Problem 1. For the following relations, decide if they are equivalence relations or not. If this is not an equivalence relation explain why. If this is an equivalence relation no need to justify but describe the equivalence classes.

1. On the set \( \mathbb{Z} \), we define the relation \( R \) by: \( x R y \) when \( x^2 \equiv y^3 \text{ mod } 4 \).

2. On the set \( \mathbb{Z} \), we define the relation \( R \) by: \( x R y \) when \( x^3 + 3 \equiv y^3 + 8 \text{ mod } 5 \).

3. On the set \( \mathbb{Z} \), we define the relation \( R \) by: \( x R y \) when \( x^3 + 3 \equiv y^3 + 7 \text{ mod } 5 \).

4. On the set \( \mathbb{R} \times \mathbb{R} \), we define the relation \( R \) by: \((x_1, y_1) R (x_2, y_2)\) when \( y_1 - x_1^2 = y_2 - x_2^2 \).

5. On the set of all functions \( \mathbb{R} \to \mathbb{R} \) we define the relation \( R \) by: \( f R g \) when there exists \( x \in \mathbb{R} \) such that \( f(x) = g(x) = 0 \).

Problem 2. Are the following partitions of the set \( A \)? Justify your answer.

1. \( A = \mathbb{R} \times \mathbb{R} \) and we consider the collection of sets \( X_n = \{(x, y) \mid n \leq x \leq n+1\} \) for \( n \in \mathbb{Z} \).

2. \( A = \mathbb{R} \times \mathbb{R} \) and we consider the collection of sets \( Y_n = \{(x, y) \mid n \leq x < n+1\} \) for \( n \in \mathbb{N} \).

3. \( A = \mathbb{R} \times \mathbb{R} \) and we consider the collection of sets \( Z_a = \{(x, y, a) \mid x, y \in \mathbb{R}\} \) for \( a \in \mathbb{R} \).

Problem 3. Let \( \mathcal{D} \) denote the set of all lines in the \( xy \)-plane which are parallel (or identical) to the line with equation \( y = 3x \). Each such line is treated as a subset of \( \mathbb{R} \times \mathbb{R} \).

1. Check for yourself that \( \mathcal{D} \) is a partition of \( \mathbb{R} \times \mathbb{R} \) (nothing to hand in).

2. Give an equivalence relation \( R \) on \( \mathbb{R} \times \mathbb{R} \) whose set of equivalence classes is \( \mathcal{D} \).

Problem 4. Let \( A \) be the set

\[
A = \{a + \sqrt{2}b : a, b \in \mathbb{Q} \text{ and } (a, b) \neq (0, 0)\}.
\]

We admit that \( \sqrt{2} \notin \mathbb{Q} \).

1. Show that \( 0 \notin A \).

2. Let \( R \) be the relation on \( A \) defined by \( xRy \iff x/y \in \mathbb{Q} \).

   (a) Show that \( R \) is an equivalence relation (not to be handed in).

   (b) Show that the equivalence class of \( \sqrt{2} \) is \( \{b\sqrt{2} : b \in \mathbb{Q} \setminus \{0\}\} \).

Problem 5. Let \( \alpha : \mathbb{R} \to \mathbb{Z} \) be the function defined the following way: for \( x \in \mathbb{R} \), \( \alpha(x) \) is the largest integer \( n \in \mathbb{Z} \) such that \( x \geq n \).
1. What is $\alpha(1)$, $\alpha(-2)$, $\alpha(-2.5)$, $\alpha(\pi)$?

2. Is $\alpha$ surjective? Justify.

3. Is $\alpha$ injective? Justify.

4. For $X$ a subset of $\mathbb{R}$, we define $\alpha(X) = \{\alpha(x) : x \in X\}$. Identify the following sets (no justification):
   \begin{enumerate}
   \item $\alpha((0, 1))$.
   \item $\alpha([0, 1])$.
   \item $\alpha([0, 1))$.
   \item $\alpha((0, \infty))$.
   \end{enumerate}

5. For $Y$ a subset of $\mathbb{Z}$, we define $\alpha^{-1}(Y) = \{x \in \mathbb{R} : \alpha(x) \in Y\}$. What is $\alpha^{-1}([1, 2] \cap \mathbb{Z})$? (no justification)

**Problem 6.** Define the function $f : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ by $f(n) = (2n + 1, n + 2)$. Verify whether this function is injective and whether it is surjective.

**Problem 7.** Consider the function $f : \{0, 1\} \times \mathbb{Z} \to \mathbb{Z}$ defined as $f(a, b) = a - 2ab + b$. Is it injective? Surjective?

**Problem 8.** Let $f : B \to C$ and $g : A \to B$ be functions.

1. Prove: if $f \circ g$ is injective then $g$ is injective. (Not to be handed in but recommended).

2. Prove: if $f \circ g$ is surjective then $f$ is surjective.

3. Give an example of $f$ and $g$ such that $f$ is injective but $f \circ g$ is not injective.

4. Give an example of $f$ and $g$ such that $f$ is surjective but $f \circ g$ is not surjective.

5. Give an example of $f$ and $g$ such that $f \circ g$ is injective but $f$ is not injective.

6. Give an example of $f$ and $g$ such that $f \circ g$ is surjective but $g$ is not surjective.

7. Show that the function $f : (0, +\infty) \to (0, +\infty)$ defined by $f(x) = \ln\left(\frac{e^x + 1}{e^x - 1}\right)$ is bijective.
   (Hint: Calculate $f \circ f$.)

**Problem 9.** (not to be handed in but recommended)

1. Show by strong induction (see HW5) that for every $n \in \mathbb{N}$, there exists $k \in \mathbb{Z}$, $k \geq 0$ such that $2^k | n$ and $\frac{n}{2^k}$ is odd.

2. Consider the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(x, y) = 2^{x-1}(2y - 1)$.
   Show that it is a bijection.