This exam has 10 questions on 9 pages, for a total of 50 points.

Duration: 2.5 hours

• This is a closed-book examination. None of the following are allowed: documents or electronic devices of any kind (including calculators, cell phones, etc.)

• If your answers are not easily readable and well organized, they may not be read and credited.

• Make sure the structure of your proof is clearly apparent. There may be some credit for the formal structure of certain proofs, independently of the mathematical content.

LAST name: __________________________________________________________

First name: : __________________________________________________________

Student Number: ______________________________________________________

Signature: ____________________________________________________________

<table>
<thead>
<tr>
<th>Question:</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Points:</td>
<td>9</td>
<td>5</td>
<td>4</td>
<td>8</td>
<td>3</td>
<td>5</td>
<td>6</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>50</td>
</tr>
<tr>
<td>Score:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
1. For each of the following statements write T or F in the box to indicate whether the statement is true or false and justify your answer for parts (b) to (e).

1 mark  
(a) \( \mathbb{N} \subseteq \mathbb{Z} \).

\[ \square \]

**Solution:** It is false since \( \mathbb{N} \) is a subset of \( \mathbb{Z} \) but not an element of \( \mathbb{Z} \).

2 marks  
(b) \( \mathbb{N} \subseteq \mathcal{P}(\mathbb{N}) \).

\[ \square \]

**Solution:** It is false. An element \( n \in \mathbb{N} \) is not a subset of \( \mathbb{N} \) so it is not an element of \( \mathcal{P}(\mathbb{N}) \).

2 marks  
(c) Let \( A, B \) and \( C \) be sets. If \( A = B - C \), then \( B = A \cup C \).

\[ \square \]

**Solution:** It is false. Counterexample: Take \( A = \{1\} \), \( B = \{1, 2, 3\} \), \( C = \{2, 3, 4\} \).

2 marks  
(d) \( (\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R}) = \mathbb{R} \times \mathbb{R} \)

\[ \square \]

**Solution:** It is false since \( (\frac{1}{2}, \frac{1}{2}) \in \mathbb{R} \times \mathbb{R} \) but \( (\frac{1}{2}, \frac{1}{2}) \notin \mathbb{R} \times \mathbb{Z} \) and \( (\frac{1}{2}, \frac{1}{2}) \notin \mathbb{Z} \times \mathbb{R} \).

2 marks  
(e) There exists a function \( f : \mathbb{R} \to \mathbb{R} \) and there exists \( X \subseteq \mathbb{R} \) such that \( f^{-1}(f(X)) \neq X \).

\[ \square \]

**Solution:** It is true. Take
\[
      f : \mathbb{R} &\to \mathbb{R} \\
      x &\mapsto 1
\]

and \( X = \{0\} \). Then \( f^{-1}(f(X)) = \mathbb{R} \).
2. 5 marks

Prove the following statement for \( n \in \mathbb{Z} \).

\( n \) is odd if and only if \( 8 \mid (n^2 - 1) \).

**Solution:** \( \Rightarrow \)

Suppose that \( n \in \mathbb{Z} \) is odd. Then there exists \( k \in \mathbb{Z} \) such that \( n = 1 + 2k \) and \( n^2 - 1 = 4k^2 + 4k = 4(k + 1) \). Since \( k \) and \( k + 1 \) are consecutive integers one of them is even so \( k(k + 1) \) is even and \( 4k(k + 1) \) is a multiple of 8.

\( \Leftarrow \)

By contrapositive. Suppose \( n \) is even. Then there exists \( k \in \mathbb{Z} \) such that \( n = 2k \) and \( n^2 - 1 = 4k^2 - 1 = 2(2k^2 - 1) + 1 \) is an odd number so it is not a multiple of 8.

3. 4 marks

Prove that for every natural number \( n \geq 8 \):

\[ 2^n > 3n^2. \]

**Solution:** By induction. For \( n \geq 8 \), denote by \( P(n) \) the statement:

\[ 2^n > 3n^2. \]

- **Base step.** Let \( n = 8 \). We have \( 2^8 - 3 \times 8^2 = (2^3)^2 \times 2^2 - 3 \times 8^2 = 8^2(4 - 3) = 8^2 > 0. \)

- **Induction step.** Let \( n \geq 8 \) and suppose that \( P(n) \) is true namely \( 2^n > 3n^2 \).

We study \( P(n+1) \):

\[
2^{n+1} - 3(n + 1)^2 = 2 \times 2^n - 3n^2 - 6n - 3 \\
> 2(3n^2) - 3n^2 - 6n - 3 = 3(n^2 - 6n - 3) - 3(n^2 - 6n - 3) = 3[(n-1)^2 - 2] \text{ by induction hypothesis}
\]

but \( 3[(n-1)^2 - 2] \geq 3(7^2 - 2) \geq 0 \) so \( 2^{n+1} - 3(n + 1)^2 > 0 \) and \( P(n+1) \) is true. We proved, for \( n \geq 8 \), that \( P(n) \implies P(n+1) \) which concludes the induction step.

4. 8 marks

Let

\[ A = \{ n \in \mathbb{Z} : 3 \mid n \}, \quad B = \{ n \in \mathbb{Z} : 4 \mid n \}, \quad C = \{ n \in \mathbb{Z} : 6 \mid n \} \]

Determine whether the following statements are true or false, and justify your answer.

1. \( \exists a \in A, \exists b \in B \) such that \( a + b \in C \)
2. \( \forall a \in A, \forall b \in B \) we have \( a + b \in C \)
3. \( \exists a \in A, \) such that \( \forall b \in B \) we have \( a + b \in C \)
4. \( \forall a \in A, \exists b \in B \) such that \( a + b \in C \)

**Solution:** See HW4.
5. Prove or disprove: There exists \((x, y) \in \mathbb{Z} \times \mathbb{Z}\) such that \(3x^2 + 2 = y^2\).

**Solution:** Proof by contradiction. Suppose there exist \((x, y) \in \mathbb{Z} \times \mathbb{Z}\) such that \(3x^2 + 2 = y^2\). Then \(y^2 \equiv 2 \mod 3\). But

- if \(y \equiv 0 \mod 3\), then \(y^2 \equiv 0 \mod 3\),
- if \(y \equiv 1 \mod 3\) or \(y \equiv -1 \mod 3\), then \(y^2 \equiv 1 \mod 3\).

so it is not possible to have \(y^2 \equiv 2 \mod 3\) therefore we found a contradiction. So there is no \((x, y) \in \mathbb{Z} \times \mathbb{Z}\) such that \(3x^2 + 2 = y^2\).

**Solution:** Proof by contradiction. Assume that there exist \((x, y) \in \mathbb{Z} \times \mathbb{Z}\) such that \(x^2 = 3y^2 + 2\) then:

- Either: \(x\) is even and \(y\) is odd. Then \(x^2\) is even, \(3y^2\) is odd and \(x^2 - 3y^2\) is odd so it contradicts \(x^2 - 3y^2 = 2\).
- Or: \(x\) is odd and \(y\) is even. Then \(x^2\) is odd, \(3y^2\) is even and \(x^2 - 3y^2\) is odd so it contradicts \(x^2 - 3y^2 = 2\).
- Or: \(x\) is even and \(y\) is even namely there are \(k, \ell \in \mathbb{Z}\) such that \(x = 2k\) and \(y = 2\ell\). But then
  \[
  4k^2 = 12\ell^2 + 2
  \]
  so
  \[
  4(k^2 - 3\ell^2) = 2
  \]
  and 4 divides 2. Contradiction.
- Or: \(x\) is odd and \(y\) is odd namely there are \(k, \ell \in \mathbb{Z}\) such that \(x = 2k + 1\) and \(y = 2\ell + 1\). But then
  \[
  4k^2 + 4k + 1 = 12\ell^2 + 12\ell + 3 + 2
  \]
  so
  \[
  4k^2 + 4k = 12\ell^2 + 12\ell + 4
  \]
  and
  \[
  k^2 + k = 3\ell^2 + 3\ell + 1 \text{ namely } k(k + 1) = 3\ell(\ell + 1) + 1.
  \]
  But \(k(k + 1)\) is an even number and \(3\ell(\ell + 1)\) is an even number so this is impossible.

In all cases we have a contradiction. So there is no \((x, y) \in \mathbb{Z} \times \mathbb{Z}\) such that \(3x^2 + 2 = y^2\).
6. We consider the following relations:

a) On the set \( \mathbb{Z} \), we consider the relation given by \( x \equiv y \mod 4 \) if \( x^2 + 6 \equiv y^2 - 2 \mod 4 \).

b) Let \( X \) denote the set of functions \( \mathbb{R} \to \mathbb{R} \). Define the relation \( S \) on \( X \) by \( f \sim g \) if there is \( x \in \mathbb{R} \) such that \( f(x) = g(x) = 0 \).

For each of them decide if it is an equivalence relation and justify your answer. If it is an equivalence relation, give the list of the equivalence classes.

**Solution:**

a) We notice that in fact we have \( x \equiv y \mod 4 \) when \( x^2 \equiv y^2 \mod 4 \).

- Reflexivity: for \( x \in \mathbb{Z} \), we have \( x^2 \equiv x^2 \mod 4 \) so \( x \sim x \).

- Symmetry: for \( x, y \in \mathbb{Z} \), we have \( x^2 \equiv y^2 \mod 4 \implies y^2 \equiv x^2 \mod 4 \) so \( x \sim y \implies y \sim x \).

- Transitivity: for \( x, y, z \in \mathbb{Z} \), we have
  
  \[(x^2 \equiv y^2 \mod 4) \land (y^2 \equiv z^2 \mod 4) \implies (x^2 \equiv y^2 \mod 4) \]
  
  so \( x \sim y \land y \sim z \implies x \sim z \).

So this is an equivalence relation. Notice

- If \( x \) is even, we have \( x^2 \equiv 0 \mod 4 \),

- If \( x \) is odd, we have \( x^2 \equiv 1 \mod 4 \).

So we have two equivalence classes for \( \sim \), the class of 0 which is made of all even integers, and the class of 1 which is made of all odd integers;

b) Not an equivalence relation. For example, \( \sim(f \circ f) \) when \( f : \mathbb{R} \to \mathbb{R} \) is defined by \( f(x) = x^2 + 1 \).

7. We consider the function

\[
    f : \mathbb{R} \longrightarrow \mathbb{R} \\
    x \mapsto \frac{x^2 - 1}{x^2 + 1}
\]

(a) Is \( f \) injective? Justify your answer.

**Solution:** It is not injective. For example \( f(1) = f(-1) = 0 \).

(b) Prove \( f([0, +\infty)) = [-1, 1) \).

**Solution:**

- We first prove \( f([0, +\infty)) = [-1, 1) \).

  Let \( y \in f([0, +\infty)) \). It means that there exists \( x \in [0, +\infty) \) such that \( f(x) = y \) namely

  \[
  \frac{x^2 - 1}{x^2 + 1} = y.
  \]
But \(-x^2 \leq x^2\) so \(-x^2 + 1 \leq x^2 + 1\) and \(-\frac{x^2-1}{x^2+1} \leq 1\) so \(y \geq -1\).

Also, \(-1 < 1\) so \(x^2 - 1 < x^2 + 1\) and \(\frac{x^2-1}{x^2+1} < 1\) so \(y < 1\).

Let \(y \in [-1,1)\). Since \(\frac{1+y}{1-y} \geq 0\) we may consider the element \(\sqrt{\frac{1+y}{1-y}}\) and we call it \(x\). Obviously we have \(x \geq 0\). Furthermore

\[
f(x) = \frac{\sqrt{1+y^2} - 1}{\sqrt{1+y^2} + 1} = \frac{1+y - 1}{1+y + 1} = \frac{y}{2} = y.
\]

So we have proved that our \(y \in [-1,1)\) can be written in the form \(f(x)\) for some \(x \in [0, +\infty)\). This means exactly that \(y \in f([0, +\infty))\).

(c) Is \(f\) surjective? Justify your answer.

\begin{center}
\textbf{Solution:} No it is not because we just proved that its range is \([-1,1)\) so its range is not equal to its target/codomain space \(\mathbb{R}\).
\end{center}

(d) Show that \(|[0, +\infty)| = |[-1,1)|\).

\begin{center}
\textbf{Solution:} The range of \(f\) is \([-1,1)\) so we may define the function

\[
g : [0, +\infty) \rightarrow [-1,1)
\]

It is surjective by construction. We prove that it is injective. Let \(x, x' \in [0, +\infty)\) such that \(g(x) = g(x')\). It means

\[
\frac{x^2 - 1}{x^2 + 1} = \frac{x'^2 - 1}{x'^2 + 1}
\]

So

\[
x^2x'^2 - x'^2 + x^2 - 1 = x^2x'^2 - x'^2 + x'^2 - 1
\]

namely

\[
2x^2 = 2x'^2
\]

so \(x^2 = x'^2\). Since both \(x \geq 0\) and \(x' \geq 0\) this implies \(x = x'\). So \(g\) is injective. We have prove that \(g\) is a bijective function \([0, +\infty) \rightarrow [-1,1)\). It implies \(|[0, +\infty)| = |[-1,1)|\).
8. Let $A$ and $B$ be sets. Prove:

$$|A - B| = |B - A|,$$

then $|A| = |B|$. 

**Solution:** See HW12 solution.

9. Let $A$ be a set and $\mathcal{F}$ be the set of all functions $f : A \rightarrow \{0, 1\}$ from $A$ to $\{0, 1\}$.

Consider the function

$$\Phi : \mathcal{F} \rightarrow \mathcal{P}(A)$$

$$f \mapsto f^{-1}(\{1\})$$

Is $\Phi$ injective? Is $\Phi$ surjective? Justify your answer.

**Solution:**

- **Injectivity:** suppose that $f, g \in \mathcal{F}$ are such that $\Phi(f) = \Phi(g)$ namely

  $$f^{-1}(\{1\}) = g^{-1}(\{1\}).$$

  Then we show that $f = g$ by showing that $\forall a \in A$, we have $f(a) = g(a)$.

  - 1st case: if $f(a) = 1$, it means that $a \in f^{-1}(\{1\})$, therefore $a \in g^{-1}(\{1\})$ so $g(a) = 1 = f(a)$.
  
  - 2nd case: if $f(a) \neq 1$, then $f(a) = 0$. Also, $a \notin f^{-1}(\{1\})$, therefore $a \notin g^{-1}(\{1\})$ so $g(a) \neq 1$ and $g(a) = 0 = f(0)$.

- **Surjectivity:** let $X \subset A$. We show that there exists $f \in \mathcal{F}$ such that $X = \Phi(f)$ namely $X = f^{-1}(\{1\})$. For this, we construct the function

  $$f : A \rightarrow \{0, 1\}$$

  $$a \mapsto \begin{cases} 1 & \text{if } a \in X \\ 0 & \text{if } a \in A \setminus X \end{cases}$$

  We have as required $f^{-1}(\{1\}) = X$.

**Solution:** Alternatively (compare with HW11 Problem 4). For each subset $X \subseteq A$, define $C_X$ to be the function

$$C_X : X \rightarrow \{0, 1\}$$

$$a \mapsto \begin{cases} 1 & \text{if } a \in X \\ 0 & \text{if } a \in A \setminus X \end{cases}$$

and consider the function

$$\Psi : \mathcal{P}(A) \rightarrow \mathcal{F}$$

$$X \mapsto C_X$$

We prove that $\Psi \circ \Phi = \text{id}_\mathcal{F}$ and $\Phi \circ \Psi = \text{id}_{\mathcal{P}(A)}$. 
• Proof that $\Psi \circ \Phi = \text{id}_F$: Let $f \in F$. By definition, $\Psi \circ \Phi(f) = \Psi(f^{-1}(\{1\}))$ is the function

$$C_{f^{-1}(\{1\})} : X \rightarrow \{0, 1\}$$

$$x \mapsto \begin{cases} 1 & \text{if } x \in f^{-1}(\{1\}) \text{ namely if } f(x) = 1 \\ 0 & \text{if } x \notin f^{-1}(\{1\}) \text{ namely if } f(x) \neq 1 \text{ namely if } f(x) = 0 \end{cases}$$

so we see that in fact $C_{f^{-1}(\{1\})} = f$ and $\Psi \circ \Phi(f) = f$.

• Proof that $\Phi \circ \Psi = \text{id}_{\mathcal{P}(A)}$: Let $X \in \mathcal{P}(A)$ namely $X \subseteq A$. By definition,

$$\Phi \circ \Psi(X) = C_X^{-1}(\{1\}) = \{a \in A : C_X(a) = 1\}.$$ But by definition of $C_X$, we have $X = \{a \in A : C_X(a) = 1\}$ ($C_X$ is precisely the function that takes value 1 at elements of $X$ and value 0 outside of $X$). So $\Phi \circ \Psi(X) = X$.

10. 2 marks

Given a sequence $(a_n)_{n \in \mathbb{N}}$ of real numbers we say $\lim_{n \to +\infty} a_n = \ell$ if

$$\forall \epsilon > 0, \exists m \in \mathbb{N} \text{ such that } \forall n \in \mathbb{N}, (n > m \implies |a_n - \ell| < \epsilon).$$

Suppose that $\lim_{n \to +\infty} a_n = \ell$. Prove that $(a_n)_{n \in \mathbb{N}}$ is bounded above, namely

there is $M \in \mathbb{R}$ such that $(\forall n \in \mathbb{N}, \text{ we have } a_n \leq M)$.

**Solution:** Suppose that $\lim_{n \to +\infty} a_n = \ell$. Then we can apply the statement

$$\forall \epsilon > 0, \exists m \in \mathbb{N} \text{ such that } \forall n \in \mathbb{N}, (n > m \implies |a_n - \ell| < \epsilon)$$

namely

$$\forall \epsilon > 0, \exists m \in \mathbb{N} \text{ such that } \forall n \in \mathbb{N}, (n > m \implies \ell - \epsilon < a_n < \ell + \epsilon)$$

for a specific choice of $\epsilon$. I pick $\epsilon = 43$. Then I know that

$$\exists m_0 \in \mathbb{N} \text{ such that } \forall n \in \mathbb{N}, (n > m_0 \implies a_n < \ell + 43)$$

This proves that for any $n \in \mathbb{N}$, we have

$$a_n \leq \max\{a_1, \ldots, a_{m_0}, \ell + 43\}$$

so $a_n$ is bounded above by $M := \max\{a_1, \ldots, a_{m_0}, \ell + 43\}$. 