Problem 1. (1) Let $k$ be a field. For $r \geq 0$, we denote by $k_r[X]$ the vector space of all polynomials in $k[X]$ with degree $\leq r$. Let $P, Q \in k[X]$ polynomials with respective degrees $p$ and $q$. Consider the map
\[ \mathcal{R} : \quad k_{q-1}[X] \times k_{p-1}[X] \rightarrow k_{p+q-1}[X] \]
\[ (A, B) \quad \rightarrow \quad AP + BQ. \]
We call resultant of $P$ and $Q$ and we denote by $\text{Res}(P, Q)$ the determinant of this linear map. Give a condition on $P$ and $Q$ which is equivalent to $\text{Res}(P, Q) = 0$. Recall that the determinant of this map is nonzero if and only if the map is injective, if and only if it is surjective. The surjectivity condition should remind you of Bezout theorem.

(2) Application: Show that the set of algebraic numbers $\Omega$ is a field. Hints.
   - The determinant of a matrix is a polynomial expression in the coefficient of the matrix.
   - If $\alpha$ is a root for $P \in \mathbb{Q}[X] \setminus \{0\}$ and $\beta$ is a root for $Q \in \mathbb{Q}[X] \setminus \{0\}$, let $\gamma := \alpha + \beta$ and $Q_\gamma := Q(\gamma - X)$. What can you say about the resultant of $P$ and $Q_\gamma$ (which are both polynomials in $\mathbb{C}[X]$)?
   - Once you find a polynomial expression in $\gamma$ which is zero, you still have to prove that this gives you a nonzero polynomial which has root $\gamma$... For this, you are probably going to use the fact that a nonzero polynomial has finitely many roots.

Given a ring $R$ and a $R$-module $M$, we say that $M$ is simple if the only submodules of $M$ are $M$ and $\{0\}$.

Problem 2. Consider the symmetric group $S_3$ and let $k$ be an arbitrary field of characteristic different from 2. Let $R$ be the group ring $k[S_3]$.

(1) Check for yourself that an $R$-module is also a $k$-vector space. Show that there exactly are two nonisomorphic $R$-modules which are 1-dimensional as $k$-vector spaces. One of them will be called the trivial character and will be denoted by $k_{triv}$ and the other one the sign character which will be denoted by $k_{sign}$ (it has something to do with the signature of $S_3$...).

(2) Consider the $k$-linear morphism of rings $k[S_3] \rightarrow M_3(k)$ sending $\sigma$ onto the corresponding permutation matrix. This endows the vector space $V_3 = ke_1 \oplus ke_2 \oplus ke_3$ with a structure of $R$-module such that $\sigma \in S_3$ acts on $e_i$ by $(\sigma, e_i) \mapsto e_{\sigma(i)}$.

(a) Does $V_3$ contain a sub-$R$-module that is isomorphic to $k_{triv}$? $k_{sign}$?

(b) Show that $\{e_1 - e_2, e_2 - e_3\}$ spans a sub-vector-space of $V_3$ which is a sub-$R$-module.

(i) If $k$ has characteristic different from 3, show that $V_3$ is the direct sum of two simple modules.

(ii) If $k$ has characteristic 3, give an inclusion of $R$-modules
\[ 0 \subset V_1 \subset V_2 \subset V_3 \]
where $V_i$ has dimension $i$. Describe the quotient representations $V_{i+1}/V_i$. Does the short exact sequence
\[ 0 \rightarrow V_1 \rightarrow V_3 \rightarrow V_3/V_1 \rightarrow 0 \]
split?

Problem 3. Let $R$ be a ring with identity element and $M$ a left $R$-module. Describe a structure of left $R$-module on $\text{Hom}_R(R, M)$. Show that
\[ M \cong \text{Hom}_R(R, M) \]
as left $R$-modules.

Problem 4. Let $m, n \in \mathbb{Z}$, both $\geq 1$ and $d := \text{gcd}(m, n)$. Show that we have an isomorphism of $\mathbb{Z}$-modules
\[ \text{Hom}_\mathbb{Z}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z} \]