Problem 6

(1) The prime ideals of \( A \) are of the form \( M = (x - a, y - b) \) for \( a, b \in \mathbb{F} \)
and \( M \supset (x^2 - y^3) \) means \( x^2 - y^3 = (x - a)P + (y - b)Q \) so \( a^2 - b^3 = 0 \)
and \( a \) is in \( \mathbb{F} \), \( b \) is in \( \mathbb{F} \).

So the prime ideals of \( A \) correspond to the \( y \)-points of the curve \( x^2 - y^3 = 0 \).

(2) We want to show that \( (x^2 - y^3) \) is a prime ideal of \( \mathbb{C}[x, y] \).

First \( \mathbb{C}[x, y] \) is a UFD it is equivalent to \( x^2 - y^3 \) being irreducible in \( \mathbb{C}[x, y] \).

Let \( \mathfrak{p} := \mathbb{C}[y] \) \( \{x^2 - y^3 \} \subseteq \mathfrak{p} \mathbb{C}[x] \) is prime.

To show that it is irreducible in \( \mathbb{C}[x, y] \) it is therefore enough to show that \( x^2 - y^3 \) is irreducible.

\[ \text{If } \mathfrak{p} = \mathbb{C}[y] \text{ then } \mathbb{C}(y)[x] = \mathbb{C}(y)[x] \text{ is integral.} \]

But \( x^2 - y^3 \in K[x] \) has degree 2. \( K \) (field)

If it was not irreducible, it would have a root in \( K \). Such a root is

\[ \frac{P(y)}{Q(y)} \in K, \quad P, Q \in \mathbb{C}[y], \quad P \neq 0. \]

and we have

\[ x^2 - y^3 \frac{P(y)}{Q(y)} = 0 \quad \text{in } \mathbb{C}[y] \]

even degree \( y \) odd degree. \text{ Contradiction.}
Let \( \varphi : \mathbb{C}[X,Y] \rightarrow \mathbb{C}[T] \)

\[ \varphi(X,Y) = T^3, \quad \varphi(T) = 1 \] 

It is a well-defined morphism of rings and its kernel contains \( X^2 - Y^3 \) so it factors through a morphism of rings

\[ \overline{\varphi} : \mathbb{C}[X,Y] \rightarrow \mathbb{C}[T^3, T^2] \]

\[ \overline{\text{Im} \varphi} = \text{Im} \overline{\varphi} = \mathbb{C}[T^3, T^2] = \text{Subring of } \mathbb{C}[T] \text{ generated by } T^2 \text{ and } T^3 \]

To show that \( \overline{\varphi} \) is injective we need to show that \( \ker \overline{\varphi} = (X^2 - Y^3) \)

Let \( P \in \mathbb{C}[X,Y] \) such that \( \varphi(P) = 0 \).

The Euclidean division of \( P \) by \( X^2 - Y^3 \) in \( \mathbb{C}[X] \) where \( \mathbb{C}[X] \) is

\[ P = Q(X^2 - Y^3) + R \]

where \( Q \in \mathbb{C}[X] \), \( R \in \mathbb{C}[X] \) (because \( X^2 - Y^3 \) is monic)

and \( \deg R \leq 1 \).

So \( R = U(Y)X + V(Y) \) where \( U, V \in \mathbb{C}[Y] \).

The kernel \( \mathbb{C}[T^3, T^2] = U(T^3)T^3 + V(T^2) = 0 \)

in \( \mathbb{C}[T] \)

But \( U(T^3)T^3 \) has odd degree

\( V(T^2) \) has even degree

so \( U = V = 0 \) and \( P = 0 \)

and \( P \in (X^2 - Y^3) \)
4) The fraction field $K$ of $A$ is the smallest field in which the non-zero elements of $A$ are invertible. Therefore $K \subseteq B(T)$.

Now $T = \frac{T^3}{T^2}$ so $T \in K$ and therefore any $P(T) \in K$ for $P \in C[T]$ and any fraction $\frac{P(T)}{Q(T)}$ where $P, Q \in C[T]$ and $Q \neq 0$ also lies in $K$.

So $K = C(T)$.

By (3) $A = C[T^3, T^2]$

$T^3$ is irreducible in $C[T^3, T^2]$.

($T^3 = P(T^3, T^2)Q(T^3, T^2)$ --- Look at the degrees)

Yet $C[T^3, T^2]/(T^3)$ is not an integral domain since $T^2 \times T^2 \subseteq (T^3)$

Yet $T^2 \notin (T^3)$.