Recall that an integer is an element of \( \mathbb{Z} \).

**Problem 1.** Determine whether the following statements are logically equivalent:

1. \( P \iff Q \) and \( \sim P \iff \sim Q \).
2. \( P \implies (Q \lor R) \) and \( P \implies ((\sim Q) \implies R) \).
3. \( P \implies (Q \lor R) \) and \( (Q \land R) \implies P \).

**Problem 2.** Let \( P, Q, R \) and \( S \) be statements. Suppose that \( P \) is false and \( [(R \implies S) \iff (P \land Q)] \) is true. Find the truth values of \( R \) and \( S \).

**Problem 3.** Let \( a \in \mathbb{Z} \). Prove the following statement:

\[
\text{if } 5 \mid 2a, \text{ then } 5 \mid a .
\]

**Problem 4.**

1. Prove the following statement. For every \( a \in \mathbb{R} \),

\[
\text{if } a \geq 4, \text{ then } -\frac{a^2}{4} + a \leq 0.
\]

2. Let \( a \in \mathbb{R} \). Prove the following statement:

\[
( \forall x \in \mathbb{R}, \ x^2 + ax + a > 0) \quad \text{if and only if} \quad (0 < a < 4).
\]

You may want to transform the expression \( x^2 + ax + a \) by completing the square.

**Problem 5.** Let \( m \in \mathbb{Z} \). Prove that if \( 5 \nmid m \), then \( m^2 \equiv 1 \pmod{5} \) or \( m^2 \equiv -1 \pmod{5} \).

**Problem 6.** Prove the following statement:

\[
\forall a \in \mathbb{Z}, \ (3 \nmid a \implies (\exists b \in \mathbb{Z} \text{ such that } ab \equiv 1 \pmod{3})).
\]

**Problem 7.**

1. Prove that the product of 5 consecutive integers is a multiple of 5.

2. (not to be handed in but recommended). Prove that the product of 2020 consecutive integers is a multiple of 2020.

**Problem 8.** We recall that given \( a, b \in \mathbb{Z} \) such that \( ab \neq 0 \), we define the gcd of \( a \) and \( b \) to be the greatest integer that divides both \( a \) and \( b \). We denote it by \( \gcd(a, b) \).

1. Let \( a, b \in \mathbb{Z} \) such that \( ab \neq 0 \). We suppose that there exists \( u, v \in \mathbb{Z} \) such that

\[
1 = au + bv.
\]

Prove that \( \gcd(a, b) = 1 \).

2. Consider the following statement:

\[
\forall a \in \mathbb{Z}, \ b \in \mathbb{Z}, \ d \in \mathbb{N},
(\exists u, v \in \mathbb{Z} \text{ such that } d = au + bv) \implies \gcd(a, b) = d.\]

(a) Write the negation of the statement.

(b) Decide if the statement or its negation is true. Justify your answer.