Homework 10

1. Prove that the set of irrational numbers is uncountable. You may assume the fact that the set of real numbers is uncountable.

2. Prove or disprove: If \( A \subseteq B \subseteq C \) and \( A \) and \( C \) are countably infinite, then \( B \) is countably infinite.

3. Prove or disprove: There exists a bijective function \( f : \mathbb{Q} \to \mathbb{R} \).

4. Prove or disprove: The set \( \mathbb{Z} \times \mathbb{Q} \) is countably infinite.

5. For every \( n \in \mathbb{N} \), define a set \( F_n \subset \mathcal{P}(\mathbb{N}) \) by

\[
F_n = \{ \{a_1, a_2, a_3, \ldots, a_n\} : a_i \in \mathbb{N} \text{ for } i \in \{1, 2, \ldots, n\} \} \subseteq \mathcal{P}(\mathbb{N}).
\]

Prove or disprove that for every \( n \in \mathbb{N} \), \(|F_n| = |\mathbb{N}|\).

6. Prove or disprove: The set \( \{(a_1, a_2, a_3, \ldots) : a_i \in \{0, 1\}\} \) of infinite sequences of 0’s and 1’s is countably infinite.

7. Suppose \( A = \{(m, n) \in \mathbb{N} \times \mathbb{R} : n = \pi m\} \). Is it true that \(|\mathbb{N}| = |A|\)?

8. Show that the two given sets have equal cardinality by describing a bijection from one to the other. Describe your bijection with a formula (not as a table).

(a) The set of even integers and the set of odd integers
(b) \( \mathbb{Z} \) and \( S = \{x \in \mathbb{R} : \sin x = 1\} \)
(c) \( \{0, 1\} \times \mathbb{N} \) and \( \mathbb{Z} \)
(d) \( \mathbb{R} \) and \( (\sqrt{2}, \infty) \)

These questions are not to turn in, but they are still interesting and we encourage you to do them as well.

9. (a) Prove that the set \( S = \{\sqrt{q} : n \in \mathbb{N}, q \in \mathbb{Q}_{\geq 0}\} \) is countable.

(b) For each \( m \in \mathbb{N} \), define the set

\[
T_m = \{s_1 + s_2 + \ldots + s_m : s_1, s_2, \ldots, s_m \in S\}.
\]

Prove that \( T_m \) is countable.

(c) Prove that the set of all numbers formed by finite sums of elements of \( S \), is countable. That is, prove that \( T_1 \cup T_2 \cup T_3 \cup \cdots \) is countable.\(^1\)

10. Describe a partition of \( \mathbb{N} \) that divides \( \mathbb{N} \) into \( \aleph_0 \) countably infinite subsets.

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\(^1\)This shows that most irrational numbers cannot be built from taking \( n \)-th roots of rational numbers.