1. Prove that the set of irrational numbers is uncountable. You may assume the fact that the set of real numbers is uncountable.

Solution 1 (Prove) Assume for a contradiction that the set of irrational numbers, call $\mathbb{I}$, is countable. We know $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$. Thus, since $\mathbb{Q}$ is countable and $\mathbb{I}$ is countable, $\mathbb{R}$ is countable, which is a contradiction. Therefore the set of irrationals is uncountable.

2. Prove or disprove: If $A \subseteq B \subseteq C$ and $A$ and $C$ are countably infinite, then $B$ is countably infinite.

Solution 2 (Prove) We have seen that any subset of a countable set is countable. Thus, since $C$ is countable we know that $B$ is countable. Moreover, since $A$ is infinite, we see that $B$ is infinite. Hence $B$ countably infinite.

3. Prove or disprove: There exists a bijective function $f : \mathbb{Q} \to \mathbb{R}$.

Solution 3 (Disprove) Assume for a contradiction that there is a bijection from the set of rational numbers to the set of real numbers, call $f$. Since we know that there is a bijection from the set of rational numbers to the set of natural numbers, call $g$, we see there is a bijection from the set of natural numbers to the set of real numbers, namely, $f \circ g^{-1}$, which is a contradiction since we showed that there can be no surjection from the set of natural numbers to the set of real numbers.

4. Prove or disprove: The set $\mathbb{Z} \times \mathbb{Q}$ is countably infinite.

Solution 4 (Prove) This is a direct consequence of a theorem we have seen in class. We have seen that the cartesian product of two countable set is countable, and since $\mathbb{Z}$ and $\mathbb{Q}$ are both countable, we see that $\mathbb{Z} \times \mathbb{Q}$ is countable.

5. For every $n \in \mathbb{N}$, define a set $F_n \subset \mathcal{P}(\mathbb{N})$ by

$$F_n = \{\{a_1, a_2, a_3, ..., a_n\} : a_i \in \mathbb{N} \text{ for } i \in \{1, 2, ..., n\}\} \subseteq \mathcal{P}(\mathbb{N}).$$

Prove or disprove that for every $n \in \mathbb{N}$, $|F_n| = |\mathbb{N}|$.

Solution 5 (Prove). Let $n \in \mathbb{N}$. We see that the set $F_n$ is the set of all subsets of $\mathbb{N}$ of size $n$. We see that $F_n$ is an infinite set. This means that we need to show that $F_n$ is countable. To show that, it is enough to show that there is an injection from $F_n$ to a countable set. We can define such injection as follows:

$$f : F_n \to \mathbb{N}^n,$$

$$f(\{a_1, a_2, a_3, \cdots, a_n\}) = (a_{k_1}, a_{k_2}, a_{k_3}, \cdots, a_{k_n}),$$

where $\{a_1, a_2, a_3, \cdots, a_n\} = \{a_{k_1}, a_{k_2}, a_{k_3}, \cdots, a_{k_n}\}$ and $a_{k_1} < a_{k_2} < a_{k_3} < \cdots < a_{k_n}$. We see that $f$ is a well-defined function since if $\{a_1, a_2, a_3, \cdots, a_n\} = \{b_1, b_2, b_3, \cdots, b_n\}$, then $(a_{k_1}, a_{k_2}, a_{k_3}, \cdots, a_{k_n}) = (b_{k_1}, b_{k_2}, b_{k_3}, \cdots, b_{k_n})$, that is, the order of in ascending order of elements is the same if the sets are the same.

Moreover, if $f(\{a_1, a_2, a_3, \cdots, a_n\}) = f(\{b_1, b_2, b_3, \cdots, b_n\})$, then
Let \((a_{k1}, a_{k2}, a_{k3}, \ldots, a_{kn}) = (b_{k1}, b_{k2}, b_{k3}, \ldots, b_{kn})\) and since \\
\(\{a_1, a_2, a_3, \ldots, a_n\} = \{a_{k1}, a_{k2}, a_{k3}, \ldots, a_{kn}\}\) and \\
\(\{b_1, b_2, b_3, \ldots, b_n\} = \{b_{k1}, b_{k2}, b_{k3}, \ldots, b_{kn}\}\) we see \\
\(\{a_1, a_2, a_3, \ldots, a_n\} = \{b_1, b_2, b_3, \ldots, b_n\}\), meaning that \(f\) is injective. \\
We have also seen in class that the finite cartesian product of countable sets is countable. \\
Thus, we see \(\mathbb{N}^n\) is countable. Therefore since \(f\) is an injection from \(F_n\) to \(\mathbb{N}^n\) and since \(\mathbb{N}^n\) is countable, we see \(F_n\) is countable.

6. Prove or disprove: The set \(\{(a_1, a_2, a_3, \ldots) : a_i \in \{0, 1\}\}\) of infinite sequences of 0’s and 1’s is countably infinite.

**Solution 6** (Disprove) We are going to show that this set, call \(A\), is uncountable by finding a bijection from \(A\) to \(\mathcal{P}(\mathbb{N})\).

We can define the bijection \(f\) as follows:

\[
f : A \to \mathcal{P}(\mathbb{N})
\]

\[
f(a_1, a_2, a_3, \ldots) = \{k \in \mathbb{N} : a_k = 1\}.
\]

c.g. \(f(0, 1, 0, 1, 0, 0, 0, 0, \ldots) = \{2, 4\}\). We see that \(f\) is a surjective since \(\forall X \subseteq \mathbb{N}\), we can construct the sequence \((x_1, x_2, x_3, \ldots)\), where \(x_k = 1\) if \(k \in \mathbb{N}\). Then, \(f(x_1, x_2, x_3, \ldots) = X\). We also see that \(f\) is injective since if \((a_1, a_2, a_3, \ldots) \neq (b_1, b_2, b_3, \ldots)\), then \(\exists m \in \mathbb{N}\) such that \(a_m = 1\) and \(b_m = 0\), in which case \(m \notin f(a_1, a_2, a_3, \ldots)\) and \(m \notin f(b_1, b_2, b_3, \ldots)\), or \(a_m = 0\) and \(b_m = 1\), in which case \(m \notin f(a_1, a_2, a_3, \ldots)\) and \(m \notin f(b_1, b_2, b_3, \ldots)\). In both cases, we see \(f(a_1, a_2, a_3, \ldots) \neq f(b_1, b_2, b_3, \ldots)\). Therefore we see that \(|A| = |\mathcal{P}(\mathbb{N})|\), which means \(A\) is uncountable.

**Soln #2 6** (Disprove) Let \(f\) be any function from \(\mathbb{N}\) to this set. Write

\[
f(1) = (a_{11}, a_{12}, a_{13}, \ldots)
\]

\[
f(2) = (a_{21}, a_{22}, a_{23}, \ldots)
\]

\[
f(3) = (a_{31}, a_{32}, a_{33}, \ldots)
\]

\[
\vdots
\]

Let \(b = (b_1, b_2, b_3, \ldots)\) where \(b_i = \begin{cases} 0 & \text{if } a_{ii} = 1 \\ 1 & \text{if } a_{ii} = 0 \end{cases}\)

Then \(b \in \{(a_1, a_2, a_3, \ldots) : a_i \in \{0, 1\}\}\). However, since \(b_i \neq a_{ii}\) for each \(i \in \mathbb{N}\), we have \(b \neq f(i)\) for each \(i \in \mathbb{N}\). Thus, \(f\) is not surjective. This shows there are no surjective functions from \(\mathbb{N}\) to this set, hence they do not have the same cardinality.

7. Suppose \(A = \{(m, n) \in \mathbb{N} \times \mathbb{R} : n = \pi m\}\). Is it true that \(|\mathbb{N}| = |A|\)?

**Solution 7** It is true that \(|A| = |\mathbb{N}|\), since we can define the function \(f : \mathbb{N} \to A, f(k) = (k, k\pi)\) for all \(k \in \mathbb{N}\). By the definition of \(A\) we see that \(f\) is injective and surjective. Therefore \(|A| = |\mathbb{N}|\).

8. Show that the two given sets have equal cardinality by describing a bijection from one to the other. Describe your bijection with a formula (not as a table).
(a) The set of even integers and the set of odd integers

Solution 8a Let $O$ denote the set of odd integers and $E$ denote the set of even integers. Then we can define the function $f : O \rightarrow E$ as $f(x) = x + 1$. We see that $f$ is injective and also for any even integer $y$, we see $y - 1$ is odd and $f(y - 1) = y$, which implies that $f$ is surjective. Therefore $f$ is bijective.

(b) $\mathbb{Z}$ and $S = \{x \in \mathbb{R} : \sin x = 1\}$

Solution 8b We see that $S = \{x \in \mathbb{R} : \sin x = 1\} = \{x \in \mathbb{R} : x = \frac{\pi}{2} + 2\pi n$ for some $n \in \mathbb{Z}\}$. Thus we can define a bijection $f : \mathbb{Z} \rightarrow S$, where $f(k) = \frac{\pi}{2} + 2\pi k$. Then, we see that $f$ is injective and, by definition of $S$, we see it is surjective too. Hence $f$ is bijective.

(c) $\{0, 1\} \times \mathbb{N}$ and $\mathbb{Z}$

Solution 8c For this question we can define the function $f : \mathbb{Z} \rightarrow (0, 1) \times \mathbb{N},$

$$f(k) = \begin{cases} (1, k + 1) & \text{if } k \geq 0 \\ (0, -k) & \text{if } k \leq -1 \end{cases}$$

Then we see that for $k, m \in \mathbb{Z}$, if $f(k) = f(m)$, then either $k, m \leq -1$ or $k, m \geq 0$. If $k, m \geq 0$ we have $(1, k + 1) = (1, m + 1)$ in which case $k = m$. If $k, m \leq -1$ we have $(0, -k) = (0, -m)$, in which case $k = m$ too. We also see that these two are the only two cases since if $k \geq 0$ and $m \leq -1$ or $m \geq 0$ and $k \leq -1$ we see $f(k) \neq f(m)$.

We also see that $f$ is surjective. If $(a, n) \in \mathbb{N}$ we have two cases: $a = 0$ or $a = 1$. If $a = 0$, we can define $k = -n$ so that $f(-k) = (0, n)$ and if $a = 1$, we can define $k = n - 1$ so that $f(k) = (1, n)$.

(d) $\mathbb{R}$ and $(\sqrt{2}, \infty)$

Solution 8d We know that the function $exp : \mathbb{R} \rightarrow (0, \infty)$ is a bijective function. Using this function we can define a bijection $f : \mathbb{R} \rightarrow (\sqrt{2}, \infty)$ as $f(x) = e^x + \sqrt{2}$. We see that $f$ is injective since exp is injective and it is surjective since exp is surjective from $\mathbb{R}$ to $(0, \infty)$.

These questions are not to turn in, but they are still interesting and we encourage you to do them as well.

9. (a) Prove that the set $S = \{\sqrt{q} : n \in \mathbb{N}, q \in \mathbb{Q}_{\geq 0}\}$ is countable.

(b) For each $m \in \mathbb{N}$, define the set

$$T_m = \{s_1 + s_2 + \ldots + s_m : s_1, s_2, \ldots, s_m \in S\}.$$

Prove that $T_m$ is countable.

(c) Prove that the set of all numbers formed by finite sums of elements of $S$, is countable. That is, prove that $T_1 \cup T_2 \cup T_3 \cup \cdots$ is countable.\(^1\)

Solution 9 In this problem we are going to use the lemma which we are going to prove at the end of this problem:

\(^1\)This shows that most irrational numbers cannot be built from taking $n$-th roots of rational numbers.
Lemma: Let $A_i$ be a denumerable (countably infinite) set $\forall i \in \mathbb{N}$ and moreover let the sets $A_i$ be mutually disjoint. Then $\bigcup_{n \in \mathbb{N}} A_n$ is denumerable.

First observe that $\mathbb{Q}_{\geq 0}$ is a countable set, and thus, we can list all the elements of it as $\mathbb{Q}_{\geq 0} = \{q_1, q_2, q_3, \ldots \}$. Then we see

\[ S = \bigcup_{n \in \mathbb{N}} S_n, \]

where $S_n = \{ \sqrt[2]{q} : q \in \mathbb{Q}_{\geq 0} \}$. These sets $S_n$ are denumerable for all $n \in \mathbb{N}$, but are not disjoint. To use the lemma, we can define the sets $B_1 = S_1$ and $B_n = S_n - \bigcup_{k=1}^{n-1} S_k$. Then we see $S = \bigcup_{n \in \mathbb{N}} B_n$. Then, by the lemma above, we can conclude that $S$ is countable. Then, since the cartesian product of finitely many countable sets is countable, we can also conclude that $S^k$ is also countable $\forall k \in \mathbb{N}$.

For part b), given $m \in \mathbb{N}$ we can define the function

\[ g_m : S^m \rightarrow T_m \]

\[ g(s_1, s_2, s_3, \ldots, s_m) = s_1 + s_2 + s_3 + \cdots + s_m. \]

By definition, we see that $g$ is a surjection. Thus, since $g$ is a surjection from a countable set $S^m$ to $T_m$, we see that $T_m$ is also countable $\forall m \in \mathbb{N}$.

Thus, similar to the reasoning above, since countable union of countable sets is countable, we get $\bigcup_{n \in \mathbb{N}} T_n$ is countable.

We still have to prove the lemma.

Proof of Lemma: We see that if we have disjoint denumerable sets $A_n$, $n \in \mathbb{N}$, then we can list the elements of these sets in infinite lists.

\[ A_1 = \{a_{11}, a_{12}, a_{13}, \ldots \}, \]
\[ A_2 = \{a_{21}, a_{22}, a_{23}, \ldots \}, \]
\[ A_3 = \{a_{31}, a_{32}, a_{33}, \ldots \}, \]

\[ \vdots \]

Then, we can list the elements of $\bigcup_{n \in \mathbb{N}} A_n$ along the diagonals of the lists above as $\bigcup_{n \in \mathbb{N}} A_n = \{a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, \ldots \}$. Therefore $\bigcup_{n \in \mathbb{N}} A_n$ is countable. \(\square\)

10. Describe a partition of $\mathbb{N}$ that divides $\mathbb{N}$ into $\aleph_0$ countably infinite subsets.

Solution 10 Here we can use the fact that the set of prime numbers is countably infinite. Let $P$ be the set of prime numbers. We have seen that $P$ is countably infinite. The we can list the elements of $P$ as $\{p_1, p_2, p_3, \ldots \}$. then we can define the sets

\[ A_i = \{p_i^n : n \in \mathbb{N} \}, \forall i \in \mathbb{N}. \]

We see by definition that $|A_i| = |\mathbb{N}|, \forall i \in \mathbb{N}$.

Moreover consider the set:
\[ A_0 = \{1\} \cup \{n \in \mathbb{N} : \exists p_i, p_j \in P, \text{ s.t. } p_i \neq p_j, \text{ and } p_i \mid n \text{ and } p_j \mid n\}. \] then we see that \( A_0 \) is also countably infinite (since, say, it contains all the multiples 6 and still a subset of the set of natural numbers).

We also observe that there are countable infinitely many \( A_k \)'s, which are all countably infinite themselves and \( \bigcup_{n \geq 0} A_n = \mathbb{N} \).

**Alternative:** We know that \( \mathbb{N} \times \mathbb{N} \) is countably infinite (Thm 14.5), namely that there exists a bijection \( f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \). For \( x \in \mathbb{N} \), consider the set

\[ A_x = f(\{x\} \times \mathbb{N}). \]

The collection of all \( A_x \) for \( x \in \mathbb{N} \) forms a partition as required. In fact we even studied such a bijection in Problem 6 HW9. With this example, for \( x \in \mathbb{N} \), the set \( A_x \) is

\[ A_x = \{2^{x-1}(2y - 1), y \in \mathbb{N}\} \]

(namely \( A_x \) is the set of natural integers which can be written as a product of \( 2^{x-1} \) by an odd natural number.).