Problem 1.  1. We define the sequence \((u_n)_{n \geq 0}\) the following way:

\[
\begin{align*}
  u_0 &= 2 \\
  u_{n+1} &= \frac{2 + u_n}{1 + u_n} \quad \text{for } n \geq 0
\end{align*}
\]

(a) Compute \(u_1\), \(u_2\) and \(u_3\).

(b) Prove that for every \(n \in \mathbb{Z}\) such that \(n \geq 0\) we have \(1 \leq u_n \leq 2\).

2. Let \(n \in \mathbb{N}\). Prove that \(\forall n \geq 7\) we have \(n! > 3^n\).

3. Let \(n \in \mathbb{N}\). Prove that \(\exists x, y, z \in \mathbb{Z}\) such that \(x \geq 2, y \geq 2,\) and \(z \geq 2\) and \(x^2 + y^2 = z^{2n+1}\).

4. Prove, using induction, that \(\forall n \in \mathbb{N}, \ 3 \mid (n^3 - n)\).

5. The Fibonacci numbers are defined by the recurrence

\[
F_1 = 1 \quad F_2 = 1 \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \quad \text{for } \ n > 2.
\]

Show that for every \(k \in \mathbb{N}\), \(F_{4k}\) is a multiple of 3.

Problem 2.  1. Prove that for every \(n \in \mathbb{Z}\), there exist \(a, b \in \mathbb{Z}\) such that \(n = 5a + 2b\).

2. Prove that for every \(n \in \mathbb{Z}\), there exist \(c, d \in \mathbb{Z}\) such that \(n = 5c + 3d\).

3. Prove, for every integer \(n \geq 4\), the following statement:

\[P(n): \text{there exist nonnegative integers } a, b \text{ such that } n = 5a + 2b.\]

Hint: We suggest two methods. Check that you are able to understand both solutions.

- Split into cases, depending on whether \(n\) is even or odd (then use induction).
- Use double induction, which is a generalization of the standard induction.

The outline here is the following:

**Base Step:** check that the statement is true for \(n = 4\) and \(n = 5\).

**Induction Step:** prove, for every \(n \geq 4\), that \((P(n) \land P(n+1)) \implies P(n+2)\).

4. Using similar methods as in question 3., prove for every integer \(n \geq 8\), the following statement:

\[Q(n): \text{there exist nonnegative integers } a, b \text{ such that } n = 5c + 3d.\]
Problem 3. In cases where proving the inductive step is harder for a proof by induction, one can use another induction method, called strong induction, and it goes as follows:

**Theorem.** A statement of the form “∀n ∈ N, P(n)” is true if

- The statement $P(1)$ is true,
  
  and,

- given $k ≥ 1$, \( (P(1) \land P(2) \land P(3) \land \ldots \land P(k)) \implies P(k + 1). \)

Use this result to prove the following statement: A chocolate bar consists of individual squares. You want to split the chocolate bar into its individual pieces by breaking it along straight lines using the minimum number of breaks. If the bar has $n$ squares, then it takes $n - 1$ breaks to split the chocolate bar into individual pieces.