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The Blowup Property of Solutions to Some Diffusion Equations with Localized Nonlinear Reactions

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In this paper we investigate the blowup property of solutions to the equation

$$u_t = \Delta u + f(u(x_0, t)),$$

where x_0 is a fixed point in the domain. We show that under certain conditions the solution blows up in finite time. Moreover, we prove that the set of all blowup points is the whole region. Furthermore, the growth rate of solutions near the blowup time is also derived. Finally, the results are generalized to the following nonlocal reaction-diffusion equation

$$u_t = \Delta u + \int_\Omega f(u)\,dx.$$

1. INTRODUCTION

This paper deals with the following diffusion equation with localized reaction,

$$u_t = \Delta u + f(u(x_0, t)), \qquad \text{in} \quad Q_T, \tag{1.1}$$

subject to either the Cauchy data

$$u(x, 0) = u_0(x), \qquad x \in \Omega, \tag{1.2}$$

or the initial and the boundary (Dirichlet or Neumann type) conditions,

$$u(x, t) = 0, \qquad \text{or} \qquad u_N(x, t) = 0, \text{ on } S_T = S \times [0, T], \tag{1.3}$$

$$u(x, 0) = u_0(x), \qquad \text{on } \bar{\Omega}, \tag{1.4}$$

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where $Q_T = \Omega \times (0, T]$, Ω is either R^n for the Cauchy problem or a bounded domain in R^n with smooth boundary $S = \partial\Omega$ while x_0 is a fixed interior point of Ω and $u_N = \partial u/\partial N$ is the outward normal derivative at the boundary S. In the sequel for convenience we shall simply call the Cauchy, initial-Dirichlet, or initial-Neumann problem as (PC), (PD), or (PN), respectively.

Equation (1.1) describes some physical phenomena in which the nonlinear reaction in a dynamical system takes place only at a single (or sometimes several) site(s). As an example, the influence of defect structures on a catalytic surface can be modelled by a similar equation. The reader can consult [15, 3] for the physical derivation. The additional motivation for our study comes from parabolic problems. In [4], the authors transformed a large class of parabolic inverse problems into the so-called nonclassical equation

$$u_t = a(u, u_x, u(x_0, t), u_x(x_0, t))\, u_{xx} + b(u, u_x, u(x_0, t), u_x(x_0, t)).$$

If we use the finite difference quotient to approximate the derivative $u_x(x_0, t)$, one obtains the same type of equation as (1.1). In the present work we are interested in the theoretical analysis, especially the blowup property of the solution. It will be seen that under the conditions similar to those for a standard reaction-diffusion equation,

$$u_t = \Delta u + f(u), \tag{1.5}$$

the solution blows up at a finite time. On the other hand, there are some other interesting properties which are different from the solution of a standard reaction-diffusion equation.

It is known (cf. Friedman and McLeod [7]) that under the certain mild restrictions on $f(s)$ the set of all blowup points for the solution of (1.5) is compact. Furthermore, in a symmetric space region, under some additional restrictions, the blowup occurs only at a single point (cf. Bebernes *et al.* [1], Friedman and McLeod [7], Weissler [18–19]). However, this is not true for our problem (PC), (PD), or (PN). We will show that the blowup set is the whole region. To understand the results, we consider the problem in one space dimension:

$$u_t = u_{xx} + f(u(x_0, t)), \qquad (x, t) \in (0, 1) \times (0, T_0], \tag{1.6}$$

$$u_x(0, t) = u_x(1, t) = 0, \qquad t \in (0, T_0], \tag{1.7}$$

$$u(x, 0) = u_0(x), \qquad x \in [0, 1]. \tag{1.8}$$

Assume $u(x, t)$ blows up at a point. Let T be the blowup time. If we differentiate the equation (1.6) with respect to x and denote $u_x(x, t)$ by $v(x, t)$, then for any $T_0 < T$, $v(x, t)$ satisfies

$$\begin{aligned} &v_t = v_{xx}, && (x, t) \in (0, 1) \times (0, T_0], \\ &v(0, t) = v(1, t) = 0, && t \in (0, T_0], \\ &v(0, t) = u_0'(x), && x \in [0, 1]. \end{aligned}$$

The maximum principle indicates that $v(x, t)$ is uniformly bounded on $[0, 1] \times [0, T_0]$ (the bound is independent of T_0). If $u(x, t)$ is uniformly bounded at some point $y^* \in [0, 1]$ for all $t \in [0, T_0]$, we have

$$u(x, t) = u(y^*, t) + \int_{y^*}^{x} v(y, t)\, dy.$$

That is, $u(x, t)$ would be uniformly bounded for any $(x, t) \in [0, 1] \times [0, T_0]$. Consequently, $\lim_{T_0 \to T} \|u\|_{C(Q_{T_0})}$ exists and is finite. It follows that we can extend our solution beyond T, which contradicts the definition of T.

It is worth noting that there is a certain connection between the localized reaction-diffusion equation (1.1) and the following nonlocal equation

$$u_t = \Delta u + \int_\Omega f(u(x, t))\, dx. \tag{1.9}$$

The mean value theorem for integrals implies that

$$\int_\Omega f(u(x, t))\, dx = f(u(x^*, t))\, |\Omega|,$$

where $x^* \in \Omega$. However, in this case the point $x^* = x^*(t)$ is a function of the time variable. Equation (1.9) in some sense is equivalent to an equation with localized reaction along a (unknown) curve $x = x^*(t)$.

The argument we use to prove the blowup property is based on the comparison principle. The key point is to construct a suitable comparison function. For the problems (PN) and (PC), this argument allows us to eliminate the assumption of the convexity on $f(s)$, which is essential in proving the finite time blowup to (1.5). Moreover, using this argument, we can deal with a much more general nonlinear reaction-diffusion (possibly degenerate) equation

$$u_t - a_{ij}(x, t, u, u_x)\, u_{x_i x_j} = f(u), \tag{1.10}$$

where the matrix (a_{ij}) is only assumed to be positive semi-definite. From this viewpoint, we also improve the classical results on the blowup property. By employing some powerful properties of the fundamental solution and Green's function along with the particular structure of Eq. (1.1), we are able to show the solution blows up everywhere. Moreover, we also derive the growth rate of the solution near the blowup time.

Section 2 deals with the local solvability and the finite time blowup. In Section 3, we shall give the profile of solutions near the blowup time. Finally, we briefly present some results for Eq. (1.9) in Section 4. All the notations used in this paper are standard.

2. Local Existence and Finite Time Blowup

Throughout this paper the following basic conditions are always assumed:

(HB) The function $f(s) \in C^2(R)$ and $f(s) \geqslant 0$; $u_0(x) \in C^{2+\alpha}(\bar{\Omega})$ is nonnegative and bounded.

The following consistency conditions hold: for (PD), $u_0(x) = 0$; and for (PN), $u_{0N}(x) = 0$ on the boundary S.

We begin with the local solvability.

THEOREM 2.1. *Under the hypothesis* (HB), *each of the problems* (PC), (PD), *and* (PN) *admits a unique classical solution for some* $T_0 > 0$. *Moreover, the solutions are nonnegative.*

Proof. The local existence can be obtained by means of a fixed point theorem (cf. [4]). The nonnegativity of the solutions follows from the maximum principle (see Lemma 2.1 below for details).

Let T be the maximal value such that the problem (PC), (PD), or (PN) is solvable on $[0, T)$. If $T = +\infty$ then we have a global solution, otherwise, as we shall now prove,

$$\lim_{t \to T} \|u(x, t)\|_0 = \infty.$$

Indeed, if $T < \infty$ and $M = \lim_{t \to T} \|u(x, t)\|_0$ is finite, then we use L_p-estimates to yield

$$\|u\|_{W_p^{2,1}} \leqslant CM.$$

The interpolation inequality with $p > n + 2$ implies

$$\|u\|_{C^{1+\alpha, (1+\alpha)/2}(\bar{Q}_T)} \leqslant C(M).$$

Thus, we apply the Schauder estimate to deduce

$$\|u\|_{C^{2+\alpha,1+\alpha/2}(Q_T)} \leqslant C(M).$$

Therefore, we can apply the local existence result to extend the solution to a larger interval $[T, T+\delta]$ with some $\delta > 0$, which contradicts the definition of T.

The interesting question is what conditions ensure that the solution blows up at a finite time. It turns out the conditions are similar or weaker to those for a regular reaction-diffusion equation (1.5) if such a property holds. Here we will use a new approach to prove this property. We first need the following version of the maximum principle which will be used in the sequel.

LEMMA 2.1 (The Maximum Principle). *Let $u(x, t)$ be a classical solution of the problem*

$$u_t - \Delta u \geqslant c(x, t)\, u(x_0, t), \qquad \text{in } Q_T,$$

$$u(x, t) = 0 \qquad \text{or} \qquad u_N(x, t) = 0, \ (x, t) \in S_T,$$

$$u(x, 0) \geqslant 0, \qquad x \in \Omega.$$

If $0 \leqslant c(x, t) \leqslant c_0$, *then*

$$u(x, t) \geqslant 0, \qquad \text{for all} \quad (x, t) \in \bar{Q}_T.$$

Proof. The proof is similar to the classical case. We omit it.

LEMMA 2.2 (The Maximum Principle for the Cauchy Problem). *Let $u(x, t)$ be a solution of the Cauchy problem*

$$u_t - \Delta u \geqslant c(x, t)\, u(x_0, t), \qquad \text{in} \quad Q_T = R^n \times (0, T], \tag{2.1}$$

$$u(x, 0) \geqslant 0, \qquad x \in R^n. \tag{2.2}$$

Then $u(x, t) \geqslant 0$ for all $(x, t) \in Q_T$.

Proof. Although one can still use the method in [9] to show the result, here we give an alternative proof. Let $u_0(x)$ be the initial value and $m(x, t)$ the compensation function which needs to be added to the right-hand side of (2.1) to convert it to an equation. By the representation of the fundamental solution, $u(x, t)$ can be expressed by

$$u(x, t) = \int_R \Gamma(x, y; t, 0)\, u_0(y)\, dy + \int_0^t \int_{R^n} \Gamma(x, y; t, \tau)[c(y, \tau)\, u(x_0, \tau) + m(y, \tau)]\, dy\, d\tau. \tag{2.3}$$

Let

$$h_0(t)=\int_{R^n}\Gamma(x_0, y; t, 0)\, u_0(y)\, dy+\int_0^t\int_{R^n}\Gamma(x_0, y; t, \tau)\, m(y, \tau)\, dy\, d\tau.$$

It is clear that $h_0(t)$ is nonnegative since $u_0(x)$ and $m(x, t)$ as well as the fundamental solution are nonnegative.

Define the operator B from $C[0, T]$ to $C[0, T]$ as

$$Bh(t)=\int_0^t\int_{R^n}[\Gamma(x_0, y; t, \tau)\, c(y, \tau)]\, h(\tau)\, dy\, d\tau.$$

We first evaluate $u(x, t)$ at $x=x_0$ in (2.3) and then regard it as an integral equation for $u(x_0, t)$. Then for small t we have the Neumann series

$$u(x_0, t)=\sum_{n=0}^{\infty} B^{(n)}h_0(t),$$

where $B^{(0)}h_0(t)=h_0(t)$ and $B^{(n+1)}h_0(t)=B[B^{(n)}h_0(t)]$, $n=1, 2, \ldots$.

As $\Gamma(x, y; t, \tau)>0$ and $c(x, t)\geqslant 0$, mathematical induction implies $B^{(n)}h_0(t)\geqslant 0$. It follows that $u(x_0, t)\geqslant 0$. Substituting this inequality into the expression (2.3) we see that $u(x, t)\geqslant 0$ for all $x\in\bar{\Omega}$ and t on a small interval. Finally, we can repeat the above procedure to obtain the desired result for the whole interval.

Now we investigate the blowup property. For the problems (PN) and (PC), we assume

(HNC) Assume $f'(s)\geqslant 0$, and $\int_{z_0}^{\infty}(1/f(s))\, ds<\infty$ for some $z_0>0$. If $f(0)>0$, we allow $z_0=0$.

THEOREM 2.2. *Let $u(x, t)$ be the solution of the Cauchy problem* (PC) *or the initial-Neumann problem* (PN). *Under the conditions* (HB) *and* (HNC), *the solution $u(x, t)$ will blow up at a finite time, provided that $u_0(x)\geqslant z_0$.*

Proof. We first consider the Neumann problem (PN). We need to construct a suitable comparison function which blows up at a finite time. To this end, we consider the auxiliary problem

$$v'(s)=f(v(s))\geqslant 0$$

$$v(0)=z_0>0.$$

The assumption (HNC) ensures that $v(s)$ is monotone increasing and blows up at a finite time $=\int_{z_0}^{\infty}(1/f(s))\, ds$. We denote this number by S_0.

Define $W(x, t) = u(x, t) - v(t)$, for $x \in \bar{\Omega}$ and $t \in [0, T]$ $(T < S_0)$. Then in Q_T, $W(x, t)$ satisfies

$$\begin{aligned} W_t - \Delta W &= f(u(x_0, t)) - v'(t) \\ &= f(u(x_0, t)) - f(v(t)) \\ &= c(t)\, W(x_0, t), \end{aligned}$$

where $c(t) = \int_0^1 f'(zu(x_0, t) + (1-z)\, v(t))\, dz$ is nonnegative and bounded while the bound depends on the upper bound of T and the known data. On the lateral boundary S_T, $W_N(x, t) = 0$. Moreover, $W(x, 0) = u_0(x) - z_0 \geqslant 0$. The maximum principle (Lemma 2.1) implies $W(x, t) \geqslant 0$ for all $(x, t) \in \bar{Q}_T$.

Since $T < S_0$ is arbitrary, we conclude that

$$u(x, t) \geqslant v(t), \qquad \text{for all} \quad (x, t) \in \bar{\Omega} \times [0, S_0).$$

Therefore, $u(x, t)$ must blow up at a finite time.

For the Cauchy problem (PC), we can take the same comparison function as above and use Lemma 2.2 to obtain the result.

Next we consider the initial-Dirichlet problem (PD). In this case, the comparison function is not easy to find because of the boundary condition. We will therefore require the convexity of $f(s)$.

(HD) In addition to the assumption (HNC), $f(s)$ is assumed to be convex.

THEOREM 2.3. *Under the condition* (HD), *the solution of* (PD) *blows up at a finite time if* $u_0(x)$ *is sufficiently large in a neighborhood of* x_0.

Proof. Without loss of generality, we may assume x_0 is the origin. Let $B_\rho(0)$ be the ball centered at the origin with the radius $\rho > 0$ such that $B_\rho(0) \subset \Omega$. Consider an auxiliary problem,

$$v_t - \Delta v = f(v(0, t)), \qquad \text{in} \quad B_\rho(0) \times (0, T], \tag{2.4}$$

$$v(x, t) = 0, \qquad (x, t) \in \partial B_\rho(0) \times (0, T], \tag{2.5}$$

$$v(x, 0) = v_0(x), \qquad x \in \Omega, \tag{2.6}$$

where $v_0(x)$ is a nonnegative smooth function which is less than $u_0(x)$ on $B_\rho(0)$. We assert that on $B_\rho(0) \times (0, T]$, $u(x, t) \geqslant v(x, t)$. Indeed, since $u(x, t) \geqslant 0$ on $\bar{Q}_T$ and $f'(s) \geqslant 0$, the maximum principle (Lemma 2.1) implies our assertion.

We now show that $v(x, t)$ blows up at a finite time. This is the following

LEMMA. *If $v_0(x)=v_0(r)$ is symmetric and $v_0'(r)\leqslant 0$, then the problem* (2.4)–(2.6) *admits a local solution $v(x,t)$ which is symmetric in the space variable, i.e., $v(x,t)=v(r,t)$ and $v_r(r,t)\leqslant 0$ for any $(r,t)\in(0,\rho)\times(0,T]$. Moreover, $v(r,t)$ blows up at a finite time if $v_0(x)$ is large enough.*

Proof. The local solvability is standard. Let $U(r,t)=v_r(r,t)$, we see that $U(r,t)$ satisfies

$$U_t-\left[\frac{n-1}{r}U+U_r\right]_r=0, \qquad 0<r<\rho, 0\leqslant t\leqslant T,$$

$$U(0,t)=0, \qquad \frac{n-1}{r}U(\rho,t)+U_r(\rho,t)=-f(v(0,t)), 0\leqslant t\leqslant T,$$

$$U(r,0)=v_0'(r),\ 0\leqslant r\leqslant\rho.$$

The strong maximum principle implies $U(r,t)=v_r(r,t)\leqslant 0$.

To show that $v(r,t)$ blows up at a finite time, we note that $v_r(r,t)\leqslant 0$ from which it follows that

$$f(v(0,t))\geqslant f(v(r,t)) \qquad \text{for all} \quad (r,t)\in B_\rho(0)\times(0,T].$$

Now we replace $f(v(0,t))$ in the right-hand side of (2.4) by $f(v(r,t))$ and denote the corresponding solution by $V(r,t)$. It is well known (cf. [1, 7]) that the blowup for the solution $V(x,t)$ occurs at a finite time. Next since $f(v(r,t))\leqslant f(v(0,t))$, the comparison principle gives $v(r,t)\geqslant V(r,t)$ and hence $v(r,t)$ blows up at a finite time. This completes our proof of the Lemma.

Finally, to finish our proof of the theorem, we let $u_0(x)$ be large enough such that $u_0(x)\geqslant v_0(r)$ on $B_\rho(0)$. Then $u(x,t)\geqslant V(r,t)$ on $B_\rho(0)\times(0,T]$ and then $u(x,t)$ will blow up in finite time.

Using the technique of the paper [12], one can improve the result of Theorem 2.3 by being more specific about the size of $u_0(x)$ sufficient to ensure the blowup. Indeed, let us define $w(x)$ to be the solution to the problem

$$\Delta w+1=0, \qquad \text{in } \Omega, \tag{2.7}$$

$$w(x)=0, \qquad \text{in } \partial\Omega. \tag{2.8}$$

Then the nonnegative steady state solutions to (2.4)–(2.6) are given by $u(x)=\alpha w(x)$, where α satisfies

$$\alpha=f(\alpha w(x)). \tag{2.9}$$

Depending upon f, x_0, and Ω, Eq. (2.9) may have two roots, $\alpha_1>\alpha_2\geqslant 0$, one root $\alpha_1 \geqslant 0$, or no roots (cf. [12]). If (2.9) has no roots or if $u_0(x) > \alpha_1 w(x)$, then the solution to (PD) blows up at a finite time (cf. [13]). Hence, we have

COROLLARY 2.4. *If Eq.* (2.9) *has no roots or* $u_0(x) > \alpha_1 w(x)$, *then the finite time blowup occurs, where* α_1 *is the largest root of* (2.9) *while* $w(x)$ *is the solution of* (2.7)–(2.8).

Remark 1. The method used in proving Theorem 2.3 also works for the problem (PC) or (PN). However, the previous approach does not require the convexity of $f(s)$.

Remark 2. The results in Theorem 2.2 are clearly valid for Eq. (1.10) with Cauchy data or initial-Neumann conditions. However, this can not be generalized to the initial-Dirichlet conditions. In [8], Friedman and McLeod proved that the problem

$$\begin{aligned} u_t &= u^2 u_{xx} + u^3, && -a < x < a, 0 < t \leqslant T, \\ u(-a, t) &= u(a, t) = 0, && 0 \leqslant t \leqslant T, \\ u(x, 0) &= u_0(x), && -a < x < a, \end{aligned}$$

admits a global solution without size restriction on $u_0(x)$ if $a < \pi/2$.

3. Profile of Solutions near the Blowup Time

This section concerns the set of blowup points and the growth rate of the solution as t tends to the blowup time. Throughout this section let T denote the blowup time. The heat operator (or a general linear parabolic operator) will be denoted by L. A point (x, T) is said to be a blowup point if there exists a sequence (x_n, t_n) $(t_n < T)$ which converges to (x, T) as n goes to ∞ such that

$$\lim_{n \to \infty} u(x_n, t_n) = \infty.$$

We denote the set of all blowup points by B.

THEOREM 3.1. *For the Cauchy problem* (PC), $B = R^n$. *For the initial-Neumann problem* (PN) *and the initial-Dirichlet problem* (PD), $B = \bar{\Omega}$.

Proof. First of all we claim that x_0 must be a blowup point. Otherwise, $u(x_0, T)$ is bounded, and so is $f(u(x_0, t))$. As before the L_p-estimate and the Schauder estimate imply that $u(x, T) \in C^{2+\alpha}(\bar{\Omega})$. The local solvability indicates that we can extend our solution beyond T, which is a contradiction. For the Cauchy problem, the solution can be represented by

$$u(x, t) = \int_{R^n} \Gamma(x, y; t, 0)\, u_0(y)\, dy + \int_0^t \int_{R^n} \Gamma(x, y; t, \tau)\, f(u(x_0, \tau))\, dy\, d\tau \tag{3.1}$$

We claim that

$$\lim_{t \to T} \int_0^t f(u(x_0, \tau))\, d\tau = \infty. \tag{3.2}$$

To prove the above assertion, we note by a direct calculation that

$$\int_{R^n} \Gamma(x, y; t, \tau)\, dy = \int_{R^n} \left[\frac{1}{\sqrt{4\pi(t-\tau)}}\right]^n \exp\left\{-\frac{|x-y|^2}{t-\tau}\right\} dy = c_0,$$

where c_0 is a positive constant. It follows that

$$u(x, t) = \int_{R^n} \Gamma(x, y; t, 0)\, u_0(y)\, dy + c_0 \int_0^t f(u(x_0, \tau))\, d\tau. \tag{3.3}$$

Since $u(x_0, t)$ tends to ∞ as t approaches T, hence

$$\lim_{t \to T} \int_0^t f(u(x_0, \tau))\, d\tau = \infty.$$

Moreover, the equality (3.3) implies that for any $x \in \Omega = R^n$,

$$\lim_{t \to T} u(x, t) = \infty,$$

i.e., $u(x, t)$ blows up on the whole space R^n.

For the initial-Neumann problem, we again use Green's representation (cf. [6, p. 694]) to obtain

$$u(x, t) = \int_\Omega G(x, y, t)\, u_0(y)\, dy$$
$$+ \int_0^t \int_\Omega G(x, y; t-\tau)\, f(u(x_0, \tau))\, dy\, d\tau, \qquad (x, t) \in Q_T, \tag{3.4}$$

where $G(x, y; t-\tau)$ is Green's function associated with the operator L

(see [6] for its construction and its properties). Moreover, the function $G(x, y; t-\tau)$ posesses the properties

$$G(x, y; t-\tau) \geqslant 0 \quad \text{and} \quad \int_\Omega G(x, y; t)\, dy = 1.$$

Using the above properties, we have

$$u(x, t) = \int_\Omega G(x, y, t)\, u_0(y)\, dy + \int_0^t f(u(x_0, \tau))\, d\tau.$$

As for the Cauchy problem, we can show the final integral in the above expression tends to ∞ as t goes to T. It follows that for any $x \in \bar{\Omega}$, $u(x, t)$ must tend to infinity as t approachs to T.

To prove the last result, let $G(x, y; t, \tau)$ be Green's function associated with the operator L along with the null Dirichlet boundary condition. Then for any $T_0 < T$ the solution can be written as

$$u(x, t) = \int_\Omega G(x, y, t)\, u_0(y)\, dy + \int_0^t \int_\Omega G(x, y; t-\tau) f(u(x_0, \tau))\, dy\, d\tau, \qquad (x, t) \in Q_{T_0}. \tag{3.5}$$

As with the preceding argument, we intend to show that

$$\int_0^t f(u(x_0, \tau))\, d\tau \to \infty \qquad \text{as} \quad t \to T.$$

For Green's function $G(x, y; t, \tau)$ we have the estimate

$$|G(x, y; t, \tau)| \leqslant C(t-\tau)^{-n/2} \exp\left\{-c\, \frac{|x-y|^2}{t-\tau}\right\}.$$

From the expression (3.5), we first evaluate $u(x, t)$ at x_0 and then calculate the integration to obtain

$$u(x_0, t) \leqslant \int_\Omega G(x, y, t)\, u_0(y)\, dy + C \int_0^t f(u(x_0, \tau))\, d\tau,$$

where C is a positive constant which depends on the upper bound of T_0 and the known data. Since x_0 is a blowup point, it follows that

$$\lim_{t \to T} \int_0^t f(u(x_0, \tau))\, d\tau = \infty.$$

The rest of the proof is similar to that in [1, Theorem 4.1]. Thus, $B = \bar{\Omega}$.

The remaining part of this section deals with the growth rate of the solution near the blowup time. The method is based on the maximum principle and similar to [2, 7].

THEOREM 3.2. *For the solution* $u(x, t)$ *of problem* (PD) *or* (PN),

$$U'(t) \leqslant f(U(t)),$$

where $U(t) = \max_{x \in \Omega} u(x, t)$.

Proof. First of all, the maximum principle indicates that $u(x, t)$ can not achieve a positive maximum on the lateral boundary S_T. Note that $f(u(x_0, t)) \leqslant f(U(t))$. The rest of the proof is exactly the same as in [7]. We omit it here.

To obtain an upper bound of the solution, we need the following

LEMMA 3.1. *Assume that the initial value* $u_0(x)$ *satisfies*

$$\Delta u_0(x) \geqslant 0, \tag{3.6}$$

then for any $(x, t) \in Q_{T_0}$ $(T_0 < T)$,

$$u_t(x, t) \geqslant 0.$$

Proof. Let $v(x, t) = u_t(x, t)$. It is easy to see that $v(x, t)$ satisfies

$$\begin{aligned} v_t &= \Delta v + f'(u(x_0, t))\, v(x_0, t), \qquad \text{in } Q_{T_0}, \\ v_N &= 0 \qquad \text{or} \qquad v(x, t) = 0,\ (x, t) \in S_T \\ v(x, 0) &= \Delta u_0(x) + f(u_0(x_0)). \end{aligned}$$

The maximum principle implies that $v(x, t) \geqslant 0$.

THEOREM 3.3. *For the solution of the problem* (PN), *under the assumption of Lemma* 3.1 *we have*

$$u_t(x, t) \geqslant \tfrac{1}{2} f(u(x_0, t)), \qquad \textit{in particular, } h'(t) \geqslant \tfrac{1}{2} f(h(t)),\ 0 \leqslant t < T,$$

where $h(t) = u(x_0, t)$.

Proof. The proof is analogous to [7]. Let $W(x, t) = u_t - \frac{1}{2} f(u(x_0, t))$. Then $W(x, t)$ satisfies

$$\begin{aligned} W_t - \Delta W &= \tfrac{1}{2} f'(u(x_0, t))\, u_t(x_0, t) \geqslant 0, \\ W_N(x, t) &= 0, \qquad (x, t) \in S_T \\ W(x, 0) &= \Delta u_0(x) + \tfrac{1}{2} f(u_0(x)), \qquad x \in \Omega. \end{aligned}$$

Hence, the desired inequality follows from the strong maximum principle.

As a direct consequence, we have:

COROLLARY 3.1. *Under the assumptions* (HB), (HNC), *and* (3.6), *let* $u(x, t)$ *be the solution to the problem* (PN) *and let the function* $h(t)$ *be defined as in Theorem* 3.3. *If* $f(s) = e^s$, *then*

$$h(t) \leqslant \ln \frac{2}{(T-t)}, \qquad \text{for any} \quad t \in (0, T).$$

For $f(s) = (s + \lambda)^p$ *with* $p > 1$ *and* $\lambda \geqslant 0$, *then*

$$h(t) + \lambda \leqslant \left[\frac{2q}{T-t}\right]^q, \qquad \text{for any} \quad t \in (0, T),$$

where $q = 1/(p-1)$.

Remark. For the solution of (PD), the growth rate near the blowup time is an open question.

4. A NONLOCAL PROBLEM

In this section we briefly generalize some results obtained in the preceding sections to the following nonlocal problem:

$$u_t = \Delta u + \int_\Omega f(u(x, t))\, dx, \qquad \text{in } Q_T \tag{4.1}$$

$$u_N(x, t) = 0, \qquad (x, t) \in S_T, \tag{4.2}$$

$$u(x, 0) = u_0(x), \qquad x \in \Omega. \tag{4.3}$$

For the local existence we refer to [5]. Here we give an alternative proof for the blowup property of solutions. We assume

$$f(s) \geqslant 0, \qquad f'(s) \geqslant 0, \qquad \text{and} \qquad f(s) \text{ is convex}, \int^\infty \frac{1}{f(s)}\, ds < \infty.$$

Moreover, we assume $\int_\Omega u_0(x)\, dx > 0$.

THEOREM 4.1. *Under the above assumptions, the solution of* (4.1)–(4.3) *blows up at a finite time.*

Proof. Let

$$h(t)=\frac{1}{|\Omega|}\int_{\Omega} u(x,t)\,dx, \qquad t\in(0,T).$$

We integrate Eq. (4.1) over Ω to obtain

$$h'(t)=\frac{1}{|\Omega|}\int_{\Omega} f(u(x,t))\,dx.$$

Jensen's inequality yields

$$\int_{\Omega} f(u)\,dx \geqslant |\Omega|\, f(h(t)), \qquad t\in[0,T).$$

It follows that

$$h'(t)\geqslant f(h(t)), \qquad t\in[0,T).$$

Hence $T\leqslant \int_{h(0)}^{\infty}(1/f(s))\,ds$ and the theorem is proved.

If we replace the Neumann boundary condition (4.2) by the Dirichlet condition,

$$u(x,t)=0, \qquad (x,t)\in S_T. \tag{4.4}$$

The blowup property can be shown by an analogous argument to that for a regular reaction-diffusion equation.

THEOREM 4.2. *Under the conditions of Theorem* 4.1, *the solution of* (4.1), (4.3)–(4.4) *blows up at a finite time if* $u_0(x)$ *is sufficiently large.*

Proof. Let $\phi(x)$ be the normalized first eigenfunction of the eigenvalue problem

$$-\Delta\phi(x)=\lambda\phi(x), \qquad \text{in } \Omega,$$

$$\phi(x)=0, \qquad x\in\Omega.$$

We multiply Eq. (4.1) by $\phi(x)$ and then integrate over Ω to have

$$a'(t)+\lambda_1 a(t)\geqslant \int_{\Omega} f(u)\,dx,$$

where $a(t)=\int_{\Omega} u(x,t)\,\phi(x)\,dx\geqslant 0$.

Jensen's inequality implies

$$\int_\Omega f(u)\,dx \geqslant |\Omega|\, f\left(\frac{1}{|\Omega|}\int_\Omega u\,dx\right).$$

Let C_0 be the maximum of $\phi(x)$ on $\bar{\Omega}$. Then

$$a(t) \leqslant C_0 \int_\Omega u\,dx.$$

Since $f(s)$ is nondecreasing, we find

$$f\left(\frac{1}{|\Omega|}\int_\Omega u\,dx\right) \geqslant f\left(\frac{1}{C_0\,|\Omega|}\,a(t)\right).$$

It follows that

$$a'(t) + \lambda_1 a(t) \geqslant |\Omega|\, f\left(\frac{1}{C_0\,|\Omega|}\,a(t)\right).$$

Consequently, when $a(0)$ is large enough, $a(t)$ blows up at a finite time. This completes our proof.

THEOREM 4.3. *For the problems* (PD) *and* (PN), *the blowup sets* $B = \bar{\Omega}$.

This can be shown exactly as Theorem 3.4. Under certain additional conditions, we can also deduce the growth rate of the solution near the blowup time.

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