Problem 1. (6 points) Write the following sets by listing their elements.

1. \( A_1 = \{ x \in \mathbb{N} : x^2 < 2 \} = \{1 \} \).
2. \( A_2 = \{ x \in \mathbb{Z} : x^2 < 2 \} = \{-1, 0, 1\} \).
3. \( A_3 = \{ x \in \mathbb{N} : (3 \mid x) \land (x \mid 216) \} = \{3, 6, 9, 12, 18, 24, 27, 36, 54, 72, 108, 216\} \).
4. \( A_4 = \left\{ x \in \mathbb{Z} : x + \frac{2}{5} \in \mathbb{Z} \right\} = \{\ldots, -12, -7, -2, 3, 8, 13, \ldots\} \).
5. \( A_5 = \{ a \in B : 6 \leq 4a + 1 < 17 \} \), where \( B = \{1, 2, 3, 4, 5, 6\} \). Then \( A_5 = \{2, 3\} \).
6. \( A_6 = \{ x \in B : 50 < xd < 100 \text{ for some } d \in D \} \), where \( B = \{2, 3, 5, 7, 11, 13, \ldots\} \) and \( D = \{5, 10\} \). Then \( A_6 = \{7, 11, 13, 17, 19\} \).

Problem 2. (3 points) Write the following sets using set builder notations:

a) \( A = \{1, 6, 11, 16, 21, \ldots\} \) can be written as
   \[
   A = \{5x + 1 : x \in \mathbb{Z}, x \geq 0\}
   \]
   or
   \[
   A = \{5(x - 1) + 1 : x \in \mathbb{N}\}.
   \]

b) \( B = \{16, 8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}\} \) can be written as
   \[
   B = \{2^k : k \in \mathbb{Z}, |k| \leq 4\}.
   \]

c) \( C = \{\ldots, -\frac{2}{5}, -\frac{1}{2}, 0, \frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}, \ldots\} \) can be written as
   \[
   \left\{ \frac{m}{m^2 + 1} : m \in \mathbb{Z} \right\}.
   \]

Problem 3. Is the emptyset \( \emptyset \) an element of the following sets? What is the cardinality of each of the following sets?

a) \( C = \{\{\emptyset\}\} \). The emptyset is not an element of \( C \) and \( C \) has cardinality 1.

b) \( D = \{\{\emptyset\}, \emptyset\} \). The emptyset is an element of \( D \) and \( D \) has cardinality 2.

c) \( E = \{\{\{\emptyset\}\}, \emptyset\} \). The emptyset is not an element of \( E \) and \( E \) has cardinality 2.

Problem 4. Recall that for \( x \in \mathbb{R} \), we define the absolute value \( |x| \) of \( x \) by

\[
|x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x \leq 0
\end{cases}
\]

For \( a \in \mathbb{R} \) we define the set \( A_a = \{ x \in \mathbb{R} : 0 \leq |x| \leq -a^2 + a + 2 \} \).
1. When $a = 1$, we have $-a^2 + a + 2 = -1^2 + 1 + 2 = 2$ so $A_1 = [-2, 2]$.

2. When $a = -1$, we have $-a^2 + a + 2 = -1^2 - 1 + 2 = 0$ so $A_{-1} = [0, 0]$ which we can simply denote by $\{0\}$.

3. When $a = 2$, we have $-a^2 + a + 2 = -2^2 + 2 + 2 = 0$ so $A_2 = [0, 0]$ which we can simply denote by $\{0\}$.

4. Is there a value of $a$ for which $A_a$ is empty? Yes for example when $a = 3$ we have $-a^2 + a + 2 = -3^2 + 3 + 2 = -4$ so $A_3 = \emptyset$. (In fact, one can check that $A_a = \emptyset$ for any $a \in (-\infty, -1)$ and for any $a \in (2, +\infty)$.)

**Problem 5.**

1. (1 point) Consider the following statement:

   If it is raining then I will take the bus, and otherwise I will ride my bicycle.

   We introduce following statements.

   
   \[
   P : \text{it is raining} \\
   Q : \text{I will take the bus} \\
   R : \text{I will ride my bicycle.}
   \]

   The statement above can be converted into: $(P \implies Q) \land (\neg P \implies R)$.

2. Given two numbers $x$ and $y$ we define the following statements:

   \[
   P : \ x = 0 \\
   Q : \ y = 0
   \]

   The statement $P \lor Q$ means that at least one of the numbers $x$ and $y$ is equal to 0. We can also say: the statement $P \lor Q$ means that $xy = 0$.

**Problem 6.** Consider the following two sets of natural numbers.

\[
A = \{2x - 1 : x \in \mathbb{N}\} = \{1, 3, 5, 7, 9, \ldots\} \\
B = \{3x : x \in \mathbb{N}\} = \{3, 6, 9, 12, 15, \ldots\}
\]

Give a description of the following four sets. A list of the first **ten** elements followed by \ldots is sufficient.

1. (1 point) $\{x \in \mathbb{N} : (x \in A) \text{ or } (x \in B)\} = \{1, 3, 5, 6, 7, 9, 11, 12, 13, 15, \ldots\}$

2. (2 points) $\{x \in \mathbb{N} : (x \in A) \implies (x \in B)\}$. It may be helpful to use the fact that $S \implies T$ is logically equivalent to $\neg S \lor T$ which can be seen by comparison of the truth tables. Therefore,

   \[
   \{x \in \mathbb{N} : (x \in A) \implies (x \in B)\} = \{x \in \mathbb{N} : (x \notin A) \text{ or } (x \in B)\} \\
   = \{x \in \mathbb{N} : x \text{ is even or } x \text{ is a multiple of } 3\} \\
   = \{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21 \ldots\}
   \]
3. (2 points) We proceed as above:
\[
\{ x \in \mathbb{N} : (x \in B) \implies (x \in A) \} = \{ x \in \mathbb{N} : (x \not\in B) \text{ or } (x \in A) \} \\
= \{ x \in \mathbb{N} : x \text{ is not a multiple of } 3 \text{ or } x \text{ is odd} \} \\
= \{ 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 19, 20 \ldots \}
\]

4. (2 points) From the two previous questions, we get:
\[
\{ x \in \mathbb{N} : (x \in A \iff x \in B) \text{ and } (x \in B \iff x \in A) \} = \{ 2, 3, 4, 8, 9, 10, 14, 15, 16, 20 \ldots \}
\]