Problem 1. Prove each of the following.

1. (2 points) The number $\sqrt[3]{2}$ is not a rational number.
2. (2 points) The number $\log_2(3)$ is not a rational number.
3. (2 points) Let $x \in \mathbb{R}$ satisfy $x^7 + 5x^2 - 3 = 0$. Then $x$ is not a rational number.
4. (2 points) Let $a, b, c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$, then $a$ or $b$ is even.
5. (2 points) There do not exist $a, n \in \mathbb{N}$ such that $a^2 + 35 = 7^n$. For $k \in \mathbb{N}$, we admit that $7 \mid k^2$ implies $7 \mid k$.

Problem 2. 1. (2 points) Show that $\sqrt{3}$ is not a rational number.
2. (2 points) Deduce that $\sqrt{3} + \sqrt{43}$ is not a rational number.

Problem 3. (2 points) Prove the equation $4x^2 + 5y^2 = 7$ has no integer solutions. (Hint: Think about this equation modulo 4.)

Problem 4. For this problem, we are going to use the following result: if $f : A \to B$

is a bijection between finite sets $A$ and $B$, then $A$ and $B$ have the same number of elements. In fact we say that they have the same cardinality and we write $|A| = |B|$.

For any set $X$, denote by $\{0, 1\}^X$ the set of all functions $X \to \{0, 1\}$. That is,

$\{0, 1\}^X = \{ f : f $ is a function from $X$ to $\{0, 1\}$ $\}$.

1. For $n \in \mathbb{N}$, define the set $X_n = \{1, \ldots, n\}$. How many elements are there in $\{0, 1\}^{X_n}$? We could actually prove this by induction but feel free to just give the answer without justification based on your intuition.

2. Let $X$ be any set. For each subset $Y \subseteq X$, define $f_Y$ to be the function

$f_Y : X \to \{0, 1\}

x \mapsto \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{if } x \in X \setminus Y \end{cases}$

This is called the characteristic of $Y$ (see how it keeps track of the elements in $Y$).

(a) (1 point) When $X = \mathbb{R}$ and $Y = [0, 1]$, draw the graph of $f_Y$.
(b) (1 point) When $X = \mathbb{R}$ and $Y = [0, +\infty)$, draw the graph of $f_Y$.
(c) (2 points) Now back to the general case of $X$. Prove that the function

$F : \mathcal{P}(X) \to \{0, 1\}^X

Y \mapsto f_Y$

is a bijection by giving the inverse of $F$.

3. Prove that $\mathcal{P}(X_n)$ has cardinality $2^n$. 

Page 1