Problem 1. We consider subsets $A, B$ and $C$ of the universe $\Omega$.

1. (2 points) Prove that $\overline{A} \subseteq B$ if and only if $A \cup B = \Omega$.

   **Solution 1:** Suppose that $\overline{A} \subseteq B$. We want to deduce that $A \cup B = \Omega$ and it is enough to deduce that $\Omega \subseteq A \cup B$ (since the other inclusion is clear). So let $x \in \Omega$. If $x \in A$ then $x \in A \cup B$. If $x \notin A$, then $x \in \overline{A}$ so $x \in B$ and therefore $x \in A \cup B$. We have proved that $x \in \Omega$ implies $x \in A \cup B$. So $\Omega \subseteq A \cup B$.

   **Solution 2:** Suppose that $A \cup B = \Omega$. Now let $x \in \overline{A}$, namely $x \notin A$. Since $x \in \Omega$ we know that $x$ is in $A$ or in $B$, and since $x \notin A$ we have $x \in B$. Therefore $\overline{A} \subseteq B$.

Using DeMorgan’s law for sets (Book page 165, 8.2).

Suppose that $\overline{A} \subseteq B$. Then

$$A \cup B = \overline{A \cup B} = \overline{A} \cap \overline{B}$$

But $\overline{A} \subseteq B$ so $\overline{A} \cap \overline{B} \subseteq B \cap \overline{B} = \emptyset$ and therefore $\overline{A} \cap \overline{B} = \emptyset$. So $\Omega = A \cup B = \emptyset$.

2. (2 points) Prove that $\overline{A} \subseteq B$ implies $(C \setminus B) \cup A = A$.

   **Solution 1:** Suppose that $\overline{A} \subseteq B$. By the previous question, note that it is equivalent to $\Omega = A \cup B$. But using the first question again, (exchanging the role of $A$ and $B$), we see that $\Omega = A \cup B$ is equivalent to $\overline{B} \subseteq A$. Therefore, $\overline{A} \subseteq B$ is equivalent to $\overline{B} \subseteq A$. Under this hypothesis, we want to check that $(C \setminus B) \cup A = A$. The inclusion $A \subseteq (C \setminus B) \cup A$ is clear. For the other inclusion, let $x \in (C \setminus B) \cup A$. It means that $x \in A$ or $x \in C \setminus B$. If $x \in C \setminus B$ then $x$ lies in $C$ but not in $B$ so in particular $x \in \overline{B}$ but $\overline{B} \subseteq A$ so $x \in A$. Therefore $x \in (C \setminus B) \cup A$ implies $x \in A$.

   **Solution 2:** First recall that $C \setminus B = C \cap \overline{B}$ so $\overline{C \setminus B} = C \cup B$. We have:

$$C \setminus B \cup A = (C \setminus B) \cup A = (C \setminus B) \cap \overline{A} = \overline{(C \setminus B) \cap \overline{A}} = \overline{(C \cup B) \cap \overline{A}} = (C \cap A) \cup (B \cap \overline{A}).$$

Now suppose that $\overline{A} \subseteq B$. Then $B \cap \overline{A} = \overline{A}$ so

$$(C \setminus B) \cup A = \overline{(C \cap A) \cup \overline{A}} = \overline{A} = A.$$

Problem 2. Let $a \in \mathbb{R}$.

1. On the $xy$-plane, draw the set $A_a = \{(x, x^2 - ax), x \in \mathbb{R}\}$ when $a = 0, a = 1$ and $a = 2$.

2. (3 points) Show that $\bigcap_{a \in \mathbb{R}} A_a = \{(0, 0)\}$.

   **Solution:** Recall that for $(x, y) \in \mathbb{R}^2$,

   $$(x, y) \in \bigcap_{a \in \mathbb{R}} A_a$$

   means

   $$\forall a \in \mathbb{R}, (x, y) \in A_a$$

   so in particular, here, it means

   $$\forall a \in \mathbb{R}, y = x^2 - ax.$$
Let \((x, y) \in \bigcap_{a \in \mathbb{R}} A_a\). In particular, we have \((x, y) \in A_0\) therefore \(y = x^2\) and \((x, y) \in A_1\) so \(y = x^2 - x\). Together these conditions imply \(x^2 - x = x^2\) so \(x = 0\) and then \(y = 0\). So \((x, y) = (0, 0)\) and \(\bigcap_{a \in \mathbb{R}} A_a \subseteq \{(0, 0)\}\).

For every \(a \in \mathbb{R}\), we have \(0 = 0^2 - a0\) therefore \((0, 0) \in A_a\). So \((0, 0) \in \bigcap_{a \in \mathbb{R}} A_a\) and \(\{(0, 0)\} \subseteq \bigcap_{a \in \mathbb{R}} A_a\).

**Problem 3.** Chapter 8 Problem 28. **Solution:** We want to prove
\[
\{12a + 25b : a, b \in \mathbb{Z}\} = \mathbb{Z}.
\]

The direct inclusion is clear.

We notice that \(1 = 12(-2) + 25(1)\) therefore for any \(x \in \mathbb{R}\) we have \(x = 12(-2x) + 25x\) and therefore \(x \in \{12a + 25b : a, b \in \mathbb{Z}\}\).

**Problem 4.** (3 points) Prove
\[
\bigcup_{n \in \mathbb{N}} \left[\frac{1}{n}, 1\right] = (0, 1].
\]

We admit: for every \(x \in \mathbb{R}\), there exists a unique \(m \in \mathbb{Z}\) such that \(x \in [m, m + 1]\). See HW5 Problem 6.

**Solution:** Recall that for \(x \in \mathbb{R}\),
\[
x \in \bigcup_{n \in \mathbb{N}} \left[\frac{1}{n}, 1\right]
\]
means
\[
\exists n \in \mathbb{N}, \text{ such that } x \in \left[\frac{1}{n}, 1\right].
\]

Let \(x \in \bigcup_{n \in \mathbb{N}} \left[\frac{1}{n}, 1\right]\). There exists \(n \in \mathbb{N}\) such that \(\frac{1}{n} \leq x \leq 1\) therefore \(0 < x \leq 1\) and \(x \in (0, 1]\).

Let \(x \in (0, 1]\). We know that there exists a unique \(m \in \mathbb{Z}\) such that \(\frac{1}{x} \in [m, m + 1]\). Notice that since \(\frac{1}{x} > 0\) we know that \(m \geq 0\) (otherwise we would have \(\frac{1}{x} < m + 1 \leq 0\) and \(\frac{1}{x} < 0\) which is not true). So \(m + 1 \in \mathbb{N}\) and \(\frac{1}{x} < m + 1\) implies \(x > \frac{1}{m + 1}\). So we have \(x \in (\frac{1}{m + 1}, 1]\) which implies \(x \in \left[\frac{1}{m + 1}, 1\right]\).

**Problem 5.** Let \(A, B\) and \(C\) be sets. For each of the following statements, either prove it is true or give a counterexample.

1. \(\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)\) **Solution:** It is false. Take \(A = \{1\}\) and \(B = \{2\}\). In that case \(A \cup B \in \mathcal{P}(A \cup B)\) but \(A \cup B \notin \mathcal{P}(A) \cup \mathcal{P}(B)\).

2. \(\mathcal{P}(A \cup B) \supseteq \mathcal{P}(A) \cup \mathcal{P}(B)\). **Solution:** It is true. Let \(X \in \mathcal{P}(A) \cup \mathcal{P}(B)\). It means \(X \subseteq A\) or \(X \subseteq B\). But \(A \subseteq A \cup B\) and \(B \subseteq A \cup B\). So in any case, \(X \subseteq A \cup B\) and \(X \in \mathcal{P}(A \cup B)\).

3. \(\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)\). **Solution:** It is true. Let \(X \in \mathcal{P}(A \cap B)\). It means \(X \subseteq A \cap B\) so in particular, \(X \subseteq A\) and \(X \subseteq B\) so \(X \in \mathcal{P}(A)\) and \(X \in \mathcal{P}(B)\).
4. $\mathcal{P}(A \cap B) \supseteq \mathcal{P}(A) \cap \mathcal{P}(B)$. **Solution**: It is true. Let $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$. It means $X \subseteq A$ and $X \subseteq B$ so $X \subseteq A \cap B$ and $X \in \mathcal{P}(A \cap B)$.

5. (1 point) $((A \cup B) \setminus C) \cup (A \cap B \cap C) \subseteq (A \setminus (B \cup C)) \cup (B \setminus (A \cup C))$. **Solution**: It is false (draw a Venn diagram to come up with a counter example). $A = \{1, 2\} \quad B = \{1, 3\} \quad C = \{1, 4\}$. We have $1 \in (A \cup B) \setminus C) \cup (A \cap B \cap C)$ but $(A \setminus (B \cup C)) = \{2\}$ and $(B \setminus (A \cup C)) = \{3\}$ so

$1 \notin (A \setminus (B \cup C)) \cup (B \setminus (A \cup C))$.

**Problem 6.** For the following relations, decide if they are equivalence relations or not and justify your answer.

1. (1 point) On the set of all lines in the $xy$ plane, we define the relation $R$ by: $DRD'$ if $D$ and $D'$ are orthogonal. **Solution**: It is not an equivalence relation as it is not reflexive.

2. (3 points) On $\mathbb{R}$, we define the relation $R$ by: $xRy$ if $\cos^2(x) + \sin^2(y) = 1$.

**Solution**: It is an equivalence relation:

- Reflexivity: for $x \in \mathbb{R}$, we have $\cos^2(x) + \sin^2(x) = 1$ so $xRx$.
- Symmetry: for $x, y \in \mathbb{R}$, suppose that $xRy$ namely $\cos^2(x) + \sin^2(y) = 1$. Then $\cos^2(y) + \sin^2(x) = (1 - \sin^2(y)) + (1 - \cos^2(x)) = 2 - (\cos^2(x) + \sin^2(y)) = 2 - 1 = 1$ so $yRx$.
- Transitivity: for $x, y, z \in \mathbb{R}$, suppose that $xRy$ and $yRz$ namely $\cos^2(x) + \sin^2(y) = 1$ and $\cos^2(y) + \sin^2(z) = 1$. Then

\[
\cos^2(x) + \sin^2(z) = \cos^2(x) + \sin^2(y) - \sin^2(y) + \sin^2(z) \\
= \cos^2(x) + \sin^2(y) - (1 - \cos^2(y)) + \sin^2(z) \\
= (\cos^2(x) + \sin^2(y)) + (\cos^2(y) + \sin^2(z)) - 1 = 1 + 1 - 1 = 1
\]

so $xRz$.

**Problem 7.** Let $R$ be the relation defined on $\mathbb{R} \times \mathbb{R}$ by

$$(x_1, y_1)R(x_2, y_2) \quad \text{if} \quad x_1^2 + y_1^2 = x_2^2 + y_2^2.$$  

1. Check that it is an equivalence relation. This should be straightforward.

2. (1 point) Describe the equivalence classes. You can make a drawing. The equivalence classes are the circles with center 0 in the $xy$-plane.

**Problem 8.** Let $D$ denote the set of all lines in the $xy$-plane which are parallel (or identical) to the line with equation $y = 3x$. Each such line is treated as a subset of $\mathbb{R} \times \mathbb{R}$.

1. Check for yourself that $D$ is a partition of $\mathbb{R} \times \mathbb{R}$. (Not be to handed in).

2. (1 point) Give an equivalence relation $\mathcal{R}$ on $\mathbb{R} \times \mathbb{R}$ whose set of equivalence classes is $D$. It is the relation $\mathcal{R}$ defined on $\mathbb{R} \times \mathbb{R}$ by

$$(x_1, y_1) \mathcal{R} (x_2, y_2) \quad \text{if} \quad y_2 - y_1 = 3(x_2 - x_1)$$