_1\frac{(x-x_0)}{(x_1-x_0)}$$

$$= f_0 + \left(\frac{f_1-f_0}{x_1-x_0}\right)(x-x_0)$$

$$= f_0 + f[x_0,x_1](x-x_0)$$

$$P_{1,1}(x) = f_1 + f[x_1,x_2](x-x_1)$$



$$P_{2,0}(x) = \frac{(x-x_0)\{f_1 + f[x_1,x_2](x-x_1)\} - (x-x_2)\{f_0 + f[x_0,x_1](x-x_0)\}}{(x_2-x_0)}$$

$$= \frac{(x-x_0)\{f_1 + f[x_1,x_2](x-x_1)\} - (x-x_0+x_0-x_2)\,f_0 - (x-x_1+x_1-x_2)\,f[x_0,x_1](x-x_0)}{(x_2-x_0)}$$

$$= f_0 + \frac{f[x_0,x_1](x-x_0)}{(x_2-x_0)}(x_1-x_0) + \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}(x-x_0)(x-x_1) - \frac{(x_1-x_2)(x-x_0)f[x_0,x_1]}{x_2-x_0}$$

$$p_{2p}(x) = f_0 + \frac{(x_1 - x_0 - x_1 + x_2)}{(x_2 - x_0)} f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

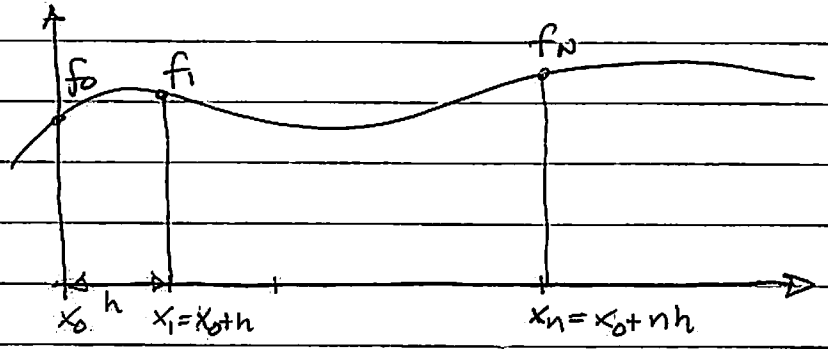
NEWTON'S DIVIDED DIFFERENCE FORMULA.

$$p_N(x) = f_0 + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots + f[x_0, x_1, \ldots, x_N](x - x_0)\ldots(x - x_{N-1})$$

DIVIDED DIFFERENCE TABLE

		f'	$f''/2!$
x_0	f_0	$\rightarrow f[x_0, x_1]$	$\rightarrow f[x_0, x_1, x_2]$
x_1	f_1	$\rightarrow f[x_1, x_2]$	$\rightarrow f[x_1, x_2, x_3]$
x_2	f_2	$\rightarrow f[x_2, x_3]$	
x_3	f_3		

POLYNOMIAL APPROXIMATIONS FOR EQUALLY SPACED MESH POINTS



$$x_n = x_0 + nh$$

OPERATORS

FORWARD DIFFERENCE: $\Delta f_n = f_{n+1} - f_n$

$$\Delta^2 f_n = \Delta f_{n+1} - \Delta f_n = f_{n+2} - 2f_{n+1} - f_n$$

BACKWARD DIFFERENCE: $\nabla f_n = f_n - f_{n-1}$

$$\nabla^2 f_n = \nabla f_n - \nabla f_n = f_n - 2f_{n-1} + f_{n-2}$$

CENTRAL DIFFERENCE: $\delta f_n = f_{n+1/2} - f_{n-1/2}$

$$\delta^2 f_n = \delta f_{n+1/2} - \delta f_{n-1/2} = f_{n+1} - 2f_n + f_{n-1}$$

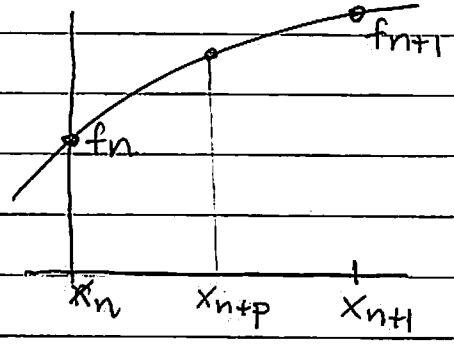
SHIFT $E f_n = f_{n+1}$

RELATIONSHIP BETWEEN SHIFT AND DIFFERENCE OPERATORS

$\Delta f_n = f_{n+1} - f_n = (E-1) f_n \qquad \Delta = E-1 \qquad E = 1+\Delta$

$\nabla f_n = f_n - f_{n-1} = (1-E^{-1}) f_n \qquad \nabla = 1-E^{-1} \qquad E^{-1} = 1-\nabla$

INTERPOLATION — GREGORY-NEWTON INTERPOLATION FORMULA



$$E^p f_n = (1+\Delta)^p f_n = \left[1 + p\Delta + \frac{p(p-1)}{2}\Delta^2 + \cdots\right] f_n$$

EG: SUM THE SERIES $S_N = \sum_{n=1}^{N} i = 1+2+\cdots+N = \frac{N(N+1)}{2}$

$N+N-1 \quad +1$

n	S_N	ΔS_N	$\Delta^2 S_N$
1	1	2	1
2	3	3	1
3	6	4	1
4	10	5	
5	15		

$$E^{p-1} f_1 = \left[1 + (p-1)\Delta + \frac{(p-1)(p-2)}{2}\Delta^2\right] f_1$$

$$= 1 + (p-1).2 + \frac{(p-1)(p-2)}{2}.1$$

$$= \frac{2 + 4p - 4 + p^2 - 3p + 2}{2} = \frac{p(p+1)}{2}$$