Lecture 12: Heat equation on a circular ring - full Fourier Series

(*Compiled 3 March 2014*)

In this lecture we use separation of variables to solve the heat equation subject on a thin circular ring with *periodic boundary conditions*. In this case we reduce the problem to expanding the initial condition function $f(x)$ in an infinite series of both cosine and sine functions, which we refer to as the Full Range Fourier Series.

Key Concepts: Heat Equation; Periodic Boundary Conditions; separation of variables; Full Fourier Series.

12.1 Heat equation on a circular Ring - Full Fourier Series

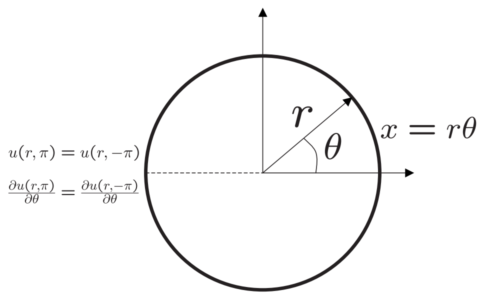

FIGURE 1. Consider a thin conducting ring with thermal conductivity α^2 that has a given initial temperature distribution

Physical Interpretation: Consider a thin circular wire in which there is no radial temperature dependence, i.e., $u(r,\theta) = u(\theta)$ so that $\dfrac{\partial u}{\partial r} = 0$. In this case the polar Laplacian reduces to

$$\begin{aligned}\Delta u &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \\ &= \frac{\partial^2 u}{\partial (r\theta)^2} \qquad (12.1)\end{aligned}$$

and if we let $x = r\theta$ then $\frac{\partial^2 u}{\partial (r\theta)^2} = u_{xx}$. In this case the heat distribution in the ring is determined by the following initial value problem with periodic boundary conditions

$$u_t = \alpha^2 u_{xx} \tag{12.2}$$

$$\text{BC:} \quad \left.\begin{array}{rcl} u(-L,t) & = & u(L,t) \\ \dfrac{\partial u}{\partial x}(-L,t) & = & \dfrac{\partial u}{\partial x}(L,t) \end{array}\right\} \text{ Periodic BC}$$

$$\text{IC:} \quad u(x,0) = f(x)$$

Assume $u(x,t) = X(x)T(t)$.

As before: $\dfrac{X''(x)}{X(x)} = \dfrac{\dot{T}(t)}{\alpha^2 T(t)} = -\lambda^2$.

IVP: $\dfrac{\dot{T}(t)}{\alpha^2 T(t)} = -\lambda^2 \Rightarrow T(t) = c\mathrm{e}^{-\lambda^2 t}$.

$r = \dfrac{2L}{2\pi} = \dfrac{L}{\pi} =$ Constant.

$$\text{BVP:} \quad \left.\begin{array}{lcl} X'' + \lambda^2 X & = & 0 \\ X(-L) & = & X(L) \\ X'(-L) & = & X'(L) \end{array}\right\} \begin{array}{l} \text{Eigenvalue Problem} \\ \text{look for } \lambda \text{ such that} \\ \text{nontrivial } x \text{ can be found.} \end{array}$$

$$\begin{aligned} X(x) &= A\cos(\lambda x) + B\sin(\lambda x) \\ X(-L) &= A\cos(\lambda L) - B\sin(\lambda L) = A\cos(\lambda L) + B\sin(\lambda L) = X(L) \\ &\text{therefore } 2B\sin(\lambda L) = 0 \\ X'(x) &= -A\lambda\sin\lambda x + B\lambda\cos(\lambda x) \\ X'(-L) &= +A\lambda\sin(\lambda L) + B\lambda\cos(\lambda L) = -A\lambda\sin(\lambda L) + B\lambda\cos(\lambda L) = X'(L) \\ &\text{therefore } 2A\lambda\sin(\lambda L) = 0 \\ \text{Therefore } \lambda_n L &= (n\pi) \quad n = 0, 1, \ldots . \end{aligned} \tag{12.3}$$

Solutions to (12.2) that satisfy the BC are thus of the form

$$u_n(x,t) = \mathrm{e}^{-\left(\frac{n\pi}{L}\right)^2 \alpha^2 t} \left\{ A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right) \right\}. \tag{12.4}$$

Superposition of all these solutions yields the general solution

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} \left\{ A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right) \right\} \mathrm{e}^{-\left(\frac{n\pi}{L}\right)^2 \alpha^2 t}. \tag{12.5}$$

In order to match the IC we have

$$f(x) = u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right). \tag{12.6}$$

As before we obtain expressions for the A_n and B_n by projecting $f(x)$ onto $\sin\left(\frac{n\pi x}{L}\right)$ and $\cos\left(\frac{n\pi x}{L}\right)$.

$$\int_{-L}^{L} f(x) \left\{ \begin{array}{c} \sin\left(\frac{m\pi x}{L}\right) \\ \cos\left(\frac{m\pi x}{L}\right) \end{array} \right\} dx = A_0 \int_{-L}^{L} \left\{ \begin{array}{c} \sin\left(\frac{m\pi x}{L}\right) \\ \cos\left(\frac{m\pi x}{L}\right) \end{array} \right\} dx \qquad (12.7)$$

$$+ \sum_{n=1}^{\infty} A_n \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \left\{ \begin{array}{c} \sin\left(\frac{m\pi x}{L}\right) \\ \cos\left(\frac{m\pi x}{L}\right) \end{array} \right\} dx$$

$$+ \sum_{n=1}^{\infty} B_n \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \left\{ \begin{array}{c} \sin\left(\frac{m\pi x}{L}\right) \\ \cos\left(\frac{m\pi x}{L}\right) \end{array} \right\} dx.$$

As in the previous example we use the orthogonality relations:

$$\begin{aligned} \int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx &= L\delta_{mn} \\ \int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx &= L\delta_{mn} \quad m \text{ and } n \neq 0 \\ &= 2L \quad m = n = 0 \\ \int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx &= 0 \quad \forall m, n. \end{aligned} \qquad (12.8)$$

Plugging these orthogonality conditions into (1.6) we obtain

$$\left. \begin{aligned} A_0 &= \frac{1}{2L} \int_{-L}^{L} f(x)\, dx = \text{ average value of } f(x) \text{ on } [-L, L] \\ A_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \text{ and } B_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx. \end{aligned} \right\} \qquad (12.9)$$

Observations:

(1) (12.6) and (12.9) represent the full Fourier Series Expansion for $f(x)$ on the interval $[-L, L]$.

(2) By defining $a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \left\{ \begin{array}{c} 2A_0 \\ \\ A_n \end{array} \right.$ and $b_n = B_n$ the Fourier Series (12.6) is often written in the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right). \qquad (12.10)$$