Problem 1. Prove the following statement:
For every $a \in \mathbb{Z}$ such that 5 does not divide $a$, there exists $b \in \mathbb{Z}$ such that $ab \equiv 1 \mod 5$.

Problem 2. We recall that given $a, b \in \mathbb{Z}$ such that $ab \neq 0$, we define the gcd of $a$ and $b$ to be the greatest integer that divides both $a$ and $b$. We denote it by $\gcd(a, b)$.

1. Let $a, b \in \mathbb{Z}$ such that $ab \neq 0$. We suppose that there exists $u, v \in \mathbb{Z}$ such that
   \[ 1 = au + bv. \]
   Prove that $\gcd(a, b) = 1$.

2. Write the negation of the following statement. Then decide if it is true or false and justify your answer.
   For every $a \in \mathbb{Z}$, $b \in \mathbb{Z}$, and $d \in \mathbb{Z}$ such that $d > 0$,
   (if there exists $u, v \in \mathbb{Z}$ such that $d = au + bv$, then $\gcd(a, b) = d$).

Problem 3. Read the Division Algorithm theorem carefully in your book (Fact 1.5 page 30 in the third edition). What is the division of

1. 167 by 8?
2. $-167$ by 8?

Problem 4. Prove the following statement:
for every $n \in \mathbb{Z}$ such that $n \geq 0$, we have $5 \mid 3^{3n+1} + 2^{n+1}$.

Problem 5. For each of the following statements:

- Negate the statement,
- Decide if the statement (prior to negation) is true or false and justify your answer.

1. $\forall a \in \mathbb{Z}$, if $6 \mid a$ and $8 \mid a$ then $48 \mid a$.
2. $\forall x \in \mathbb{R}$, $\forall y \in \mathbb{R}$, if $xy \geq 0$ then $x + y \geq 0$.
3. For every house $h$ in Vancouver, there is a positive integer $c$ such that (if the number of cats in $h$ is $> c$, then there is no mouse in the house).

Problem 6. Explain what is wrong in the following proof and write a correct version of it.
For any real number $b$ such that $1 < b < 5$, prove that $b^3 - 6b^2 + 5b < 0$.

Proof. Suppose that $1 < b < 5$.
$b^3 - 6b^2 + 5b = b(b^2 - 6b + 5) < 0$ so, since $b > 0$ we have $b^2 - 6b + 5 < 0$. But we notice that $b^2 - 6b + 5 = (b-1)(b-5)$. Since $1 < b < 5$, we know that $b-1 > 0$ and $b-5 < 0$ so indeed $b^2 - 6b + 5 = (b-1)(b-5) < 0$. 

\[ \square \]