Lecture 2: Series solutions to ODE with variable coefficients

In this lecture we will introduce series methods for the solution of variable coefficient ODE. We introduce the concepts of ordinary points about which Taylor series solutions are obtained and singular points about which more general solutions are required.

Key Concepts: Variable coefficient ODE, Series Solutions, Ordinary Points and Taylor Series, Singular Points, radius of convergence of power series.

2 Series Solution of ODEs

2.1 Power Series:

\[ f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \]

polynomial approximation.

Idea: Extend the polynomial to include infinitely many terms.

\[ f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{Power Series} \]

(2.1)

Example 1: \( e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \).

More General Power Series:

\[ f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots \]

(2.2)
2.2 Taylor series

If we know all the derivatives of a function $f(x)$ at a single point $x_0$ then we can solve for the coefficients of a power series that represents the function at neighboring points $x$ as follows:

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots$$

$$f'(x) = a_1 + 2a_2 (x - x_0) + 3a_3 (x - x_0)^2 + \cdots + na_n (x - x_0)^{n-1} + \cdots$$

$$\Rightarrow f'(x_0) = a_1$$

$$f''(x) = 2a_2 + 3 \cdot 2a_3 (x - x_0) + \cdots + n(n-1)a_n (x - x_0)^{n-2} + \cdots$$

$$\Rightarrow f''(x_0) = 2a_2$$

$$f^{(3)}(x) = 3!a_3 + 4 \cdot 3 \cdot 2 (x - x_0) + \cdots + n(n-1)(n-2)a_n (x - x_0)^{n-3} + \cdots$$

$$\Rightarrow f^{(3)}(x_0) = 3!a_3$$

Therefore $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$ (2.3)

### Alternative Form of Taylor Series:

$$f(x_0 + h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} h^n$$ (2.4)

### Example 2: Taylor-Maclaurin expansions of common functions

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$
$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

$$e^i\theta = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots$$
$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i \left(\theta - \frac{\theta^3}{3!} + \cdots\right)$$

$$= \cos \theta + i \sin \theta$$ (2.6)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$= \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \left(x + \frac{x^3}{3!} + \cdots\right)$$
$$= \cosh x + \sinh x$$

$$e^{-x} = \cosh x - \sinh x$$
$$\cosh x = \frac{e^x + e^{-x}}{2}$$
$$\sinh x = \frac{e^x - e^{-x}}{2}$$
2.3 Series Solution to a constant coefficient ODE:

Example 3: In this example we use power series to solve the linear ODE

\[ y' + 2y = 0 \] (2.7)

which we solved by integrating factor in the previous lecture. Since the unknown solution \( y(x) \) and all its derivatives are defined implicitly by the ODE \( y' = -2y \), let us look for a series solution of the form:

\[ y(x) = \sum_{n=0}^{\infty} a_n x^n. \]

Therefore \( y' + 2y = \sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n = 0 \)

In the first sum let

\[
\begin{align*}
m &= n - 1 \\
n &= m + 1
\end{align*}
\]

Therefore \( \sum_{n=0}^{\infty} a_{m+1}(m+1)x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0 \)

\( n \leftrightarrow m : \sum_{m=0}^{\infty} \{a_{m+1}(m+1) + 2a_m\} x^m = 0 \) (2.9)

\[ a_{m+1} = -\frac{2}{(m+1)} a_m \]

\[ a_1 = -2a_0, a_2 = -\frac{2}{3} \cdot \frac{2}{3} a_0, a_3 = -\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} a_0 = (-1)^3 \cdot \frac{2^3}{3^3} a_0, \]

\[ \ldots, a_m = (-1)^m \frac{2^m}{m!} a_0 \]

Therefore \( y(x) = a_0 \sum_{m=0}^{\infty} \frac{(-2x)^m}{m!} = a_0 e^{-2x} \)

2.4 Variable coefficient ODE: ordinary and singular points

Example 4: We consider the following Cauchy-Euler equation:

\[ (x-1)y'' + y' = 0 \] (2.11)

whose solution is obtained as follows

\[ y = (x-1)' \Rightarrow r(r-1) + r = r^2 = 0 \quad r = 0, 0. \]

So that \( y(x) = A + B \ln |x-1| \) is the general solution (2.12)

Method I: The first method we consider for obtaining a series solution to (2.11) is to use the ODE to calculate all the derivatives of \( y(x) \) by direct differentiation and then to substitute these derivatives into Taylor’s formula. Although this method can, in principle, be applied to any suitable ODE, we will see that the computations can rapidly become tedious. However, this method does highlight when the power series method will fail.
Assume that \(y(0)\) and \(y'(0)\) are given, then
\[
y'' = -\frac{y'}{x-1} \Rightarrow y''(0) = y'(0)
\]
\[
y''' = -\frac{y''}{x-1} + \frac{y'}{(x-1)^2} \Rightarrow y'''(0) = y''(0) + y'(0) = 2y'(0)
\]
Substituting this into Taylor’s formula
\[
y(x) = y(0) + xy'(0) + \frac{x^2}{2} y''(0) + \frac{x^3}{3!} y'''(0) + \cdots
\]
(2.13)

We observe that this process works for equation (2.11) using the expansion point \(x_0 = 0\); but will not work for \(x_0 = 1\); which is called a singular point. In fact, a power series expansion is possible for all points \(x_0 \neq 1\); which are called ordinary points.

Method II: We now consider an alternative, and more convenient, method for determining the coefficients of a power series solution to the ODE, by substituting the power series into the ODE and determining recursion formulae for these coefficients. Expand around the ordinary point \(x_0 = 0\):
\[
y(x) = \sum_{n=0}^{\infty} c_n x^n, \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}
\]
(2.14)
\[
(x-1) \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} n c_n x^{n-1} = 0
\]
\[
- \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=2}^{\infty} c_n \{n(n-1) + n\} x^{n-1} + c_1 = 0
\]
(2.15)
\[
m - 1 = n - 2 \Rightarrow m = n - 1 \quad n = 2 \Rightarrow m = 1 \quad n = m + 1
\]
\[
-c_2 \cdot 2 \cdot 1 + c_1 + \sum_{m=2}^{\infty} \left[-c_{m+1}(m+1)m + c_m m^2\right] x^{m-1} = 0
\]
where \(c_0\) and \(c_1\) are arbitrary:
\[
c_{m+1} = \frac{m}{m+1} c_m \quad m \geq 2 \quad c_2 = \frac{c_1}{2}
\]
\[
c_3 = \frac{2}{3} c_2 = \frac{c_1}{3} \quad c_4 = \frac{3}{4} c_3 = \frac{c_1}{4} \quad \cdots \quad c_n = \frac{c_1}{n}
\]
Therefore
\[
y(x) = c_0 + c_1 \sum_{n=1}^{\infty} \frac{x^n}{n}
\]
(2.16)

We observe that the solutions (2.13) and (2.16) obtained by the two different methods are identical.

Recall
\[
y(x) = A + B \ln|x-1|
\]
(2.17)
Thus the series solution is identical to the solution (2.12) provided \(|x| \leq 1\). We note that the radius of convergence for the power series (2.13) is 1, which corresponds to the distance between the expansion point \(x_0 = 0\) and the nearest singular point \(x = 1\).
Power series solution of general variable coefficient linear ODE:

Consider solving variable coefficient linear ODEs of the form

\[ P(x)y'' + Q(x)y' + R(x)y = 0 \]  

Homogeneous Eq. (2.18)

Divide through by \( P(x) \):

\[ Ly = y'' + p(x)y' + q(x)y = 0 \quad p(x) = Q/P, \quad q(x) = R/P \]  

(2.19)

In order to calculate the higher derivatives of \( y(x) \) to substitute into Taylor’s formula, we rewrite (2.19) as follows

\[ y'' = -p(x)y' - q(x)y \]

If \( y(x_0) \) and \( y'(x_0) \) are given, then \( y''(x_0) \) can be obtained directly from the ODE. Higher derivatives of \( y \) can, in turn, be obtained by differentiating the ODE repeatedly. This process will be successful provided \( p(x) \) and \( q(x) \) are infinitely differentiable at \( x = x_0 \). In this case \( p(x) \) and \( q(x) \) are said to be analytic at \( x_0 \) and have Taylor expansions of the form

\[ p(x) = p_0 + p_1(x-x_0) + \cdots = \sum_{k=0}^{\infty} p_k(x-x_0)^k \]

\[ q(x) = q_0 + q_1(x-x_0) + \cdots = \sum_{k=0}^{\infty} q_k(x-x_0)^k \]

Ordinary points: The expansion point \( x_0 \) is said to be an ordinary point of (2.19) if \( p(x) = Q/P \) and \( q(x) = R/P \) are analytic at \( x_0 \). If \( x_0 \) is an ordinary point it is possible to obtain power series expansions of the solution \( y(x) \) of the form:

\[ y(x) = \sum_{n=0}^{\infty} c_n(x-x_0)^n. \]  

(2.20)

The idea is to substitute the expansion (2.20) into (2.19) and solve for the unknown coefficients \( c_n \) in order to determine a solution.

Observations:

- If \( P, Q \) and \( R \) are polynomials then a point \( x_0 \) such that \( P(x_0) \neq 0 \) is an ordinary point.
- If \( x_0 = 0 \) is an ordinary point then we assume

\[ y = \sum_{n=0}^{\infty} c_n x^n, \quad y' = \sum_{n=1}^{\infty} c_n nx^{n-1}, \quad y'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} \]

\[ 0 = Ly = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \left( \sum_{n=0}^{\infty} p_n x^n \right) \sum_{n=1}^{\infty} n c_n x^{n-1} \]

\[ + \left( \sum_{n=0}^{\infty} q_n x^n \right) \left( \sum_{n=0}^{\infty} c_n x^n \right) \]

\[ \sum_{m=0}^{\infty} \left\{ (m+2)(m+1)c_{m+2} + (p_0(m+1)c_{m+1} + \cdots + p_m c_1) \right\} x^m = 0 \]

(2.21)

(2.22)

yields a non-degenerate recursion for the \( c_m \). At an ordinary point \( x_0 \) we can obtain two linearly independent solutions of the form (2.20).
Singular Points: If \( p(x) \) or \( q(x) \) are not analytic at \( x_0 \), then \( x_0 \) is said to be a singular point of (2.19). For example if \( P, Q \) and \( R \) are polynomials and \( P(x_0) = 0 \) and \( Q(x_0) \neq 0 \) or \( R(x_0) \neq 0 \) then \( x_0 \) is a singular point. Or if \( p(x) = \sqrt{x} \) and \( q(x) = 2 \), then \( x_0 = 0 \) is a singular point because \( p(x) \) is not differentiable at \( x = 0 \).

The radius of convergence of (2.20) is at least as large as the radius of convergence of each of the series expansions for \( p(x) = Q/P \) and \( q(x) = R/P \), i.e., up to the closest singularity to \( x_0 \).

Example 5 The Airy equation: Consider the Airy equation, which arises in Quantum Mechanics:

\[
Ly = y'' - xy = 0
\]  
(2.23)

We observe that \( x = 0 \) is an ordinary point.

\[
y = \sum_{n=0}^{\infty} c_n x^n, \quad y' = \sum_{n=1}^{\infty} c_n n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}
\]

\[
\sum_{m=2}^{\infty} c_n n(n-1) x^{n-2} = \sum_{n=0}^{\infty} c_n x^{n+1}
\]

\[
m + 1 = n - 2 \quad n = m + 3 \quad n = 2 \Rightarrow m = -1
\]

\[
c_2 2x^0 + \sum_{m=0}^{\infty} [c_{m+3}(m+3)(m+2) - c_m] x^{m+1} = 0
\]

\[
c_2 = 0 \quad c_{m+3} = \frac{c_m}{(m+3)(m+2)} \quad m = 0, 1, \ldots
\]

(1) \( c_0 \rightarrow c_3 \rightarrow c_6 \).

\[
c_3 = \frac{c_0}{3.2}, \quad c_6 = \frac{c_3}{6.5} = \frac{c_0}{6.5 \cdot 3.2}, \quad c_9 = \frac{c_6}{9.8 \cdot 6.5 \cdot 3.2}
\]

\[
c_{3n} = \frac{c_0}{(3n)(3n-1)(3n-3)(3n-4) \ldots 9.8 \cdot 6.5 \cdot 3.2}
\]

\[
y_0(x) = 1 + \frac{x^4}{3.2} + \frac{x^6}{6.5 \cdot 3.2} + \cdots + \frac{x^{3n}}{(3n)(3n-1) \ldots 3.2} + \cdots
\]

(2) \( c_1 \rightarrow c_4 \rightarrow c_7 \rightarrow \).

\[
c_4 = \frac{c_1}{4.3}, \quad c_7 = \frac{c_4}{7.6 \cdot 4.3}, \quad c_{10} = \frac{c_7}{(10.9)(7.6)(4.3)}
\]

\[
c_{3n+1} = \frac{c_1}{(3n+1)(3n)(3n-2)(3n-3) \ldots 7.6 \cdot 4.3}
\]

\[
y_1(x) = x + \frac{x^4}{4.3} + \frac{x^7}{7.6 \cdot 4.3} + \cdots + \frac{x^{3n+1}}{(3n+1)(3n) \ldots 4.3}
\]

Radius of Convergence:

\[
\lim_{m \to \infty} \frac{c_{m+3}}{c_m} |x|^3 = \lim_{m \to \infty} \frac{|x|^3}{(m+3)(m+2)} = 0 < 1 \quad \rho = \infty.
\]

See B&D for expansion of Airy Solution about \( x_0 = 1 \); i.e. \( y(x) = \sum a_n (x-1)^n \). It is useful to write \( x = (x - 1) + 1 \).

\[
y'' = (x - 1)y + y
\]

(2.29)
Series solutions to ODE with variable coefficients

Example 6  The Hermite Equation

Consider the Hermite equation, which has application in Quantum mechanics and numerical analysis:

\[ Ly = y'' - 2xy' + \lambda y = 0 \]  (2.30)

Since \( x = 0 \) is an ordinary point let \( y(x) = \sum_{n=0}^{\infty} a_n x^n \) then

\[ L y = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - 2 \sum_{n=1}^{\infty} a_n n x^n + \lambda \sum_{n=0}^{\infty} a_n x^n = 0. \]  (2.31)

\[ m = n - 2 \rightarrow n = m + 2 \quad m \leftarrow n \quad m \leftarrow n \]
\[ n = 2 \rightarrow m = 0 \]

Therefore

\[ \sum_{m=1}^{\infty} [a_{m+2}(m+2)(m+1) - 2a_m m + \lambda a_m] x^m + [a_2 \lambda + a_0] x^0 = 0. \]  (2.32)

\[ x^0 : \quad a_2 = -\lambda a_0 / 2 \]  (2.33)

\[ x^m : \quad a_{m+2} = \frac{(2m - \lambda)a_m}{(m+1)(m+2)} \quad m \geq 1 \]  (2.34)

\[ a_0 : \]
\[ a_2 = -\frac{\lambda}{2}a_0, a_4 = \frac{(4-\lambda)}{4.3}a_2 = \frac{(4-\lambda)(-\lambda)}{4.3.2}a_0, a_6 = \frac{(8-\lambda)(4-\lambda)(-\lambda)}{6.5.4.3.2}a_0 \]
\[ a_{2k} = \frac{[4(k-1) - \lambda][4(k-2) - \lambda] \ldots [-\lambda]}{(2k)!}a_0 \]
\[ y_0 = a_0 \left[ 1 - \frac{\lambda}{2}x^2 + \frac{(\lambda - 4)\lambda}{4!}x^4 + \frac{(8-\lambda)(4-\lambda)(-\lambda)}{6!}x^6 + \ldots \right] \]  (2.35)

\[ a_1 : \]
\[ a_3 = \frac{(2-\lambda)}{3!}a_1; a_5 = \frac{(6-\lambda)}{5!}a_3 = \frac{(2-\lambda)(6-\lambda)(-\lambda)}{3!}a_1 \ldots \]
\[ y_1 = a_1 \left[ x + \frac{(2-\lambda)}{3!}x^3 + \frac{(6-\lambda)(2-\lambda)}{5!}x^5 + \frac{(10-\lambda)(6-\lambda)(2-\lambda)}{7!}x^7 + \ldots \right] \]  (2.36)

The general solution is of the form

\[ y(x) = Ay_0(x) + By_1(x) \]  (2.37)

Note:

(a) If \( \lambda = 2n \) then the recursion yields \( a_{m+2} = 0 = a_{m+4} = \ldots \) for \( m = n \). Thus if \( n \) is an even integer then the series solution \( y_0 \) will terminate and become a polynomial of degree \( n \).

In this case:

\[ y_0(x) = a_0 \left[ 1 - nx^2 + n(n-2)x^4/4! - n(n-2)(n-4)x^6/6! + \ldots + (-1)^{n/2}n(n-2)\ldots 2(2^{n/2}x^n/n!) \right]. \]  (2.38)
On the other hand if $n$ is an odd integer then the series solution $y_1(x)$ will terminate and become a polynomial of degree $n$. In this case

$$y_1(x) = a_1 \left[ x - 2(n - 1) \frac{x^3}{3!} + 2^2(n - 1)(n - 3) \frac{x^5}{5!} 
- (n - 1)(n - 3)(n - 5) \frac{x^7}{7!} + \cdots \right] \tag{2.39}$$

$$+ (n - 1)(n - 3) \ldots 3.1(-2)^{\frac{n-1}{2}} \frac{x^n}{n!} \right]$$

(b) For example in the special case $\lambda = 4 = 2n$ then $n = 2$.

$$y_0(x) = a_0[1 - 2x^2]. \tag{2.40}$$