Problem 1. 1. (1 point mostly for presentation since these things are in the book). Given two statements $S$ and $T$:

(a) (De Morgan’s Laws) See Textbook 2.6. From the truth tables (you had to give them), one sees that statements $\neg(S \land T)$ and $(\neg S) \lor (\neg T)$ are logically equivalent.

(b) After giving the truth table, one observes that $\neg(S \implies T)$ is logically equivalent to $S \land \neg T$.

2. Consider the following statement: If it is raining then I will take the bus, and otherwise I will ride my bicycle.

(a) (2 points) We introduce following statements.

\[
\begin{align*}
P &: \text{it is raining} \\
Q &: \text{I will take the bus} \\
R &: \text{I will ride my bicycle.}
\end{align*}
\]

The statement above can be converted into: $(P \implies Q) \land (\neg P \implies R)$.

(b) Negation of the above statement:

i. (2 points) with symbols. Using Question 1 (a), we know that $\neg[(P \implies Q) \land (\neg P \implies R)]$ is logically equivalent to

$$(\neg P \implies Q) \lor (\neg(\neg P \implies R)).$$

Using Question 1 (b), we know that $\neg(P \implies Q)$ is logically equivalent to $P \land \neg Q$, and $\neg(\neg P \implies R)$ is logically equivalent to $\neg P \land \neg R$. So $\neg[(P \implies Q) \land (\neg P \implies R)]$ is logically equivalent to

$$(P \land \neg Q) \lor (\neg P \land \neg R).$$

ii. (2 points) in plain English. It rains and I do not take the bus, or it does not rain and I do not ride my bicycle.

Problem 2. Consider the following two sets of natural numbers.

\[
A = \{2x - 1 : x \in \mathbb{N}\} = \{1, 3, 5, 7, 9, \ldots\}
\]

\[
B = \{3x : x \in \mathbb{N}\} = \{3, 6, 9, 12, 15, \ldots\}
\]

Give a description of the following two sets. A list of the first ten elements followed by \ldots is sufficient.

1. (1 point) $\{x \in \mathbb{N} : (x \in A) \text{ or } (x \in B)\} = \{1, 3, 5, 6, 7, 9, 11, 12, 13, 15, \ldots\}$

2. (2 points) $\{x \in \mathbb{N} : (x \in A) \implies (x \in B)\}$. It may be helpful to use the fact that $S \implies T$ is logically equivalent to $\neg S \lor T$ which can be seen by comparison of the truth tables. Therefore,

\[
\begin{align*}
\{x \in \mathbb{N} : (x \in A) \implies (x \in B)\}
&= \{x \in \mathbb{N} : (x \notin A) \text{ or } (x \in B)\} \\
&= \{x \in \mathbb{N} : x \text{ is even or } x \text{ is a multiple of 3}\} \\
&= \{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21 \ldots\}
\end{align*}
\]
3. (2 points) We proceed as above:

\[ \{ x \in \mathbb{N} : (x \in B) \implies (x \in A) \} = \{ x \in \mathbb{N} : (x \not\in B) \text{ or } (x \in A) \} = \{ x \in \mathbb{N} : x \text{ is not a multiple of } 3 \text{ or } x \text{ is odd} \} = \{ 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 19, 20 \ldots \} \]

4. (2 points) From the two previous questions, we get:

\[ \{ x \in \mathbb{N} : (x \in A \implies x \in B) \text{ and } (x \in B \implies x \in A) \} = \{ 2, 3, 4, 8, 9, 10, 14, 15, 16 \ldots \} \]

**Problem 3.** (4 points) Let \( x \in \mathbb{R} \). Suppose that \( |x| > 10 \). Then

- either \( x > 10 \) and then \( x - 10 > 0 \) and \( x - 4 > 0 \), therefore \( (x - 10)(x - 4) > 0 \). But \( (x - 10)(x - 4) = x^2 - 14x + 40 \) so we have proved that \( x^2 + 40 > 14x \).
- or \( x < -10 \) and then \( 14x < 0 \) while \( x^2 + 40 > 0 \). So \( 14x < 0 < x^2 + 40 \) and therefore \( x^2 + 40 > 14x \).

**Problem 4.** Let \( a, b, \) and \( c \) be integers. Consider the statements:

\[ P: c \text{ divides } ab \]
\[ Q: c \text{ divides } a \]
\[ R: c \text{ divides } b \]

1. (2 points) Write the statement \( P \implies Q \lor R \) in words: \( c \) divides \( ab \) implies that \( c \) divides \( a \) or \( c \) divides \( b \).

2. (1 point) Give an example of integers \( a, b \) and \( c \) for which the statement in part 1. is false. When \( c = 6 \), \( a = 2 \) and \( b = 3 \) we have \( c \) divides \( ab \) and yet \( c \) does not divide \( a \) and \( c \) does not divide \( b \).

**Problem 5.** (4 points) Let \( n \in \mathbb{Z} \). We want to prove:

If \( 4 \) divides \( n - 1 \), then \( n \) is odd and \((-1)^{\frac{n-1}{2}} = 1\).

Suppose that \( 4 \) divides \( n - 1 \). It means that there is \( m \in \mathbb{Z} \) such that \( n - 1 = 4m \). Therefore we have: \( n = 4m + 1 = 2(2m) + 1 \) is an odd number. Furthermore \( \frac{n-1}{2} = 2m \) is an even number. This implies \((-1)^{\frac{n-1}{2}} = (-1)^{2m} = ((-1)^2)^m = 1\).

**Problem 6.** 1. (3 points) Let \( n \in \mathbb{Z} \). Prove that if \( 5n \) is even then \( n \) is even.

- By direct proof. Suppose that \( 5n \) is even. It means that there is \( m \in \mathbb{Z} \) such that \( 5n = 2m \). Then \( n = 5n - 4n = 2m - 4n = 2(m - 2n) \). Since \( m - 2n \in \mathbb{Z} \), this proves that \( n \) is an even number.
• By contrapositive. We are going to prove that
if \( n \) is not even, then \( 5n \) is not even

namely

if \( n \) is odd then \( 5n \) is odd.

Suppose that \( n \) is odd. It means that there is \( m \in \mathbb{Z} \) such that \( n = 2m + 1 \). Then
\[
5n = 5(2m + 1) = 10m + 5 = 2(5m + 2) + 1.
\]
Since \( 5m + 2 \in \mathbb{Z} \), this proves that \( 5n \) is odd.

2. (3 points) Let \( n \in \mathbb{Z} \). Prove that if 5 divides \( n \) and 2 divides \( n \), then 10 divides \( n \).

Suppose that 5 divides \( n \) and 2 divides \( n \). The first hypothesis means that there is \( m \in \mathbb{Z} \) such that \( n = 5m \). The second hypothesis means that \( n \) is even, namely that \( 5m \) is even. We know by the previous question that "5m is even" implies that "\( m \) is even" namely there is \( k \in \mathbb{Z} \) such that \( m = 2k \). So we have \( n = 5m = 10k \) with \( k \in \mathbb{Z} \). Therefore 10 divides \( n \).

3. (2 points) Is the following statement true?

For \( n \in \mathbb{Z} \), if 6 divides \( n \) and 2 divides \( n \), then 12 divides \( n \).

This statement means that for all possible \( n \in \mathbb{Z} \), if 6 and 2 both divide \( n \) then necessarily 12 divides \( n \). If we pick \( n = 6 \), then we do have "6 and 2 both divide \( n \" and yet "12 does not divide \( n \". Therefore the statement is false because it fails for at least one \( n \in \mathbb{Z} \) (namely \( n = 6 \)).

You were not asked to provide as many details as what follows. But please read the discussion because later we will work on negating statements and it may be a challenging topic. So this is a good preparation.

More formally, for \( n \in \mathbb{Z} \), we introduce the statements:

\[
P : 6 \text{ divides } n
\]
\[
Q : 2 \text{ divides } n
\]
\[
R : 12 \text{ divides } n.
\]

We are working on the statement:

For any/all \( n \in \mathbb{Z} \), we have \((P \land Q) \implies R\).

We wonder: is it true that for all possible \( n \in \mathbb{Z} \) we have \((P \land Q) \implies R\)?

To prove that it is false, we just need to provide one example of \( n \in \mathbb{Z} \) for which

• we do not have \((P \land Q) \implies R\), namely
• for which we have \(\neg[(P \land Q) \implies R]\), namely
• for which we have \((P \land Q) \land \neg R\).

This is what we did above: for \( n = 6 \), the statements \( P \) and \( Q \) are both true so \((P \land Q)\) is true, and yet \( R \) is not true. So for \( n = 6 \) we have \((P \land Q) \land \neg R\).