Problem 1. For \( R \) a subring of \( \mathbb{C} \) and \( z \in \mathbb{C} \) we denote by \( R[z] \) the image of the morphism of rings
\[
R[X] \longrightarrow \mathbb{C} \\
P \longmapsto P(z)
\]
where \( R[X] \) is the ring of polynomials with coefficients in \( R \).

1. Show that \( R[z] \) is the subring of \( \mathbb{C} \) generated by \( R \) and \( z \).
   
   This is somewhat formal. You can denote by \( A \) the subring of \( \mathbb{C} \) generated by \( R \) and \( z \) and explain why the image \( R[z] \) of the above map is equal to \( A \).

2. Find a quotient ring of \( \mathbb{Q}[X] \) which is isomorphic to
   
   (a) \( \mathbb{Q}[i\sqrt{7}] \),
   
   (b) \( \mathbb{Q}\left[\frac{1 + \sqrt{5}}{2}\right] \).

3. Find a quotient ring of \( \mathbb{Z}[X] \) which is isomorphic to
   
   (a) \( \mathbb{Z}[i\sqrt{7}] \),
   
   (b) \( \mathbb{Z}\left[\frac{1 + \sqrt{5}}{2}\right] \).

4. Let \( A = \mathbb{Z}[\sqrt{10}] \) and \( K = \mathbb{Q}[\sqrt{10}] \).
   
   (a) Describe the elements of \( A \) and the elements of \( K \). Justify your answer.
   
   (b) For an element \( x \in K \) consider the multiplication \( m_x : K \rightarrow K \). Check that it is a \( \mathbb{Q} \)-linear map on the 2-dimensional \( \mathbb{Q} \)-vector space \( K \). Give a basis of \( K \) as a \( \mathbb{Q} \)-vector space. If \( x = a + b\sqrt{10} \) with \( a, b \in \mathbb{Q} \), write the matrix of \( m_x \) in the basis you just gave. Denote by \( T(x) \) its trace and by \( N(x) \) its determinant. Explain without calculation why \( N(xy) = N(x)N(y) \). What happens when \( x \in A \) ?
   
   (c) Show that \( 2 \) is irreducible in \( A \) namely that if \( 2 = xy \) with \( x, y \in A \) then \( x \) or \( y \) is a unit of \( A \).
   
   (d) Show that \( (2) \) is not a prime ideal of \( A \).
Problem 2. (1) What are the prime ideals of
(a) \( A = \mathbb{C}[X] \),
(b) \( A = \mathbb{R}[X]/(X^2 + X + 1) \),
(c) \( A = \mathbb{R}[X]/(X^3 - 6X^2 + 11X - 6) \),
(d) \( A = \mathbb{R}[X]/(X^4 - 1) \).
(2) Explain why these rings are also vector spaces over \( \mathbb{R} \).
(3) (not to be handed in) Determine the morphisms of \( R \)-algebras from these rings into \( \mathbb{C} \) (respectively into \( \mathbb{R} \)), namely the morphisms of rings \( A \to \mathbb{C} \) (respectively \( A \to \mathbb{R} \)) which fix \( \mathbb{R} \).

Problem 3. Let \( k \) be a field with characteristic different from 2 and \( G = \{ e, g \} \) the group with two elements (the element \( e \) is the identity in \( G \) and in \( A \)). We consider the group ring \( A = k[G] \) (see Section 7.2).
(1) What are the ideals of \( A \)?
You can notice that \( A \) is a 2-dimensional vector space. Then check that an ideal of \( A \) is also a sub-vector-space of \( A \), therefore it can have dimension 0, 1 or 2. Then among the 1-dimensional vector subspaces of \( A \), find the ones which are also ideals.
(2) Is \( A \) principal?
(3) What are the nilpotent elements of \( A \), namely the elements \( a \) such that there is \( n \geq 1 \) satisfying \( a^n = 0 \).
This question is a bit more difficult. Two kinds of approaches :
- Find two orthogonal idempotents \( x \) and \( y \) in \( A \) such that \( x + y = e \) (namely two elements \( x \) and \( y \) in \( A \) such that \( xy = yx = 0 \) and \( x^2 = x \), \( y^2 = y \)). It implies (prove it) that \( A \) is isomorphic to the product of rings \( A_x \times A_y \) (where \( A_x \) has \( x \) as identity element and \( A_y \) has \( y \) as identity element). Conclude.....
- Prove that \( A \) is isomorphic to \( k[X]/(X^2 - 1) \) and work with the latter ring, which most likely you understand better than \( A \).
Remark : if you understand both approaches, you can wonder how they relate to each other. Can you write \( k[X]/(X^2 - 1) \) as a product of two rings ? What happens if \( k \) has characteristic 2 ? These are very good questions to think of, but they are not part of the problem set.
(4) What is the intersection of all prime ideals of \( A \)?

Problem 4. Let \( A \) be an integral domain and \( a, b \in A \) such that \( (a) = (b) \). What can you say about \( a \) and \( b \)?

Problem 5. We admit the following result known as Eisenstein Criterion.
Let \( f \in \mathbb{Q}[X] \) a monic polynomial with degree \( m \geq 1 \)
\[ f = X^m + a_{m-1}X^{m-1} + \cdots + a_1X + a_0. \]
Suppose that
(i) \( a_0, \ldots, a_{m-1} \in \mathbb{Z} \),
(ii) there is a prime number \( p \) that divides \( a_0, \ldots, a_{m-1} \) and
(iii) \( p^2 \) does not divide \( a_0 \).
Then \( f \) is irreducible over \( \mathbb{Q} \) namely if \( f = gh \) with \( g, h \in \mathbb{Q}[X] \) then \( g \) or \( h \) is a nonzero constant polynomial.

Let \( p \) be a prime number and \( \epsilon \) a \( p^{th} \) primitive root of 1 in \( \mathbb{C} \). Let \( A = \mathbb{Z}[\epsilon] \) be the subring of \( A \) generated by \( \epsilon \), namely the intersection of all subrings of \( \mathbb{C} \) containing \( \epsilon \). Note that \( \mathbb{Z} \) is a subring of \( A \).
(1) Show that the polynomial \( \Phi_p = 1 + X + \ldots + X^{p-1} \) is irreducible over \( \mathbb{Q} \).
(2) Deduce that \( \Phi_p = 1 + X + \ldots + X^{p-1} \) is a generator for the ideal \( \{ P \in \mathbb{Q}[X], P(\epsilon) = 0 \} \).
(3) Show that the map
\[ Z^{p-1} \to A \]
\[ (x_0, \ldots, x_{p-2}) \mapsto \sum_{i=0}^{p-2} x_i \epsilon^i \]
is an isomorphism of additive groups.
END OF PROBLEMSET 2. THE REST OF THIS PROBLEM AND PROBLEM 6 WILL BE PART OF PROBLEMSET 3.

(4) Show that the intersection of $\mathbb{Z}$ with the ideal $(1 - \epsilon)A$ is equal to the ideal $p\mathbb{Z}$ of $\mathbb{Z}$. (You can consider the Euclidean division of $\Phi_p$ by $(X - 1)$).

(5) Deduce $A/(1 - \epsilon)A \simeq \mathbb{Z}/p\mathbb{Z}$.

(6) What can we say about the ideal $(1 - \epsilon)A$?

**Problem 6.** Problem 33 of Section 7.4 except for question (d).

**Problem 7.** Some recommended problems, not to be handed in:
Problems 9, 15, 16, 17 of Section 7.4
Problem 1 Section 7.6