Solutions to homework 1:

Problem 1 (1 point). \(\{x \in \mathbb{Z} : |2x| < 5\} = \{-2, -1, 0, 1, 2\}\).

Problem 2 (2 points).
\[
\{5x \in \mathbb{Z} : x \in \mathbb{Z}, |2x| \leq 8\} = \{5 \times (-4), 5 \times (-3), 5 \times (-2), 5 \times (-1), 5 \times 0, 5 \times 1, 5 \times 2, 5 \times 3, 5 \times 4\}
\]
\[
= \{-20, -15, -10, -5, 0, 5, 10, 15, 20\}.
\]

Problem 3.  

a) (2 points) \(A = \{1, 6, 11, 16, 21, \ldots\}\) can be written as
\[
A = \{5x + 1 : x \in \mathbb{Z}, x \geq 0\}.
\]
It can also be written as
\[
A = \{5x - 4 : x \in \mathbb{N}\}.
\]

b) (2 points) \(B = \{-\frac{1}{16}, -\frac{1}{8}, -\frac{1}{4}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}\}\). The set \(B\) can be written as
\[
B = \{x \in \mathbb{Q} : \text{there is } y \in \{1, 2, 3, 4\} \text{ such that } |x| = \frac{1}{2^y}\}.
\]
Alternatively, we will see later that one can write
\[
B = \left\{ \frac{1}{2^y} : y \in \{1, 2, 3, 4\} \right\} \cup \left\{ -\frac{1}{2^y} : y \in \{1, 2, 3, 4\} \right\}.
\]
There are many other ways to write it. For example,
\[
B = \left\{ \frac{1}{x} : \text{there is } y \in \{1, 2, 3, 4\} \text{ such that } (x - 2^y)(x + 2^y) = 0 \right\}.
\]
and lastly as several of you ingeniously suggested:
\[
B = \left\{ \frac{|x|}{x^2} : x \in \mathbb{Z}, 1 \leq |x| \leq 4 \right\}.
\]

Problem 4 (3 points). Is the emptyset \(\emptyset\) an element of the following sets:

a) \(C = \{\emptyset\}\). NO. Here the set \(C\) contains only one element and this element is the set containing the emptyset.

b) \(D = \{\emptyset, \emptyset\}\). YES
c) \( E = \{\{\emptyset\}\}, \{\emptyset\} \). NO.

**Problem 5** (4 points). For \( a \in \mathbb{R} \) we define the set \( A_a = \{ x \in \mathbb{R}, 0 \leq |x| \leq -a^2 + a + 2 \} \).

1. When \( a = 1 \), we have \(-a^2 + a + 2 = -1^2 + 1 + 2 = 2\) so \( A_1 = [-2, 2] \).

2. When \( a = -1 \), we have \(-a^2 + a + 2 = -1^2 - 1 + 2 = 0\) so \( A_1 = [0, 0] \) which we can simply denote by \( \{0\} \).

3. When \( a = 2 \), we have \(-a^2 + a + 2 = -2^2 + 2 + 2 = 0\) so \( A_2 = [0, 0] \) which we can simply denote by \( \{0\} \).

4. Is there a value of \( a \) for which \( A_a \) is empty? Yes for example when \( a = 3 \) we have \(-a^2 + a + 2 = -3^2 + 3 + 2 = -4\) so \( A_3 = \emptyset \). (In fact, one can check that \( A_a = \) for any \( a \in (-\infty, 1) \) and for any \( a \in (2, +\infty) \).)

**Problem 6.** (2 points) Given two numbers \( x \) and \( y \) we define the following statements:

\[ P : \quad x = 0 \quad \quad \quad Q : \quad y = 0 \]

The statement \( P \lor Q \) means at least one of the numbers \( x \) and \( y \) equals 0.