Problem 1. Let 
\[ f : A \to B \]
\[ x \mapsto f(x) \]
be a function. We recall the following definitions:
- For \( Y \) a subset of \( B \), we define \( f^{-1}(Y) = \{ x \in A : f(x) \in Y \} \).
- For \( X \) a subset of \( A \), we define \( f(X) = \{ f(x) : x \in X \} \).

1. For \( x \in X \), what is \( f(\{ x \}) \)?

2. If \( b \in B \setminus f(A) \), what is \( f^{-1}(\{ b \}) \)?

3. For each of the following statement, decide if it is equivalent to “\( f \) is injective”. If it is equivalent, no justification is necessary. If it is not equivalent, justify your answer.

   (a) \( \forall a, a' \in A, f(a) = f(a') \implies a = a' \).
   
   Solution: Yes, this is equivalent.

   (b) \( \forall b \in B \), there is exactly one element in \( A \) such that \( f(a) = b \).
   
   Solution: No, this is not equivalent (this statement means that \( f \) is bijective).

   (c) \( \forall b \in B \), the set \( f^{-1}(\{ b \}) \) is empty or contains exactly one element.
   
   Solution: Yes, this is equivalent.

   (d) \( \forall b \in f(A) \), the set \( f^{-1}(\{ b \}) \) contains exactly one element.
   
   Solution: Yes, this is equivalent.

   (e) \( \forall a \in A \), \( f^{-1}(\{ a \}) \) contains \( a \).
   
   Solution: No, this is not equivalent (this statement is true for any function).

   (f) \( \forall a \in A \), \( f^{-1}(\{ a \}) \) contains at most one element.
   
   Solution: Yes, this is equivalent.

   (g) \( \forall a \in A \), \( f^{-1}(\{ a \}) \) contains exactly one element.
   
   Solution: Yes, this is equivalent.

4. Likewise, for each of the following statement, decide if it is equivalent to “\( f \) is surjective”. If it is equivalent, no justification is necessary. If it is not equivalent, justify your answer.

   (a) \( \forall a \in A \), \( f(a) \in B \).
   
   Solution: No, this is not equivalent (this statement is true for any function).

   (b) \( \forall b \in B \), there is exactly \( a \in A \) such that \( f(a) = b \).
   
   Solution: No, this is not equivalent (this statement means that \( f \) is bijective).

   (c) \( f(A) = B \).
   
   Solution: Yes, this is equivalent.

   (d) \( f^{-1}(B) = A \).
   
   Solution: No, this is not equivalent (this statement is true for any function).

   (e) \( f^{-1}(f(A)) = A \).
   
   Solution: No, this is not equivalent (this statement is true for any function).

   (f) \( f(f^{-1}(B)) = B \).
Problem 2. (3 points) Consider the function

\[ f : \mathbb{N} \to \mathbb{Z} \]
\[ n \mapsto \begin{cases} (n-1)/2 & \text{if } n \text{ is odd} \\ -n/2 & \text{if } n \text{ is even} \end{cases} \]

Show that \( f \) is bijective by computing its inverse function.

Solution: We define the function

\[ g : \mathbb{Z} \to \mathbb{N} \]
\[ n \mapsto \begin{cases} 2n+1 & \text{if } n \geq 0 \\ -2n & \text{if } n < 0. \end{cases} \]

(Notice that indeed these values land in \( \mathbb{N} \)). We compute \( f \circ g \) for \( n \in \mathbb{Z} \):

- If \( n \geq 0 \), we have \( f \circ g(n) = f(g(n)) = f(2n+1) = ((2n+1)-1)/2 = n \).
- If \( n \geq 0 \), we have \( f \circ g(n) = f(g(n)) = f(-2n) = -(-2n)/2 = n \).

Therefore \( f \circ g = \text{id}_{\mathbb{Z}} \).

We compute \( g \circ f(n) \) for \( n \in \mathbb{N} \):

- If \( n \) is odd, we have \( g \circ f(n) = g(f(n)) = g((n-1)/2) = 2((n-1)/2) + 1 = n \).
- If \( n \) is odd, we have \( g \circ f(n) = g(f(n)) = g(-n/2) = -2(-n/2) = n \).

Therefore \( g \circ f = \text{id}_{\mathbb{N}} \). So \( f \) has an inverse function (namely \( g \)) and \( f \) is bijective.


1. (2 points) There exists a function \( f : \mathbb{R} \to \mathbb{R} \) and a subset \( X \) of \( \mathbb{R} \) such that \( f^{-1}(f(X)) \neq X \).

Solution: True. Take for example \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = x(x-1)(x-2) \). Let \( X = \{0\} \). Then \( f(X) = \{0\} \) and \( f^{-1}(f(X)) = \{0,1,2\} \neq X \). (I take here on purpose a function which is not injective (it is surjective but it does not matter). Why?)

- You could have taken \( f(x) = x^2 \). Which choice for \( X \) then?
- Is there a function \( f : \mathbb{R} \to \mathbb{R} \) and a subset \( Y \) of \( \mathbb{R} \) such that \( f(f^{-1}(Y)) \neq Y \)? Pick a non surjective function like \( f(x) = e^x \).

2. (2 points) For any function \( f : A \to B \) and any subset \( Y \) of \( B \), we have \( f(f^{-1}(Y)) \subseteq Y \).

Solution: True. Let \( y \in f(f^{-1}(Y)) \). It means that there exists \( x \in f^{-1}(Y) \) such that \( f(x) = y \). But \( x \in f^{-1}(Y) \) means that \( f(x) \in Y \). So \( y \in Y \). Extra question: prove that if \( f \) is surjective then these sets are equal.

3. (2 points) For any function \( f : A \to B \) and any subset \( X \) of \( A \), we have \( f^{-1}(f(X)) \subseteq X \).

Solution: False. See question 1. Extra question: prove that if \( f \) is injective then these sets are equal.

4. (6 points) The range of the function

\[ f : [0, +\infty) \to \mathbb{R} \]
\[ x \mapsto \frac{x^2 - 1}{x^2 + 1} \]
is $[-1,1)$ and there exists a bijective function $[0, +\infty) \rightarrow [-1,1)$. Prove $f([0, +\infty)) = [-1,1)$.

**Solution:** True.

- We first prove $f([0, +\infty)) = [-1,1)$.

  Let $y \in f([0, +\infty))$. It means that there exists $x \in [0, +\infty)$ such that $f(x) = y$ namely
  
  $\frac{x^2 - 1}{x^2 + 1} = y$.

  But $-x^2 \leq x^2$ so $-x^2 + 1 \leq x^2 + 1$ and $-\frac{x^2 - 1}{x^2 + 1} \leq 1$ so $y \geq -1$.

  Also, $-1 < 1$ so $x^2 - 1 < x^2 + 1$ and $\frac{x^2 - 1}{x^2 + 1} < 1$ so $y < 1$.

- Let $y \in [-1,1)$. Since $\frac{1+y}{1-y} \geq 0$ we may consider the element $\sqrt{\frac{1+y}{1-y}}$ and we call it $x$.

  Obviously we have $x \geq 0$. Furthermore
  
  $f(x) = \sqrt{\frac{1+y}{1-y}}^2 - 1 = \frac{\frac{1+y}{1-y} - 1}{\sqrt{\frac{1+y}{1-y}}^2} = \frac{1+y - 1 + y}{1+y + 1 - y} = \frac{2y}{2} = y$.

  So we have proved that our $y \in [-1,1)$ can be written in the form $f(x)$ for some $x \in [0, +\infty)$.

  This means exactly that $y \in f([0, +\infty))$.

- This above proves that the range of $f$ is $[-1,1)$ so we can define the function

  $g: [0, +\infty) \rightarrow [-1,1)$

  $x \mapsto \frac{x^2 - 1}{x^2 + 1}$

  It is surjective by construction. We prove that it is injective. Let $x, x' \in [0, +\infty)$ such that $g(x) = g(x')$. It means

  $\frac{x^2 - 1}{x^2 + 1} = \frac{x'^2 - 1}{x'^2 + 1}$

  So

  $x^2 x'^2 - x'^2 + x^2 - 1 = x^2 x'^2 - x'^2 + x'^2 - 1$ namely

  $2x^2 = 2x'^2$

  so $x^2 = x'^2$. Since both $x \geq 0$ and $x' \geq 0$ this implies $x = x'$. So $g$ is injective. We have prove that $g$ is a bijective function $[0, +\infty) \rightarrow [-1,1)$.

**Problem 4.** Consider the function

$f: \mathbb{R} \rightarrow [-1, +\infty)$

$x \mapsto x^2 + 2x$

1. Show that $f$ is well defined namely that: $\forall x \in \mathbb{R}$, we have $x^2 + 2x \geq -1$.

2. (3 points) What is $f^{-1}([0])$? $f^{-1}([-4])$? $f^{-1}([-1])$?

3. (3 points) Show that Range($f$) = $[-1, +\infty)$.

4. (3 points) Is the function $f$ injective, surjective, bijective?
5. (1 point) Now we consider the function

\[ g : \mathbb{R} \rightarrow (-1, +\infty) \quad x \mapsto f(e^x) \]

Show that the function \( g \) is not surjective.

Solution:

1. For any \( x \in \mathbb{R} \), we have \((x + 1)^2 = x^2 + 2x + 1 \geq 0\) so \( x^2 + 2x \geq -1 \).

2. \( f^{-1}([0]) = \{0, -2\} \). \( f^{-1}([-4]) = \emptyset \). \( f^{-1}([-1]) = \{-1\} \).

3. Let \( y \in [-1, +\infty) \). We want to show that there exists \( x \in \mathbb{R} \) such that \( f(x) = y \) namely \( x^2 + 2x - y = 0 \). To study this equation, we compute its discriminant \( \Delta = 4 + 4y \) which is \( \geq 0 \) since \( y \geq 1 \). Therefore, this equation has a solution (in fact it has two solutions \( x = -1 + \sqrt{1+y} \) or \( x = -1 - \sqrt{1+y} \)). This proves that \( y \in \text{Range}(f) \).

So for any \( y \in [-1, +\infty) \) we have \( y \in \text{Range}(f) \) so \([-1, +\infty) \subseteq \text{Range}(f) \). Therefore \( \text{Range}(f) = [-1, +\infty) \).

4. Surjective (Question 3), not injective (because \( f^{-1}([0]) \) has more than 1 element).

5. We show that \( g \) is not surjective.

   Method 1: We prove that \( \text{Range}(g) \neq [-1, +\infty) \) (namely in fact we prove that \( \text{Range}(g) \) is "smaller" than \([1, +\infty)\)) by direct proof. Let \( y \in \text{Range}(g) \). It means that there exists \( x \in \mathbb{R} \) such that \( y = f(e^x) \) namely \( y = e^{2x} + 2e^x \) so \( y > 0 \). So the elements in \( \text{Range}(g) \) are all \( > 0 \) and \( g \), which has target space \([-1, +\infty)\), is not surjective.

   Method 2: We prove that \( \text{Range}(g) \neq [-1, +\infty) \) by contradiction. Suppose that \( \text{Range}(g) = [-1, +\infty) \). Then \(-1 \in \text{Range}(g) \). So there exists \( x \in \mathbb{R} \) such that \( g(x) = -1 \). It means that \( f(e^x) = -1 \) namely \( e^x \in f^{-1}([-1]) \). But we computed earlier that \( f^{-1}([-1]) = \{-1\} \). So \( e^x = -1 \). This cannot be true so there is no \( x \) in \( \mathbb{R} \) such that \( g(x) = -1 \). And we proved \( \text{Range}(g) \neq [-1, +\infty) \).