On the pro-p Iwahori Hecke Ext-algebra of $\mathrm{SL}_2(\mathbb{Q}_p)$

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Abstract

Let $G = \mathrm{SL}_2(\mathfrak{F})$ where $\mathfrak{F}$ is a finite extension of $\mathbb{Q}_p$. We suppose that the pro-p Iwahori subgroup I of G is a Poincaré group of dimension d. Let k be a field containing the residue field of $\mathfrak{F}$.

In this article, we study the graded Ext-algebra $E^* = \mathrm{Ext}^*_{\mathrm{Mod}(G)}(k[G/I], k[G/I])$. Its degree zero piece E^0 is the usual pro-p Iwahori-Hecke algebra H.

We study E^d as an H-bimodule and deduce that for an irreducible admissible smooth representation of G, we have $H^d(I, V) = 0$ unless V is the trivial representation.

When $\mathfrak{F} = \mathbb{Q}_p$ with $p \geq 5$, we have $d = 3$. In that case we describe E^* as an H-bimodule and give the structure as an algebra of the centralizer in E^* of the center of H. We deduce results on the values of the functor $H^*(I, \_)$ which attaches to a (finite length) smooth k-representation V of G its cohomology with respect to I. We prove that $H^*(I, V)$ is always finite dimensional. Furthermore, if V is irreducible, then V is supersingular if and only if $H^*(I, V)$ is a supersingular H-module.

Contents

1 Introduction

Let $\mathfrak{F}$ be a locally compact nonarchimedean field with residue characteristic p, and let G be the group of $\mathfrak{F}$-rational points of a connected reductive group $\mathbf{G}$ over $\mathfrak{F}$. We suppose that $\mathbf{G}$ is $\mathfrak{F}$-split.

Let k be a field of characteristic p and let $\mathrm{Mod}(G)$ denote the category of all smooth representations of G in k-vector spaces. For a general $\mathbf{G}$ and $\mathfrak{F}$ this category is still poorly understood. One way of approaching it consists in considering the Hecke algebra H of the pro-p Iwahori subgroup $I \subset G$. In this case the natural left exact functor

$$\begin{aligned} \mathfrak{h} : \mathrm{Mod}(G) &\longrightarrow \mathrm{Mod}(H) \\ V &\longmapsto V^I = \mathrm{Hom}_{k[G]}(\mathbf{X}, V) \end{aligned}$$

sends a nonzero representation onto a nonzero module. Its left adjoint is

$$\begin{aligned} \mathfrak{t} : \mathrm{Mod}(H) &\longrightarrow \mathrm{Mod}^I(G) \subseteq \mathrm{Mod}(G) \\ M &\longmapsto \mathbf{X} \otimes_H M \ . \end{aligned}$$

Here $\mathbf{X}$ denotes the space of k-valued functions with compact support on G/I with the natural left action of G. The functor $\mathfrak{t}$ has values in the category $\mathrm{Mod}^I(G)$ of all smooth k-representations of G generated by their I-fixed vectors. This category, which a priori has no reason to be an abelian subcategory of $\mathrm{Mod}(G)$, contains all irreducible representations. But in general $\mathfrak{t}$ is not an equivalences of categories and little is known about $\mathrm{Mod}^I(G)$ and $\mathrm{Mod}(G)$ unless $G = \mathrm{GL}_2(\mathbb{Q}_p)$ or $G = \mathrm{SL}_2(\mathbb{Q}_p)$ ([Koz1], [Oll1], [OS2], [Pas]).

The functor $\mathfrak{h}$, although left exact, is not right exact since p divides the pro-order of I. It is therefore natural to consider the derived functor. In [Sch-DGA] the following result is shown: When $\mathfrak{F}$ is a finite extension of $\mathbb{Q}_p$ and I is a torsion free pro-p group, there exists a derived version of the functors $\mathfrak{h}$ and $\mathfrak{t}$ providing an equivalence between the derived category $D(G)$ of smooth representations of G in k-vector spaces and the derived category of differential graded modules over a certain differential graded pro-p Iwahori-Hecke algebra $H^\bullet$.

The article [OS3] opened up the study of the Hecke differential graded algebra $H^\bullet$ by giving the first results on its cohomology algebra $E^* := \mathrm{Ext}^*_{\mathrm{Mod}(G)}(\mathbf{X}, \mathbf{X})$. This is the pro-$p$ Iwahori Hecke Ext-algebra we refer to in the title of the current article. We suppose in this introduction that I is a torsion free p-adic Lie group which forces $\mathfrak{F}$ to be a finite extension of $\mathbb{Q}_p$. We denote by d the dimension of I as a Poincaré group. The Ext algebra E^* is supported in degrees 0 to d.

When $\mathbf{G}$ is almost simple and simply connected, the ideal $\mathfrak{J}H$ which controls the supersingularity (see §2.1) has finite codimension in H. We show that we have an isomorphism of H-bimodules

$$\mathrm{Ext}^d_{\mathrm{Mod}(G)}(\mathbf{X}, \mathbf{X}) \cong \chi_{triv} \oplus \mathrm{Inj}((H/\mathfrak{J}H)^\vee) \tag{1}$$

where χ_{triv} is the trivial character of H and $\mathrm{Inj}((H/\mathfrak{J}H)^\vee)$ is an injective envelope of the dual module $(H/\mathfrak{J}H)^\vee$. When $\mathbf{G} = \mathrm{SL}_2$, the center of H contains a polynomial algebra $k[\zeta]$ and $\mathfrak{J}H = \zeta H$. The large injective module inside of $\mathrm{Ext}^d_{\mathrm{Mod}(G)}(\mathbf{X}, \mathbf{X})$ is ξ-divisible for any $\xi \in H$ which is a non-zero-divisor. This, together with the decomposition (1), allows us to prove (Cor. 2.19) that given Q a nonzero polynomial in $k[X]$, we have $H^d(I, \mathbf{X}/\mathbf{X}Q(\zeta)) = 0$ unless $Q(1) = 0$ in which case $H^d(I, \mathbf{X}/\mathbf{X}Q(\zeta)) \cong \chi_{triv}$. But we remark that every irreducible admissible representation of $\mathrm{SL}_2(\mathfrak{F})$ is a quotient $\mathbf{X}/\mathbf{X}Q(\zeta)$ for some Q as above and we prove:

Proposition. *(Proposition 2.20). We have $H^d(I, V) = 0$ for any irreducible admissible representation of $\mathrm{SL}_2(\mathfrak{F})$ except when V is the trivial representation in which case $H^d(I, k_{triv}) \cong \chi_{triv}$ as an H-bimodule.*

In Sections 3 and 4, we move on to the study of E^1 and E^{d-1} respectively. Here we fully use the Frobenius reciprocity recalled in §2.2 which allows to identify E^i with $H^i(I, \mathbf{X})$. We decompose the latter, via the Shapiro isomorphism, as a direct sum

$$\oplus_{w \in \widetilde{W}} H^i(I_w, k)$$

where w ranges over $\widetilde{W}$ (defined at the beginning of Section 2, see also §2.4.1) and $I_w = I \cap wIw^{-1}$. We explain in §3.2 that we see elements of $H^1(I_w, k)$ as triples. This is valid for $G = \mathrm{SL}_2(\mathfrak{F})$ with no restriction on $\mathfrak{F}$ and stems from the computation of the Frattini quotient of I_w. When I is a Poincaré group of dimension d, we use the duality between E^1 and E^{d-1} (§14) to also express elements of $H^{d-1}(I_w, k)$ as triples in §4.1. When $G = \mathrm{SL}_2(\mathbb{Q}_p)$, $p \geq 5$, Remark §3.2 points out that the triples of $H^1(I_w, k)$ are simply the elements in

$$\mathrm{Hom}(\mathbb{Z}_p/p\mathbb{Z}_p, k) \times \mathrm{Hom}((1+p\mathbb{Z}_p)/(1+p^2\mathbb{Z}_p), k) \times \mathrm{Hom}(\mathbb{Z}_p/p\mathbb{Z}_p, k)$$

hence by duality the triples of $H^2(I_w, k)$ are the elements in

$$\mathbb{Z}_p/p\mathbb{Z}_p \otimes_{\mathbb{F}_p} k \times ((1+p\mathbb{Z}_p)/(1+p^2\mathbb{Z}_p)) \otimes_{\mathbb{F}_p} k \times \mathbb{Z}_p/p\mathbb{Z}_p \otimes_{\mathbb{F}_p} k \ .$$

In this context, the full left action of H on the triples of E^1 and of E^2 can be found in §3.6 and §4.3 (the proof of the most technical formulas is postponed to the appendix). The right action of H on the triples can be deduced using the anti-involution $\mathcal{J}$ of E^* (see §2.2.3 and Lemmas 3.7 and 4.1). We are especially interested in the left and right action of the central element $\zeta \in H$ (which is fixed by $\mathcal{J}$).

In Section 5 we study the $k[\zeta]$-torsion on the left in certain graded pieces of E^* when $G = \mathrm{SL}_2(\mathfrak{F})$, with various restrictive conditions on $\mathfrak{F}$ depending on the graded piece in question. Only for the computation of the $k[\zeta]$-torsion in E^2 do we use the explicit formulas for the action of ζ hence we have to restrict ourselves to $G = \mathrm{SL}_2(\mathbb{Q}_p)$, $p \geq 5$.

Contemplating the formulas for the action of ζ on E^1 and E^2 (still when $G = \mathrm{SL}_2(\mathbb{Q}_p)$, $p \geq 5$) emphasizes the role of the operators

$$f := \zeta \cdot \mathrm{id}_{E^*} \cdot \zeta - \mathrm{id}_{E^*} \quad \text{and} \quad g := \zeta \cdot \mathrm{id}_{E^*} - \mathrm{id}_{E^*} \cdot \zeta$$

as introduced in §6.1. The kernel of f is a $k[\zeta^{\pm 1}]$-bimodule. Describing its structure as an H-bimodule requires the technical paragraph 3.7.3.2 (then see Propositions 6.8, 6.19 and Lemma 6.2). On the other hand, as the centralizer in E^* of ζ, the kernel of g is naturally a subalgebra of E^*. We describe this kernel precisely in §6.2.1 and §6.3.1 (and Lemma 6.2) and conclude in Proposition 8.1 that it actually coincides with the centralizer $\mathcal{C}_{E^*}(Z)$ of the whole center $Z := Z(H)$ of H in E^*. The product in this natural subalgebra of E^* is explicitly given in Section 8. (Note that the center of H is no longer central in E^*).

Proposition 6.13 says that E^2 is, as an H-bimodule, isomorphic to the direct sum of the kernels of the operators f and g (restricted to E^2) and Proposition 6.10 says that it is also (essentially) the case for E^1. This allows us to completely determine the structure of E^* as a left and right $k[\zeta]$-module (Proposition 7.2) and to establish results such as Proposition 7.6 where we study the $k[\zeta]$-torsion on the left in spaces of the form $H^*(I, \mathbf{X}/\mathbf{X}Q(\zeta))$ for $Q \in k[X]$. This in particular leads to the following theorem:

Theorem. *(Theorem 7.11). Let $G = \mathrm{SL}_2(\mathbb{Q}_p)$ with $p \neq 2, 3$. For any representation of finite length in* $\mathrm{Mod}(G)$ *we have:*

i. The k-vector space $H^*(I, V)$ is finite dimensional;

ii. if V is generated by its subspace of I-fixed vectors V^I and $Q(\zeta)V^I = 0$ for some nonzero polynomial $Q \in k[X]$, then the left H-module $H^*(I, V)$ is $P(\zeta)$-torsion for the polynomial $P(X) := Q(X)Q(\frac{1}{X})X^{\deg(Q)}$.

The most interesting consequence of this theorem is that, under the same hypotheses, an irreducible representation V in $\mathrm{Mod}(G)$ is supersingular if and only if the left H-module $H^*(I,V)$ is supersingular (this is Corollary 7.12 which uses the theorem in the case when $Q = X$). This strongly indicates that the notion of supersingularity for general G can be extended to objects in the derived category $D(G)$ by introducing a theory of supports via the dg algebra $H^\bullet$. We hope to return to this in another paper.

In [OS2] §3.5 we studied the representation theoretic meaning of the localization H_ζ of the Hecke algebra in the central element ζ. Despite the fact that ζ is no longer central in E^* it turns out (Remark 7.7) that $\zeta^{\mathbf{N}_0}$ is a left and right Ore set in E^*, so that the localization E^*_ζ does exist. We will show elsewhere that E^*_ζ again is a Yoneda Ext-algebra and will investigate its meaning for the nonsupersingular $\mathrm{SL}_2(\mathbb{Q}_p)$-representations.

After this paper was finished E. Bodon ([Bod]) gave in his thesis, building very much on the computational methods developed in the present paper, two further structural results in the case $G = \mathrm{SL}_2(\mathbb{Q}_p)$ with $p \neq 2, 3$. He describes explicitly the full graded center of E^*. Even more remarkably he shows that the algebra E^* as an algebra over H is finitely presented.

In forthcoming work of the second author with K. Ardakov we develop a general theory of central spaces for a certain class of Grothendieck categories which refines the notion of the center of an abelian category. It was shown in [AS] that the usual center of the category $\mathrm{Mod}(G)$ is very small. For example, if $\mathbf{G} = \mathrm{SL}_2$ then this center is the group ring $k[Z(G)]$ of the center of G. In contrast the central space in this case is a projective variety over k which is a quotient of the affine variety $\mathrm{Spec}(Z(H))$ by a relation which is given by the annihilator ideal of the $Z \otimes_k Z$-bimodule E^*. The results of the present paper allow to compute this ideal and therefore this projective variety explicitly. Therefore we strongly believe that this bimodule and its support variety play a basic role for the computation of the central space of $\mathrm{Mod}(G)$ for general groups G.

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2 Notations, preliminaries and results on the top cohomology

Throughout the paper we fix a locally compact nonarchimedean field $\mathfrak{F}$ (for now of any characteristic) with ring of integers $\mathfrak{O}$, its maximal ideal $\mathfrak{M}$, and a prime element π. The residue field $\mathfrak{O}/\pi\mathfrak{O}$ of $\mathfrak{F}$ is $\mathbb{F}_q$ for some power $q = p^f$ of the residue characteristic p. We choose the valuation $\mathrm{val}_{\mathfrak{F}}$ on $\mathfrak{F}$ normalized by $\mathrm{val}_{\mathfrak{F}}(\pi) = 1$ We let $G := \mathbf{G}(\mathfrak{F})$ be the group of $\mathfrak{F}$-rational points of a connected reductive group $\mathbf{G}$ over $\mathfrak{F}$ which we always assume to be $\mathfrak{F}$-split. We will very soon specialize to the case when $\mathbf{G}$ is almost simple and simply connected (starting Section 2.3) and in fact the core of this article (starting Section 3) will focus on the case when $\mathbf{G} = \mathrm{SL}_2$ and $\mathfrak{F} = \mathbb{Q}_p$ with $p \neq 2, 3$.

We fix an $\mathfrak{F}$-split maximal torus $\mathbf{T}$ in $\mathbf{G}$, put $T := \mathbf{T}(\mathfrak{F})$, and let T^0 denote the maximal compact subgroup of T and T^1 the pro-p Sylow subgroup of T^0. We also fix a chamber C in the apartment of the semisimple Bruhat-Tits building $\mathscr{X}$ of G which corresponds to $\mathbf{T}$. The stabilizer $\mathcal{P}^\dagger_C$ of C contains an Iwahori subgroup J. Its pro-p Sylow subgroup I is called the pro-p Iwahori subgroup. We have $T \cap J = T^0$ and $T \cap I = T^1$. If $N(T)$ is the normalizer of T in G, then we define the group $\widetilde{W} := N(T)/T^1$. In particular, it contains $\Omega := T^0/T^1$. The quotient $W := N(T)/T^0 \cong \widetilde{W}/\Omega$ is the extended affine Weyl group. The finite Weyl group is $W_0 := N(T)/T$. The length on W pulls back to a length function ℓ on $\widetilde{W}$ (see [OS3] §2.1.4).

For any compact open subset $A \subseteq G$ we let char_A denote the characteristic function of A.

The coefficient field for all representations in this paper is an arbitrary field k of characteristic $p > 0$.

For any open subgroup $U \subseteq G$ we let $\mathrm{Mod}(U)$ denote the abelian category of smooth representations of U in k-vector spaces.

2.1 The pro-p-Iwahori Hecke algebra

We consider the compact induction $\mathbf{X} := \mathrm{ind}_I^G(1)$ of the trivial I-representation. It can be seen as the space of compactly supported functions $G \to k$ which are constant on the left cosets mod I. It lies in $\mathrm{Mod}(G)$. For Y a compact subset of G which is right invariant under I, the characteristic function char_Y is an element of $\mathbf{X}$. Equivalently one may view $\mathbf{X} = k[G/I]$ as the k-vector space with basis the cosets $gI \in G/I$. The pro-p Iwahori-Hecke algebra is defined to be the k-algebra

$$H := \mathrm{End}_{k[G]}(\mathbf{X})^{\mathrm{op}} .$$

We often will identify H, as a right H-module, via the map $H \to \mathbf{X}^I, h \mapsto (\mathrm{char}_I)h$ with the submodule $\mathbf{X}^I$ of I-fixed vectors in $\mathbf{X}$. The Bruhat-Tits decomposition of G says that G is the disjoint union of the double cosets IwI for $w \in \widetilde{W}$. Hence we have the I-equivariant decomposition

$$\mathbf{X} = \oplus_{w \in \widetilde{W}} \mathbf{X}(w) \quad \text{with} \quad \mathbf{X}(w) := \mathrm{ind}_I^{IwI}(1) , \tag{2}$$

where the latter denotes the subspace of those functions in $\mathbf{X}$ which are supported on the double coset IwI. In particular, we have $\mathbf{X}(w)^I = k\tau_w$ where $\tau_w := \mathrm{char}_{IwI}$ and hence $H = \oplus_{w \in \widetilde{W}} k\tau_w$ as a k-vector space.

The defining (braid and quadratic) relations of H are recalled in [OS3] §2.2. They ensure in particular that we have a well defined trivial character of H denoted by χ_{triv} and defined by ([OS3] §2.2.2):

$$\chi_{triv} : \tau_w \longmapsto 0,\ \tau_\omega \longmapsto 1, \text{ for any } w \in \widetilde{W} \text{ with } \ell(w) \geq 1 \text{ and } \omega \in \widetilde{W} \text{ with } \ell(\omega) = 0. \tag{3}$$

To define the notion of supersingularity for H-modules, we refer to [OS3] §2.3. Recall that there is a a central subalgebra $\mathcal{Z}^0(H)$ of H which is isomorphic to the affine semigroup algebra $k[X_*^{dom}(T)]$, where $X_*^{dom}(T)$ denotes the semigroup of all dominant cocharacters of T. The cocharacters $\lambda \in X_*^{dom}(T) \setminus (-X_*^{dom}(T))$ generate a proper ideal of $k[X_*^{dom}(T)]$, the image of which in $\mathcal{Z}^0(H)$ is denoted by $\mathfrak{J}$. We call an H-module M supersingular if any element in M is annihilated by a power of $\mathfrak{J}$.

2.2 The Ext-algebra

We refer to [OS3] §3. We form the graded Ext-algebra

$$E^* := \mathrm{Ext}^*_{\mathrm{Mod}(G)}(\mathbf{X}, \mathbf{X})^{\mathrm{op}}$$

over k with the multiplication being the (opposite of the) Yoneda product. Obviously

$$H := E^0 = \mathrm{Hom}_{\mathrm{Mod}(G)}(\mathbf{X}, \mathbf{X})^{\mathrm{op}}$$

is the usual pro-p Iwahori-Hecke algebra over k. By using Frobenius reciprocity for compact induction and the fact that the restriction functor from $\mathrm{Mod}(G)$ to $\mathrm{Mod}(I)$ preserves injective objects we obtain the identification

$$E^* = \mathrm{Ext}^*_{\mathrm{Mod}(G)}(\mathbf{X}, \mathbf{X})^{\mathrm{op}} = H^*(I, \mathbf{X}) . \tag{4}$$

The only part of the multiplicative structure on E^* which is still directly visible on the cohomology $H^*(I, \mathbf{X})$ is the right multiplication by elements in $E^0 = H$, which is functorially induced by the right action of H on $\mathbf{X}$. In [OS3], we made the full multiplicative structure visible on $H^*(I,\mathbf{X})$. We recall that for $*=0$ the above identification is given by $H \xrightarrow{\cong} \mathbf{X}^I, \tau \longmapsto (\mathrm{char}_I)\tau$.

Noting that the cohomology of profinite groups commutes with arbitrary sums, we obtain from the I-equivariant decomposition (2) a decomposition of vector spaces

$$H^*(I,\mathbf{X}) = \oplus_{w\in\widetilde{W}} H^*(I,\mathbf{X}(w)) \ . \tag{5}$$

For $w \in \widetilde{W}$, we let $I_w := I \cap wIw^{-1}$ (see [OS3] §2.1.5). We call Shapiro isomorphism and denote by Sh_w the composite map

$$\mathrm{Sh}_w : H^*(I,\mathbf{X}(w)) \xrightarrow{\mathrm{res}} H^*(I_w,\mathbf{X}(w)) \xrightarrow{H^*(I_w,\mathrm{ev}_w)} H^*(I_w,k) \tag{6}$$

where $\mathrm{ev}_w : \mathbf{X}(w) \longrightarrow k,\ f \longrightarrow f(w)$ (see also [OS3] §3.2).

2.2.1 The cup product

Recall from [OS3] §3.3 that there is a naive product structure on the cohomology $H^*(I,\mathbf{X})$. By multiplying maps we obtain the G-equivariant map $\mathbf{X}\otimes_k \mathbf{X} \longrightarrow \mathbf{X},\ f\otimes f' \longmapsto ff'$. It gives rise to the cup product

$$H^i(I,\mathbf{X}) \otimes_k H^j(I,\mathbf{X}) \xrightarrow{\cup} H^{i+j}(I,\mathbf{X}) \tag{7}$$

which has the property that $H^i(I,\mathbf{X}(v)) \cup H^j(I,\mathbf{X}(w)) = 0$ whenever $v \neq w$. On the other hand, since $\mathrm{ev}_w(ff') = \mathrm{ev}_w(f)\,\mathrm{ev}_w(f')$ and since the cup product is functorial and commutes with cohomological restriction maps, we have the commutative diagrams

$$\begin{array}{ccc} H^i(I,\mathbf{X}(w)) \otimes_k H^j(I,\mathbf{X}(w)) & \xrightarrow{\cup} & H^{i+j}(I,\mathbf{X}(w)) \\ \downarrow{\scriptstyle \mathrm{Sh}_w\otimes \mathrm{Sh}_w} & & \downarrow{\scriptstyle \mathrm{Sh}_w} \\ H^i(I_w,k)\otimes_k H^j(I_w,k) & \xrightarrow{\cup} & H^{i+j}(I_w,k) \end{array} \tag{8}$$

for any $w \in \widetilde{W}$, where the bottom row is the usual cup product on the cohomology algebra $H^*(I_w,k)$. In particular, the cup product (7) is anticommutative.

2.2.2 The Yoneda product

The Yoneda product in E^* ([OS3] §4.2) satisfies the following property:

$$H^i(I,\mathbf{X}(v)) \cdot H^j(I,\mathbf{X}(w)) \subseteq H^{i+j}(I, \mathrm{ind}_I^{IvI\cdot IwI}(1)) \text{ for } v,w\in\widetilde{W}. \tag{9}$$

The product of $a \in H^i(I,\mathbf{X}(v))$ by $b \in H^j(I,\mathbf{X}(w))$ is explicitly described in [OS3] Prop. 5.3. We record here the following results.

Proposition 2.1. *Let $v,w\in\widetilde{W}$ and $a \in H^i(I,\mathbf{X}(v))$, $b\in H^j(I,\mathbf{X}(w))$.*

- *if $\ell(vw)=\ell(v)+\ell(w)$, then*

$$a\cdot b = (a\cdot\tau_w)\cup(\tau_v\cdot b) \in H^{i+j}(I,\mathbf{X}(vw)) \ ; \tag{10}$$

- *if $\ell(v)=1$ and $\ell(vw)=\ell(w)-1$, then $a\cdot b$ lies in $H^{i+j}(I,\mathbf{X}(vw)) \oplus \bigoplus_{\omega\in T^0/T^1} H^{i+j}(I,\mathbf{X}(\omega w))$. If furthermore $\mathbf{G}$ is semisimple and simply connected, then*

$$a\cdot b - (a\cdot\tau_w)\cup(\tau_v\cdot b) \in H^{i+j}(I,\mathbf{X}(vw)) \ . \tag{11}$$

Proof. The first point is [OS3] Cor. 5.5. We prove the second point in §9.1 of the Appendix. $\square$

2.2.3 Anti-involution

We refer to [OS3] §6. The graded algebra E^* is equipped with an an involutive anti-automorphism. It is defined the following way. For $w \in \widetilde{W}$, we have $I_{w^{-1}} = w^{-1} I_w w$ and a linear isomorphism $(w^{-1})_* : H^i(I_w, k) \overset{\cong}{\to} H^i(I_{w^{-1}}, k)$, for all $i \geq 0$. Via the Shapiro isomorphism (6), this induces the linear isomorphism $\mathcal{J}_w$:

$$\begin{array}{ccc} H^i(I, \mathbf{X}(w)) & \xrightarrow[\cong]{\mathcal{J}_w} & H^i(I, \mathbf{X}(w^{-1})) \\ \downarrow{\scriptstyle \mathrm{Sh}_w} & & \cong \downarrow{\scriptstyle \mathrm{Sh}_{w^{-1}}} \\ H^i(I_w, k) & \xrightarrow[\cong]{(w^{-1})_*} & H^i(I_{w^{-1}}, k) \end{array} \tag{12}$$

Summing over all $w \in \widetilde{W}$, the maps $(\mathcal{J}_w)_{w \in \widetilde{W}}$ induce a linear isomorphism

$$\mathcal{J} : H^i(I, \mathbf{X}) \xrightarrow{\cong} H^i(I, \mathbf{X}) .$$

and it is proved in [OS3] Prop. 6.1 that $\mathcal{J}$ is an anti-automorphism of the graded algebra E^*. Restricted to $E^0 = H$, the map $\mathcal{J}$ coincides with the anti-involution $\tau_g \mapsto \tau_{g^{-1}}$ for any $g \in G$ of the algebra H.

We may twist the action of H on a left, resp. right, module Y by $\mathcal{J}$ and thus obtain the right, resp. left module $Y^{\mathcal{J}}$, resp. ${}^{\mathcal{J}}Y$, with the twisted action of H given by $(y, h) \mapsto \mathcal{J}(h)y$, resp. $(h, y) \mapsto y\mathcal{J}(h)$. If Y is an H-bimodule, then we may define the twisted H-bimodule ${}^{\mathcal{J}}Y^{\mathcal{J}}$ the obvious way and we recall that $({}^{\mathcal{J}}Y^{\mathcal{J}})^\vee \cong {}^{\mathcal{J}}(Y^\vee)^{\mathcal{J}}$ ([OS3] Rmk. 7.1), where $(-)^\vee = \mathrm{Hom}_k(-, k)$.

2.2.4 Filtrations

Let $i \geq 0$. We define on E^i two filtrations:

- a decreasing filtration $(F^n E^i)_{n \geq 0}$ where $F^n E^i := \oplus_{w \in \widetilde{W},\, \ell(w) \geq n} H^i(I, \mathbf{X}(w))$;
- an increasing filtration $(F_n E^i)_{n \geq 0}$ where $F_n E^i := \oplus_{w \in \widetilde{W},\, \ell(w) \leq n} H^i(I, \mathbf{X}(w))$.

When $i = 0$, we will often write $F^n H$ (resp. $F_n H$) instead of $F^n E^0$ (resp. $F_n E^0$). Recall that $(F^n H)_{n \geq 0}$ is a filtration of H as an H-bimodule.

Moreover, $F_n E^*$ is an algebra filtration, which means that $F_n E^i \cdot F_m E^j \subseteq F_{n+m} E^{i+j}$. This follows from (9) together with the fact ([OS3] Cor. 2.5-ii and Remark 2.10) that

$$IvI \cdot IwI \begin{cases} = IvwI & \text{if } \ell(vw) = \ell(v) + \ell(w), \\ \subseteq \bigcup_{\ell(v') < \ell(v) + \ell(w)} Iv'I & \text{if } \ell(vw) < \ell(v) + \ell(w). \end{cases} \tag{13}$$

2.2.5 Duality

Recall that, given a vector space Y, we denote by $Y^\vee$ the dual space $Y^\vee := \mathrm{Hom}_k(Y, k)$ of Y. For Y a vector space which decomposes into a direct sum $Y = \oplus_{w \in \widetilde{W}} Y_w$, we denote by $Y^{\vee,f}$ the so-called finite dual of Y which is defined to be the image in $Y^\vee = \prod_{w \in \widetilde{W}} Y_w^\vee$ of $\oplus_{w \in \widetilde{W}} Y_w^\vee$.

In this paragraph, we always **assume** that the pro-p Iwahori group I is a torsion free p-adic Lie group. This forces the field $\mathfrak{F}$ to be a finite extension of $\mathbb{Q}_p$ with $p \geq 5$. Then I, as well as every subgroup I_w for $w \in \widetilde{W}$, is a Poincaré group of dimension d where d is the dimension of G as a p-adic Lie group. It implies that $H^d(I, k)$ is one-dimensional. Let $\eta : H^d(I, k) \cong k$ a fixed isomorphism (we will make a specific choice for η when $G := \mathrm{SL}_2(\mathbb{Q}_p)$, $p \neq 2, 3$ in §3.2.3). Furthermore the Ext-algebra is supported in degrees 0 to d. We refer to [OS3] §7.2. There is a duality between its i^{th} and $d-i^{\text{th}}$ pieces ([OS3] §7.2.4) which we recall here. Let $\mathcal{S} \in \mathbf{X}^\vee$ be the linear map given by $\mathcal{S} := \sum_{g \in G/I} \mathrm{ev}_g$. It is easy to check that $\mathcal{S} : \mathbf{X} \to k$ is G-equivariant when k is endowed with the trivial action of G. We denote by $\mathcal{S}^i := H^i(I, \mathcal{S})$ the maps induced on cohomology. By [OS3] Prop. 7.18, the map

$$\Delta^i : E^i = H^i(I, \mathbf{X}) \longrightarrow H^{d-i}(I, \mathbf{X})^\vee = (E^{d-i})^\vee$$
$$a \longmapsto l_a(b) := \eta \circ \mathcal{S}^d(a \cup b)$$

induces an injective homomorphism of H-bimodules $E^i \longrightarrow ({}^{\mathcal{J}}(E^{d-i})^{\mathcal{J}})^\vee$ with image $({}^{\mathcal{J}}(E^{d-i})^{\mathcal{J}})^{\vee,f}$. Here we consider (as in §2.2.3) the twisted H-bimodule ${}^{\mathcal{J}}(E^{d-i})^{\mathcal{J}}$, namely the space E^{d-i} with the action of H on $b \in E^{d-i}$ given by $(\tau, b, \tau') \mapsto \mathcal{J}(\tau') \cdot b \cdot \mathcal{J}(\tau)$ for $\tau, \tau' \in H$. The anti-involution $\mathcal{J}$ was introduced in §2.2.3. We still denote by Δ^i the isomorphism

(14) $$\Delta^i : E^i \longrightarrow ({}^{\mathcal{J}}(E^{d-i})^{\mathcal{J}})^{\vee,f} \ .$$

Recall that the choice of η defines naturally a basis for E^d, namely, as in [OS3] §8, we single out the unique element $\phi_w \in H^d(I, \mathbf{X}(w))$ such that (see also Rmk. 7.4 *loc. cit.*)

(15) $$\eta \circ \mathcal{S}^d(\phi_w) = \eta \circ \mathrm{cores}_I^{I_w} \circ \mathrm{Sh}_w(\phi_w) = 1.$$

2.2.6 Automorphisms of the pair $(G, \mathbf{X})$

For U a locally compact and totally disconnected group let $\mathrm{Mod}(U)$ be the abelian category of smooth U-representations in k-vector spaces. It has enough injective objects.

We consider now a continuous group homomorphism $\xi : U' \to U$ between two such groups. Any object M in $\mathrm{Mod}(U)$ can be viewed via ξ as an object $\xi^* M$ in $\mathrm{Mod}(U')$. An equivariant map $f : M \to M'$ between an object M in $\mathrm{Mod}(U)$ and an object M' in $\mathrm{Mod}(U')$ is, by definition, a morphism $f : \xi^* M \to M'$ in $\mathrm{Mod}(U')$. In other words $f : M \to M'$ is a k-linear map such that $f(\xi(g')m) = g' f(m)$ for any $m \in M$ and $g' \in U'$. We observe the following: Let $M \xrightarrow{\sim} \mathcal{I}_M^\bullet$ and $M' \xrightarrow{\sim} \mathcal{I}_{M'}^\bullet$ be injective resolutions in $\mathrm{Mod}(U)$ and $\mathrm{Mod}(U')$, respectively. Then $\xi^* M \xrightarrow{\sim} \xi^* \mathcal{I}_M^\bullet$ is a resolution in $\mathrm{Mod}(U')$ and f extends to a unique homotopy class of maps of resolutions $\xi^* \mathcal{I}_M^\bullet \xrightarrow{\tilde{f}} \mathcal{I}_{M'}^\bullet$ in $\mathrm{Mod}(U')$. This means that we may derive f to a map between any appropriate cohomological functors on $\mathrm{Mod}(U)$ and $\mathrm{Mod}(U')$.

We will apply this in the following two contexts. First suppose that U and U' are profinite groups. Then f extends to a map on cohomology

$$(\xi, f)^* : H^i(U, M) \longrightarrow H^i(U', M') \ .$$

Secondly, let U and U' be general again. For any further object L in $\mathrm{Mod}(U)$ we obtain natural maps

$$(\xi, f)^* : \mathrm{Ext}^i_{\mathrm{Mod}(U)}(L, M) \longrightarrow \mathrm{Ext}^i_{\mathrm{Mod}(U')}(\xi^* L, M')$$
$$(\mathcal{I}_L^\bullet \to \mathcal{I}_M^\bullet[i]) \longmapsto (\xi^* \mathcal{I}_L^\bullet \to \xi^* \mathcal{I}_M^\bullet[i] \xrightarrow{\tilde{f}[i]} \mathcal{I}_{M'}^\bullet[i])$$

and, in particular,

$$\xi^* := (\xi, \mathrm{id}_M)^* : \mathrm{Ext}^i_{\mathrm{Mod}(U)}(L, M) \longrightarrow \mathrm{Ext}^i_{\mathrm{Mod}(U')}(\xi^* L, \xi^* M) \ .$$

The latter map is evidently compatible with the Yoneda product, since in the derived category it is simply the composition product. Now suppose that ξ and f are isomorphisms. Then we have the "conjugation" homomorphism

$$\mathrm{Ext}^i_{\mathrm{Mod}(U)}(M, M) \longrightarrow \mathrm{Ext}^i_{\mathrm{Mod}(U')}(M', M')$$
$$(\mathcal{I}_M^\bullet \xrightarrow{\tau} \mathcal{I}_M^\bullet[i]) \longmapsto (\mathcal{I}_{M'}^\bullet \xrightarrow{\widetilde{f^{-1}}} \xi^* \mathcal{I}_M^\bullet[i] \xrightarrow{\xi^* \tau} \xi^* \mathcal{I}_M^\bullet[i] \xrightarrow{\tilde{f}[i]} \mathcal{I}_{M'}^\bullet[i]) \ ,$$

which again is compatible with the Yoneda product.

We now return to our group G and suppose given an automorphism $\xi : G \xrightarrow{\cong} G$ with the property that $\xi(I) = I$. It induces the G-equivariant bijection $\mathcal{X} : \xi^*\mathbf{X} \xrightarrow{\cong} \mathbf{X}$ which sends gI to $\xi^{-1}(g)I$. We therefore obtain the k-linear graded bijections

$$\Gamma_\xi : E^* \xrightarrow{\cong} E^* \qquad \text{and} \qquad \Gamma_\xi : H^*(I, \mathbf{X}) \xrightarrow{\cong} H^*(I, \mathbf{X})$$

which correspond to each other under the identification (4). The left hand one is an algebra automorphism. Both are involutions provided we have $\xi^2 = \mathrm{id}_G$. In terms of elements of $\mathbf{X}$ as functions we have $\mathcal{X}(f) = f \circ \xi$. This immediately implies that Γ_ξ is compatible with the cup product (7) on $H^*(I, \mathbf{X})$. In the following we list further properties, but for which we assume in addition that $\xi(T) = T$. Then $\xi(N(T)) = N(T)$, so that ξ induces an automorphism ξ of $\widetilde{W}$.

1. For all $w \in \widetilde{W}$, Γ_ξ induces a map

   (16) $$H^*(I, \mathbf{X}(w)) \longrightarrow H^*(I, \mathbf{X}(\xi^{-1}(w))) \ .$$

   Since $\xi(I_w) = I_{\xi(w)}$ we correspondingly have a map

   (17) $$H^*(I_w, \mathbf{X}(w)) \longrightarrow H^*(I_{\xi^{-1}(w)}, \mathbf{X}(\xi^{-1}(w))) \ .$$

2. Because $\mathrm{ev}_{\xi^{-1}(w)} \circ \mathcal{X}_{|\mathbf{X}(w)} = \mathrm{ev}_w$, the above maps are compatible with the Shapiro isomorphism in the sense that the following diagram

   (18)
   $$\begin{array}{ccccc} H^*(I, \mathbf{X}(w)) & \xrightarrow{\mathrm{res}^I_{I_w}} & H^*(I_w, \mathbf{X}(w)) & \xrightarrow{H^*(I_w, \mathrm{ev}_w)} & H^*(I_w, k) \\ \downarrow{\scriptstyle (16)} & & \downarrow{\scriptstyle (17)} & & \downarrow \\ H^*(I, \mathbf{X}(\xi^{-1}(w))) & \xrightarrow{\mathrm{res}^I_{I_{\xi^{-1}(w)}}} & H^*(I_{\xi^{-1}(w)}, \mathbf{X}(\xi^{-1}(w))) & \xrightarrow{H^*(I_{\xi^{-1}(w)}, \mathrm{ev}_{\xi^{-1}(w)})} & H^*(I_{\xi^{-1}(w)}, k) \end{array}$$

   commutes. Its horizontal arrows are the Shapiro isomorphisms (6) and the right hand side vertical arrow is induced by the isomorphism $I_{\xi^{-1}(w)} \xrightarrow[\cong]{\xi} I_w$.

3. Γ_ξ commutes with $\mathcal{J}$ defined in (12); more precisely, each diagram

   (19)
   $$\begin{array}{ccc} H^*(I, \mathbf{X}(w)) & \xrightarrow{\mathcal{J}} & H^*(I, \mathbf{X}(w^{-1})) \\ \downarrow{\scriptstyle (16)} & & \downarrow{\scriptstyle (16)} \\ H^*(I, \mathbf{X}(\xi^{-1}(w))) & \xrightarrow{\mathcal{J}} & H^*(I, \mathbf{X}(\xi^{-1}(w)^{-1})) \end{array}$$

   is commutative.

4. We have noted already the compatibility of Γ_ξ with the cup product on $H^*(I, \mathbf{X})$. It now holds in the more precise form of the commutativity of the diagrams

   (20)
   $$\begin{array}{ccc} H^i(I, \mathbf{X}(w)) \otimes_k H^j(I, \mathbf{X}(w)) & \xrightarrow{\cup} & H^{i+j}(I, \mathbf{X}(w)) \\ \downarrow{\scriptstyle (16)\otimes(16)} & & \downarrow{\scriptstyle (16)} \\ H^i(I, \mathbf{X}(\xi^{-1}(w))) \otimes_k H^j(I, \mathbf{X}(\xi^{-1}(w))) & \xrightarrow{\cup} & H^{i+j}(I, \mathbf{X}(\xi^{-1}(w))). \end{array}$$

2.3 The top cohomology E^d when $\mathbf{G}$ is almost simple simply connected

Without extra conditions on $\mathbf{G}$ or on $\mathfrak{F}$, we have the following. The ideal $\mathfrak{J}$ (§2.1) generates a two-sided ideal $\mathfrak{J}H$ in H. Recall that we denote by $V^\vee$ the k-linear dual of a k-vector space V. We consider the obvious inclusion of H-bimodules

$$(H/\mathfrak{J}H)^\vee \longrightarrow I((H/\mathfrak{J}H)^\vee) := \bigcup_m (H/\mathfrak{J}^m H)^\vee \ .$$

Lemma 2.2.

- *$I((H/\mathfrak{J}H)^\vee)$ is an injective H-module on the left and on the right.*
- *If furthermore $\mathbf{G}$ is semisimple, then $I((H/\mathfrak{J}H)^\vee)$ is an injective hull of $(H/\mathfrak{J}H)^\vee$ as a left as well as a right H-module.*

Proof. The following argument arose from a discussion with K. Ardakov. The other case being entirely analogous we prove the statement as left H-modules.

Step1: We show that the left H-module $I((H/\mathfrak{J}H)^\vee)$ is injective. By Baer's criterion it suffices to consider test diagrams of the form

$$\begin{array}{ccc} L & \xrightarrow{\ \subseteq\ } & H \\ {\scriptstyle\alpha}\downarrow & & \\ I((H/\mathfrak{J}H)^\vee) & & \end{array}$$

where $L \subseteq H$ is a left ideal. The ring H being noetherian the left ideal L is finitely generated. Hence the image of α is contained in $(H/\mathfrak{J}^a H)^\vee$ for any sufficiently large a. The homomorphism then must factorize through a homomorphism $\bar{\alpha} : L/\mathfrak{J}^a L \to (H/\mathfrak{J}^a H)^\vee$. Furthermore, since the ideal $\mathfrak{J}H$ in the noetherian ring H is centrally generated it has the Artin-Rees property (cf. [MCR] Prop. 4.2.6). This implies that we find an integer $b \geq a$ such that $\mathfrak{J}^b H \cap L \subseteq \mathfrak{J}^a L$. This reduces us to finding the broken arrow in the diagram

$$\begin{array}{ccc} L/\mathfrak{J}^b H \cap L & \longrightarrow & H/\mathfrak{J}^b H \\ {\scriptstyle \mathrm{pr}}\downarrow & & \\ L/\mathfrak{J}^a L & & \\ {\scriptstyle \bar{\alpha}}\downarrow & & \\ (H/\mathfrak{J}^a H)^\vee & & \\ {\scriptstyle \subseteq}\downarrow & \swarrow\!\!\text{- - -} & \\ (H/\mathfrak{J}^b H)^\vee. & & \end{array}$$

We note that the horizontal arrow is injective and that this is a diagram of $H/\mathfrak{J}^b H$-modules. So it suffices to show that that the $H/\mathfrak{J}^b H$-module $(H/\mathfrak{J}^b H)^\vee$ is injective. But the computation

$$\mathrm{Hom}_{H/\mathfrak{J}^b H}(M, (H/\mathfrak{J}^b H)^\vee) = \mathrm{Hom}_k(H/\mathfrak{J}^b H \otimes_{H/\mathfrak{J}^b H} M, k) = \mathrm{Hom}_k(M, k)$$

shows that these functors are exact in the $H/\mathfrak{J}^b H$-module M.

Step 2: Assume that the group $\mathbf{G}$ is semisimple. Then $H/\mathfrak{J}^m H$ is finite dimensional over k for any $m \geq 1$. We show that the inclusion $(H/\mathfrak{J}H)^\vee \subseteq I((H/\mathfrak{J}H)^\vee)$ is essential, i.e., that any nonzero H-submodule Y of $I((H/\mathfrak{J}H)^\vee)$ has nonzero intersection with $(H/\mathfrak{J}H)^\vee$. It, of course, suffices to consider the case when Y is a cyclic module. We then have $Y \subseteq (H/\mathfrak{J}^m H)^\vee$ for some large m. Let $Y^\perp \subseteq H/\mathfrak{J}^m H$ be the orthogonal complement of Y. Suppose that $Y \cap (H/\mathfrak{J}H)^\vee = 0$. This means that $Y^\perp + \mathfrak{J}H/\mathfrak{J}^m H = H/\mathfrak{J}^m H$

But $\mathfrak{J}H/\mathfrak{J}^m H$ is contained in the Jacobson radical of $H/\mathfrak{J}^m H$. Hence the Nakayama lemma implies that $Y^\perp = H/\mathfrak{J}^m H$, which gives rise to the contradiction that $Y = 0$. $\square$

Remark 2.3. The anti-involution $\mathcal{J} : H \to H$ yields an isomorphism of H-bimodules $H \cong {}^{\mathcal{J}}H^{\mathcal{J}}$. By [OS3] Remark 6.3, it preserves the central ideal $\mathfrak{J}$, as well as the central ideal $\mathfrak{J}^m$ for any $m \geq 1$. Therefore, we have an isomorphism of H-bimodules $H/\mathfrak{J}^m H \cong {}^{\mathcal{J}}(H/\mathfrak{J}^m H)^{\mathcal{J}}$. By [OS3] Remark 7.1, we also have $(H/\mathfrak{J}^m H)^\vee \cong {}^{\mathcal{J}}((H/\mathfrak{J}^m H))^\vee)^{\mathcal{J}}$.

Until the end of this paragraph, we assume as in §2.2.5 that the pro-p Iwahori group I is torsion free. Therefore it is a Poincaré group of dimension d. The map $\mathcal{S}^d : H^d(I, \mathbf{X}) \to k$ was introduced in §2.2.5. Assume also that $\mathbf{G}$ is almost simple and simply connected. Then in [OS3] §8, we studied E^d using the isomorphism

$$E^d \xrightarrow{\cong} ({}^{\mathcal{J}}E^{0\,\mathcal{J}})^{\vee,f} \tag{21}$$

recalled in (14). (Notice that some of the results there are true under weaker hypotheses than the ones of the current context). By Prop. 8.6 *loc. cit.*, we have an isomorphism of H-bimodules

$$E^d \cong \ker(\mathcal{S}^d) \oplus \chi_{triv}. \tag{22}$$

Proposition 2.4. *Suppose that* $\mathbf{G}$ *is almost simple and simply connected. Then we have an isomorphism of* H*-bimodules*

$$\ker(\mathcal{S}^d) \cong \bigcup_m (H/\mathfrak{J}^m H)^\vee.$$

In particular, $\ker(\mathcal{S}^d)$ *is an injective hull of the left (resp. right)* H*-module* $(H/\mathfrak{J}H)^\vee$ *and is supersingular as a left (resp. right)* H*-module.*

Proof. In fact, via (21), we have the isomorphism $\ker(\mathcal{S}^d) \cong ({}^{\mathcal{J}}\ker(\chi_{triv})^{\mathcal{J}})^{\vee,f}$ where $(\ker(\chi_{triv}))^{\vee,f}$ is the image of $(E^0)^{\vee,f}$ in the natural restriction map $(E^0)^\vee \to (\ker(\chi_{triv}))^\vee$. This gives the alternate description of $\ker(\mathcal{S}^d)$ as an H-bimodule:

$${}^{\mathcal{J}}(\ker(\mathcal{S}^d))^{\mathcal{J}} \cong \bigcup_m (\ker(\chi_{triv})/F^m H \cap \ker(\chi_{triv}))^\vee. \tag{23}$$

Recall indeed that $\mathbf{G}$ being semisimple, $H/F^m H$ is a finite dimensional vector space. On the other hand, the character χ_{triv} is not supersingular ([OS3] Remark 2.12.iv and Lemma 2.13) and therefore we have $\mathfrak{J}^m H + \ker(\chi_{triv}) = H$ for any $m \geq 1$. Hence

$$\bigcup_m (H/\mathfrak{J}^m H)^\vee = \bigcup_m (\ker(\chi_{triv})/\mathfrak{J}^m H \cap \ker(\chi_{triv}))^\vee . \tag{24}$$

But, since $\mathbf{G}$ is almost simple simply connected, [OS3] Lemma 2.14 says that

$$\mathfrak{J}^m H \cap \ker(\chi_{triv}) = \mathfrak{J}^m \cdot \ker(\chi_{triv}) \subseteq F^m H \subseteq \ker(\chi_{triv}) \qquad \text{for any } m \geq 1$$

(the left equality coming from $\mathfrak{J}^m H + \ker(\chi_{triv}) = H$). Furthermore, the braid relations imply that $F^{jm}H \subseteq (F^j H)^m$.

Fact 2.5. *There is a* $j \geq 1$ *such that* $F^j H \subseteq \mathfrak{J}H$*.*

Proof. By a finite base extension of k we may assume that $\mathbb{F}_q \subseteq k$. Then any simple supersingular H-module is a character ([OS2] Lemma 3.8). But any supersingular character of H must vanish on τ_s for at least one simple affine reflection s. There is a sufficiently large integer $r \geq 1$ such that in a reduced decomposition of an element w of length $\geq r$ every simple affine reflection occurs. This implies that $F^r H$ is contained in the intersection $\mathfrak{R}$ of all the supersingular characters.

But $\mathfrak{R}/\mathfrak{J}H$ is the Jacobson radical of the artinian ring $H/\mathfrak{J}H$. In any artinian ring the Jacobson radical is nilpotent. Hence we find an $n \geq 1$ such that $\mathfrak{R}^n \subseteq \mathfrak{J}H$. Now take $j := nr$. □

The fact implies that $F^{jm}H \subseteq \mathfrak{J}^m H$ for any $m \geq 1$. It follows that the two filtrations $\mathfrak{J}^m H \cap \ker(\chi_{triv})$ and $F^m H \cap \ker(\chi_{triv})$ of $\ker(\chi_{triv})$ are cofinal. Hence, the right hand sides of (23) and of (24) are isomorphic and we have ${}^{\mathcal{J}}(\ker(\mathcal{S}^d))^{\mathcal{J}} \cong \bigcup_m (H/\mathfrak{J}^m H)^\vee$ as H-bimodules. Now using Remark 2.3:

$$\ker(\mathcal{S}^d) \cong \bigcup_m (H/\mathfrak{J}^m H)^\vee$$

as H-bimodules and by Lemma 2.2 we have proved that $\ker(\mathcal{S}^d)$ is an injective hull of the left (resp. right) H-module $(H/\mathfrak{J}H)^\vee$.

□

2.4 The pro-p-Iwahori Hecke algebra of SL_2

For §2.4.1–2.4.6 we refer to [OS2] §3.

2.4.1 Root datum

To fix ideas we consider $I = \begin{pmatrix} 1+\mathfrak{M} & \mathfrak{O} \\ \mathfrak{M} & 1+\mathfrak{M} \end{pmatrix}$ (by abuse of notation, here and later in this paragraph, all matrices are understood to have determinant one). We let $T \subseteq G$ be the torus of diagonal matrices, T^0 its maximal compact subgroup, T^1 its maximal pro-p subgroup, and $N(T)$ the normalizer of T in G. We choose the positive root with respect to T to be $\alpha(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}) := t^2$, which corresponds to the Borel subgroup of upper triangular matrices. The affine Weyl group W sits in the short exact sequence

$$0 \longrightarrow \Omega = T^0/T^1 \longrightarrow \widetilde{W} = N(T)/T^1 \longrightarrow W = N(T)/T^0 \longrightarrow 0 \ .$$

Let $s_0 := s_\alpha := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $s_1 := \begin{pmatrix} 0 & -\pi^{-1} \\ \pi & 0 \end{pmatrix}$, and $\theta := \begin{pmatrix} \pi & 0 \\ 0 & \pi^{-1} \end{pmatrix}$, such that $s_0 s_1 = \theta$. The images of s_0 and s_1 in W are the two reflections corresponding to the two vertices of the standard edge fixed by I in the tree of G. They generate W, i.e., we have $W = \langle s_0, s_1 \rangle = \theta^{\mathbb{Z}} \dot\cup s_0 \theta^{\mathbb{Z}}$ (by abuse of notation we do not distinguish in the notation between a matrix and its image in W or $\widetilde{W}$). We let ℓ denote the length function on W corresponding to these generators as well as its pull-back to $\widetilde{W}$. One has

$$\ell(\theta^i) = |2i| \qquad \text{and} \qquad \ell(s_0\theta^i) = |1-2i|.$$

Remark 2.6. Consider $\mathbf{SL_2}(\mathfrak{F})$ as a subgroup of $\mathbf{GL_2}(\mathfrak{F})$. Then the matrix $\varpi := \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$ normalizes I; furthermore, $s_1 = \varpi s_0 \varpi^{-1}$.

2.4.2 Generators and relations

The characteristic functions $\tau_w := \mathrm{char}_{IwI}$ of the double cosets IwI form a k-basis of H when w ranges over $\widetilde{W}$. Let $e_1 := -\sum_{\omega\in\Omega} \tau_\omega$. The relations in H are

$$\tau_v \tau_w = \tau_{vw} \qquad \text{whenever } \ell(w) + \ell(v) = \ell(wv) \quad \text{and} \tag{25}$$

$$\tau_{s_i}^2 = -e_1\tau_{s_i} \qquad \text{for } i = 0, 1. \tag{26}$$

The elements τ_ω, τ_{s_i}, for $\omega \in \Omega$ and $i = 0, 1$, generate H as a k-algebra. Note that the k-algebra $k[\Omega]$ identifies naturally with a subalgebra of H via $\omega \mapsto \tau_\omega$.

The trivial character of H (see (3)) may be defined by

$$\chi_{triv} : \tau_s \longmapsto 0,\ \tau_\omega \longmapsto 1,\ \text{for } s \in \{s_0, s_1\} \text{ and } \omega \in \Omega. \tag{27}$$

The sign character χ_{sign} of H, which can be introduced in general as in [OS3] §2.2.2, is easy to describe in the current context when $\mathbf{G} = \mathrm{SL}_2$:

$$\chi_{sign} : \tau_s \longmapsto -1,\ \tau_\omega \longmapsto 1,\ \text{for } s \in \{s_0, s_1\} \text{ and } \omega \in \Omega \tag{28}$$

2.4.3 The involution ι

There is an involutive automorphism ι of H satisfying

$$\iota(\tau_s) = -e_1 - \tau_s \text{ for } s \in \{s_0, s_1\} \text{ and } \iota(\tau_\omega) = \tau_\omega \text{ for } \omega \in \Omega \tag{29}$$

(see [OS1] §4.8). For $\epsilon = 0, 1$, the following sequence of left H-modules is exact:

$$0 \longrightarrow H\tau_{s_\epsilon} \longrightarrow H \longrightarrow H\iota(\tau_{s_\epsilon}) \longrightarrow 0 \tag{30}$$

(see the remark after the proof of [OS2] Prop. 3.54).

For a left (resp. right) H-module M, we denote by ιM (resp. $M\iota$) the H-module on the space M with the action of H twisted by ι.

2.4.4 The central element ζ

We refer to [OS2] §3.2.2. Consider the element

$$\zeta := (\tau_{s_0} + e_1)(\tau_{s_1} + e_1) + \tau_{s_1}\tau_{s_0} = (\tau_{s_1} + e_1)(\tau_{s_0} + e_1) + \tau_{s_0}\tau_{s_1}. \tag{31}$$

Notice that $\mathcal{J}(\zeta) = \zeta$ and that $\chi_{triv}(\zeta) = \chi_{sign}(\zeta) = 1$. The element ζ is central in H, and the subalgebra $k[\zeta]$ of H generated by ζ is the algebra of polynomials in the variable ζ. Furthermore, ζ is not a zero divisor in H and the k-algebra $H/H\zeta$ is finite dimensional (see for example [OS2] Lemma 1.3). We will denote by H_ζ the algebra obtained by localizing H in ζ. The anti-involution $\mathcal{J}$ extends to H_ζ. The involution ι also fixes ζ and induces an involutive automorphism of H_ζ.

For $\epsilon = 0, 1$, define H_ϵ to be the subalgebra of H generated by τ_{s_ϵ}, τ_ω, $\omega \in \Omega$. The following result is [OS2] Cor. 3.4.

Lemma 2.7. *Let $\epsilon = 0$ or 1; the morphism of $(H_\epsilon, k[\zeta])$-bimodules*

$$\begin{array}{ccccc} H_\epsilon \otimes_k k[\zeta] & \oplus & H_\epsilon \otimes_k k[\zeta] & \longrightarrow & H \\ 1 \otimes 1 & & & \longmapsto & 1 \\ & & 1 \otimes 1 & \longmapsto & \tau_{s_{1-\epsilon}} \end{array}$$

is an isomorphism. In particular, H is a free and finitely generated $k[\zeta]$-module of rank $4(q-1)$.

Fact 2.8. *Suppose that $\mathbb{F}_q \subseteq k$ and that $p \neq 2$ or $\mathfrak{F} = \mathbb{Q}_p$. Then for V an irreducible quotient of $\mathbf{X}e_1/\mathbf{X}e_1(\zeta - 1)$ we have $V^I \cong \chi_{triv}$ or $V^I \cong \chi_{sign}$ as a left H-module.*

Proof. A basis of $He_1/He_1(\zeta-1)$ is given by the image in the quotient of

$$\iota(\tau_{s_0})\tau_{s_1}e_1,\ \tau_{s_0}\iota(\tau_{s_1})e_1,\ \iota(\tau_{s_0})e_1,\ \tau_{s_0}e_1$$

(compare with Lemma 2.7). The elements $\iota(\tau_{s_0})\tau_{s_1}e_1$ and $\tau_{s_0}\iota(\tau_{s_1})e_1$ support respectively the characters χ_{triv} and χ_{sign}. This follows from using repeatedly $\iota(\tau_{s_0})\tau_{s_1}e_1+\iota(\tau_{s_1})\tau_{s_0}e_1=(-\zeta+1)e_1\equiv 0$ in $He_1/He_1(\zeta-1)$ and likewise $\tau_{s_0}\iota(\tau_{s_1})e_1+\tau_{s_1}\iota(\tau_{s_0})e_1\equiv 0$ in $He_1/He_1(\zeta-1)$. Then it is easy to see that in the resulting quotient, $\iota(\tau_{s_0})e_1$ and $\tau_{s_0}e_1$ support respectively the characters χ_{triv} and χ_{sign}. So we have an exact sequence of left H-modules

$$0\to\chi_{triv}\oplus\chi_{sign}\to He_1/He_1(\zeta-1)\to\chi_{triv}\oplus\chi_{sign}\to 0\ . \tag{32}$$

All the modules in question are annihilated by $\zeta-1$ so they are H_ζ-modules. Suppose furthermore that $\mathbb{F}_q\subseteq k$ and that $\mathfrak{F}=\mathbb{Q}_p$ or $p\neq 2$. We may apply [OS2] Thm. 3.33 which ensures that the functor $\mathbf{X}\otimes_H\_$ is exact on (32), provides an exact sequence of G representations

$$0\to\mathbf{X}\otimes_H\chi_{triv}\oplus\mathbf{X}\otimes_H\chi_{sign}\to\mathbf{X}e_1/\mathbf{X}e_1(\zeta-1)\to\mathbf{X}\otimes_H\chi_{triv}\oplus\mathbf{X}\otimes_H\chi_{sign}\to 0$$

and that for $\chi\in\{\chi_{sign},\chi_{triv}\}$ we have $(\mathbf{X}\otimes_H\chi)^I\cong\chi$ and therefore $\mathbf{X}\otimes_H\chi$ is an irreducible representation of G. Therefore any irreducible quotient of $\mathbf{X}e_1/\mathbf{X}e_1(\zeta-1)$ is isomorphic to $\mathbf{X}\otimes_H\chi_{triv}$ or $\mathbf{X}\otimes_H\chi_{sign}$. $\square$

Remark 2.9. After localizing (30) in ζ we get an exact sequence of left H_ζ-modules

$$0\longrightarrow H_\zeta\tau_{s_\epsilon}\longrightarrow H_\zeta\longrightarrow H_\zeta\iota(\tau_{s_\epsilon})\longrightarrow 0\ . \tag{33}$$

Notice that the map $h\mapsto\zeta^{-1}h\tau_{s_{1-\epsilon}}\tau_{s_\epsilon}$ splits the inclusion $H_\zeta\tau_{s_\epsilon}\longrightarrow H_\zeta$ because $\zeta\tau_{s_\epsilon}=\tau_{s_\epsilon}\tau_{s_{1-\epsilon}}\tau_{s_\epsilon}$ (compare with the proof of [OS2] Lemma 3.30). So we have $H_\zeta\cong H_\zeta\tau_{s_\epsilon}\oplus H_\zeta\iota(\tau_{s_\epsilon})$ as left H_ζ-modules.

Remark 2.10. The element ζ depends on the choice of the uniformizer π. Let $u\in\mathfrak{O}^\times$. We verify that if we pick $u\pi$ as a uniformizer, the new corresponding central element ζ_u is

$$\zeta_u:=\tau_{\omega_{u^{-1}}}(\tau_{s_0}+e_1)(\tau_{s_1}+e_1)+\tau_{\omega_u}\tau_{s_1}\tau_{s_0}=\tau_{\omega_u}(\tau_{s_1}+e_1)(\tau_{s_0}+e_1)+\tau_{\omega_{u^{-1}}}\tau_{s_0}\tau_{s_1} \tag{34}$$

where ω_u is the element $\left(\begin{smallmatrix}u^{-1}&0\\0&u\end{smallmatrix}\right)T^1\in\Omega$. Of course we have $\zeta=\zeta_1$. A system of generators of the center Z of H as a k-vector space is given by the set of all ζ_u for u ranging over a system of representatives of $(\mathfrak{O}/\mathfrak{M})^\times$ (to which one has to add τ_1 if $p=2$) (see [OS2] (24) in Remark 3.5).

We have the formula: $\zeta_{u_1}\zeta_{u_2}=\zeta_{u_1u_2}\zeta$ for any $u_1,u_2\in\mathfrak{O}^\times$. In particular

$$\zeta_u\,\zeta_{u^{-1}}=\zeta^2\quad\text{and}\quad\zeta_{u^2}\,\zeta=\zeta_u^2\ . \tag{35}$$

These identities ensure that the localized algebra H_ζ does not depend on the choice of the uniformizer.

2.4.5 Supersingularity

In the current context where $\mathbf{G}=\mathrm{SL}_2$, the ideal $\mathfrak{J}$ introduced in §2.1 is the central ideal $\zeta k[\zeta]$. Following the definition introduced in that paragraph, an H-module M is called supersingular if any element in M is annihilated by a power of ζ.

Remark 2.11. Let $u\in\mathfrak{O}^\times$. From (35), one easily deduces that an element in M is annihilated by a power of ζ if and only if it is annihilated by a power of ζ_u. Therefore, even if ζ does depend on the choice of a uniformizer, the notion of supersingularity does not.

2.4.6 Idempotents

The element e_1 is a central idempotent in H. More generally, to any k-character $\lambda : \Omega \to k^\times$ of Ω, we associate the following idempotent in H:

$$e_\lambda := -\sum_{\omega \in \Omega} \lambda(\omega^{-1})\tau_\omega \ . \tag{36}$$

Note that $\mathcal{J}(e_\lambda) = e_{\lambda^{-1}}$ and $e_\lambda \tau_\omega = \tau_\omega e_\lambda = \lambda(\omega) e_\lambda$ for any $\omega \in \Omega$. We parameterize Ω by the isomorphism

$$\begin{aligned} (\mathfrak{O}/\mathfrak{M})^\times &\xrightarrow{\cong} \Omega \\ u &\longmapsto \omega_u := \begin{pmatrix} [u]^{-1} & 0 \\ 0 & [u] \end{pmatrix} T^1 \ , \end{aligned} \tag{37}$$

where $[u]$ is a lift in $\mathfrak{O}$ for $u \in (\mathfrak{O}/\mathfrak{M})^\times$, and we pick the multiplicative Teichmüller lift.

Remark 2.12. Given a homomorphism of groups $\Lambda : (\mathfrak{O}/\mathfrak{M})^\times \to k^\times$, we may consider the character $\lambda : \Omega \to k^\times$ obtained via composition with the inverse of (37) and the corresponding idempotent as in (36). We will then use the shortcut e_Λ to denote the latter. This will be used in the following context:

If $q = p$ we have the homomorphism $\mathrm{id} : (\mathfrak{O}/\mathfrak{M})^\times = \mathbb{F}_p^\times \xrightarrow{\subseteq} k^\times$, which will play an important role later on. For $m \in \mathbb{Z}$, we will consider the idempotent element

$$e_{\mathrm{id}^m} \in k[\Omega] \tag{38}$$

with the above convention. When $m = 0$ this is consistent with the notation e_1 in §2.4.2.

Suppose for a moment that $\mathbb{F}_q \subseteq k$. Then all simple modules of $k[\Omega]$ are one dimensional. The set $\widehat{\Omega}$ of all k-characters of Ω has cardinality $q-1$ which is prime to p. This implies that the family $\{e_\lambda\}_\lambda \in \widehat{\Omega}$ is a family of orthogonal idempotents with sum equal to 1. It gives the ring decomposition $k[\Omega] = \prod_{\lambda \in \widehat{\Omega}} k e_\lambda$. Let $\Gamma := \{\{\lambda, \lambda^{-1}\} : \lambda \in \widehat{\Omega}\}$ denote the set of s_0-orbits in $\widehat{\Omega}$. To $\gamma \in \Gamma$ we attach the element $e_\gamma := e_\lambda + e_{\lambda^{-1}}$ (resp. $e_\gamma := e_\lambda$) if $\gamma = \{\lambda, \lambda^{-1}\}$ with $\lambda \neq \lambda^{-1}$ (resp. $\gamma = \{\lambda\}$). Using the braid relations, one sees that e_γ is a central idempotent in H and we have the ring decomposition $H = \prod_{\gamma \in \Gamma} H e_\gamma$. If $q = p$ then the idempotent

$$e_{\gamma_0} := e_{\mathrm{id}} + e_{\mathrm{id}^{-1}} \tag{39}$$

will be of particular importance (see (38)).

2.4.7 Certain H-modules

For later purposes we construct in this section certain families of H-modules. The reader may skip this at first reading coming back to it only when needed. We fix a homomorphism of k-algebras $\kappa : H \to R$ as well as an element $z \in Z(R)$ in the center of R. Let $M_2(R)$ denote, as usual, the algebra of 2 by 2 matrices over R. We also fix a character $\mu : \Omega \to k^\times$. With these choices we define the matrices

$$M_0 := \begin{pmatrix} -\kappa(e_\mu) & 0 \\ z\kappa(\tau_{s_1}) & 0 \end{pmatrix} \ , \ M_1 := \begin{pmatrix} 0 & z\kappa(\tau_{s_0}) \\ 0 & -\kappa(e_{\mu^{-1}}) \end{pmatrix} \ , \text{ and } M_\omega := \begin{pmatrix} \mu^{-1}(\omega)\kappa(\tau_\omega) & 0 \\ 0 & \mu(\omega)\kappa(\tau_\omega) \end{pmatrix} \text{ for } \omega \in \Omega.$$

It is straightforward to check that these matrices satisfy the relations

$$M_i^2 = \sum_{\omega \in \Omega} M_\omega M_i \ , \ M_\omega M_i = M_i M_{\omega^{-1}} \ , \text{ and } M_\omega M_{\omega'} = M_{\omega\omega'} \ .$$

Hence we obtain a k-algebra homomorphism $\kappa_2 : H \to M_2(R)$ by sending τ_{s_i} to M_i and τ_ω to M_ω. By using this homomorphism to equip the left R-module $R \oplus R$ with a right H-module structure we obtain an (R, H)-bimodule denoted by $(R \oplus R)[\kappa, z, \mu]$.

2.4.8 Frobenius extensions

The space $\mathrm{Hom}_{k[\zeta]}(H, k[\zeta])$ is naturally an H-bimodule via $(h, \Lambda, h') \mapsto \Lambda(h'\_h)$.

Proposition 2.13. *We have an isomorphism of H-bimodules*

$$\iota H \cong H\iota \cong \mathrm{Hom}_{k[\zeta]}(H, k[\zeta]) \ .$$

Proof. The first isomorphism is given by the map $\iota : H \to H$. From Lemma 2.7 we know that H is a free $k[\zeta]$-module with basis the set of all τ_w for w ranging over the set

$$\omega,\ \omega s_0,\ \omega s_1,\ \omega s_0 s_1 \text{ when } \omega \in \Omega. \tag{40}$$

We define in $\mathrm{Hom}_{k[\zeta]}(H, k[\zeta])$ the dual basis, namely for each $x \in$ (40), we define the map $\Lambda_x \in \mathrm{Hom}_{k[\zeta]}(H, k[\zeta])$ which sends each τ_y with $y \in$ (40) to 0 except $\Lambda_x(\tau_x) = 1 \in k[\zeta]$. We check that

$$\Lambda_{s_0 s_1}(\tau\tau') = \Lambda_{s_0 s_1}(\iota(\tau')\tau) \tag{41}$$

which ensures that

$$f : \iota H \longrightarrow \mathrm{Hom}_{k[\zeta]}(H, k[\zeta]) \tag{42}$$

$$\tau \longmapsto f(\tau)(\tau') := \Lambda_{s_0 s_1}(\tau\tau') \tag{43}$$

defines a homomorphism of H-bimodules.

Let $w, w' \in \widetilde{W}$ and $\tau := \tau_w$, $\tau' := \tau_{w'}$. Since $\Lambda_{s_0 s_1}$ is $k[\zeta]$-linear it is enough to verify (41) when $w, w' \in$ (40). And in fact it is easy to see that both sides of (41) are then zero except possibly in the following cases. Let $\omega, \omega' \in \Omega$. The verifications below rely on the quadratic formulas (26) and the expression $\zeta = (\tau_{s_0} + e_1)(\tau_{s_1} + e_1) + \tau_{s_1}\tau_{s_0} = (\tau_{s_1} + e_1)(\tau_{s_0} + e_1) + \tau_{s_0}\tau_{s_1}$. We spell out a few of them.

- If $w = \omega s_0$ and $w' = \omega' s_1$, we have $\Lambda_{s_0 s_1}(\tau\tau') = \Lambda_{s_0 s_1}(\tau_{\omega\omega'^{-1}}\tau_{s_0 s_1})$ which is equal to 1 if $\omega = \omega'$ and to 0 otherwise.
  We have $\Lambda_{s_0 s_1}(\iota(\tau')\tau) = -\Lambda_{s_0 s_1}(\tau_{\omega'\omega^{-1}}(\tau_{s_1} + e_1)\tau_{s_0}) = -\Lambda_{s_0 s_1}(\tau_{\omega'\omega^{-1}}(\zeta - (\tau_{s_1} e_1 + e_1) - \tau_{s_0 s_1})) = \Lambda_{s_0 s_1}(\tau_{\omega'\omega^{-1} s_0 s_1})$ which is also equal to 1 if $\omega = \omega'$ and to 0 otherwise.

- If $w = \omega s_1$ and $w' = \omega' s_0$, we easily check that both $\Lambda_{s_0 s_1}(\tau\tau')$ and $\Lambda_{s_0 s_1}(\iota(\tau')\tau)$are equal to -1 if $\omega = \omega'$ and to 0 otherwise.

- If $w = \omega s_0$ and $w' = \omega' s_0 s_1$, we have $\Lambda_{s_0 s_1}(\tau\tau') = -\Lambda_{s_0 s_1}(\tau_{\omega\omega'^{-1}} e_1 \tau_{s_0 s_1}) = -\Lambda_{s_0 s_1}(e_1 \tau_{s_0 s_1})$ which is equal to 1.

  We compute

  $$\begin{aligned}\Lambda_{s_0 s_1}(\iota(\tau')\tau) &= \Lambda_{s_0 s_1}(\tau_{\omega'\omega}(\tau_{s_0} + e_1)(\tau_{s_1} + e_1)\tau_{s_0}) = \Lambda_{s_0 s_1}(\tau_{\omega'\omega}(\zeta - \tau_{s_1 s_0})\tau_{s_0}) \\ &= \Lambda_{s_0 s_1}(\tau_{\omega'\omega} e_1 \tau_{s_1 s_0}) = \Lambda_{s_0 s_1}(e_1 \tau_{s_1 s_0}) = -\Lambda_{s_0 s_1}(\sum_{u \in \Omega} \tau_u \tau_{s_1 s_0})\end{aligned}$$

  which is equal to 1 (see the previous case).

- If $w = \omega s_0 s_1$ and $w' = \omega' s_1$, we check that $\Lambda_{s_0 s_1}(\tau\tau') = \Lambda_{s_0 s_1}(\iota(\tau')\tau) = 1$.

- If $w = \omega s_1$ and $w' = \omega' s_0 s_1$, we have $\Lambda_{s_0 s_1}(\tau\tau') = \Lambda_{s_0 s_1}(\tau_{\omega\omega'^{-1}}\tau_{s_1 s_0 s_1}) = \Lambda_{s_0 s_1}(\tau_{\omega\omega'^{-1}}\zeta\tau_{s_1}) = 0$. We have $\Lambda_{s_0 s_1}(\iota(\tau')\tau) = \Lambda_{s_0 s_1}(\tau_{\omega'\omega}(\tau_{s_0} + e_1)(\tau_{s_1} + e_1)\tau_{s_1}) = 0$.

- If $w = \omega s_0 s_1$ and $w' = \omega' s_0$, we have likewise $\Lambda_{s_0 s_1}(\tau\tau') = \Lambda_{s_0 s_1}(\iota(\tau')\tau) = 0$.

- If $w = \omega s_0 s_1$ and $w' = \omega' s_0 s_1$, we have $\Lambda_{s_0s_1}(\tau\tau') = \Lambda_{s_0s_1}(\tau_{\omega\omega'}\tau_{s_0s_1s_0s_1}) = \Lambda_{s_0s_1}(\tau_{\omega\omega'}\zeta\tau_{s_0s_1})$ which is equal to ζ if $\omega' = \omega^{-1}$ and to 0 otherwise. We have
$\Lambda_{s_0s_1}(\iota(\tau')\tau) = \Lambda_{s_0s_1}(\tau_{\omega'\omega}(\tau_{s_0}+e_1)(\tau_{s_1}+e_1)\tau_{s_0s_1}) = \Lambda_{s_0s_1}(\tau_{\omega'\omega}(\zeta-\tau_{s_1s_0})\tau_{s_0s_1}) = \Lambda_{s_0s_1}(\tau_{\omega'\omega}(\zeta\tau_{s_0s_1} - \zeta e_1\tau_{s_1}))$ which is also equal to ζ if $\omega' = \omega^{-1}$ and to 0 otherwise.

To prove that (42) is surjective, we verify the following. We have

a) $-\tau_{s_0} \cdot \Lambda_{s_0s_1} = \Lambda_{s_1}$.

b) $(\tau_{s_1} + e_1) \cdot \Lambda_{s_0s_1} = \Lambda_{s_0}$.

c) $-(\tau_{s_1} + e_1)\tau_{s_0} \cdot \Lambda_{s_0s_1} = \Lambda_1$.

d) for all $w \in$ (40) and $\omega \in \Omega$, we have $\Lambda_w \cdot \tau_{\omega^{-1}} = \Lambda_{\omega w}$.

Property d) is immediate. The other properties are easily verified by evaluating explicitly the left hand side at all elements of the form τ_w for $w \in$ (40). For example $-(\tau_{s_1}+e_1)\tau_{s_0}\cdot\Lambda_{s_0s_1}(\tau_\omega) = -\Lambda_{s_0s_1}(\tau_\omega(\tau_{s_1}+e_1)\tau_{s_0})$ which we already computed above is equal to 1 if $\omega = 1$ and to 0 otherwise.

Once it is proved that (42) is surjective, the injectivity is immediate since both spaces are free $k[\zeta]$-modules of the same rank. □

Using a free resolution of any arbitrary left (resp. right) $k[\zeta]$-module, and since H is finitely generated free hence projective over $k[\zeta]$, it follows immediately:

Corollary 2.14. *Let M be a left, resp. right, $k[\zeta]$-module. We have an isomorphism of left, resp. right, H-modules*

$$H \otimes_{k[\zeta]} M \cong \iota H \otimes_{k[\zeta]} M \cong \mathrm{Hom}_{k[\zeta]}(H, M) \quad \textit{resp.} \quad M \otimes_{k[\zeta]} H \cong M \otimes_{k[\zeta]} H\iota \cong \mathrm{Hom}_{k[\zeta]}(H, M)\ .$$

Proof. For the left hand isomorphisms note that ιH (resp. $H\iota$) is naturally isomorphic to H as an $(H, k[\zeta])$-bimodule (resp as a $(k[\zeta], H)$-bimodule) since ι fixes ζ. □

Corollary 2.15. *For $a \in k$, the finite dimensional k-algebra $H/(\zeta - a)H$ is Frobenius.*

Proof. The isomorphism of H-bimodules (42) clearly factors through an isomomorphism of $H/(\zeta-a)H$-bimodules

$$\iota(H/(\zeta - a)H) \cong \mathrm{Hom}_{k[\zeta]}(H/(\zeta - a)H, k[\zeta]/(\zeta - a)) \cong \mathrm{Hom}_k(H/(\zeta - a)H, k)\ . \tag{44}$$

□

2.4.9 Finite duals

We consider the finite dual $H^{\vee,f}$ of H (see §2.2.5) with basis $(\tau_w^\vee)_{w\in\widetilde{W}}$ defined to be the dual of $(\tau_w)_{w\in\widetilde{W}}$. When I is a Poincaré group of dimension d, we have an isomorphism between E^d and the twisted H-bimodule ${}^{\jmath}(H^{\vee,f})^{\jmath}$ given by (14). In §2.2.5, just like in[OS3] §8, we denoted by ϕ_w the element of E^d corresponding to $\tau_w^\vee$ and we computed in Prop. 8.2 *loc. cit* that the structure of H-bimodule of ${}^{\jmath}(H^{\vee,f})^{\jmath}$ is given by the following formulas. Let $w \in \widetilde{W}$, $\omega \in \widetilde{\Omega}$ and $s \in \{s_0, s_1\}$.

$$\tau_w^\vee \cdot \tau_\omega = \tau_{w\omega}^\vee\ ,\ \tau_\omega \cdot \tau_w^\vee = \tau_{\omega w}^\vee, \tag{45}$$

$$\tau_w^\vee \cdot \tau_s = \begin{cases} \tau_{ws}^\vee - \tau_w^\vee \cdot e_1 & \text{if } \ell(ws) = \ell(w) - 1, \\ 0 & \text{if } \ell(w\tilde{s}) = \ell(w) + 1, \end{cases} \qquad \tau_s \cdot \tau_w^\vee = \begin{cases} \tau_{sw}^\vee - e_1 \cdot \tau_w^\vee & \text{if } \ell(sw) = \ell(w) - 1, \\ 0 & \text{if } \ell(sw) = \ell(w) + 1. \end{cases} \tag{46}$$

Remark 2.16. For all $w \in \widetilde{W}$ with length ≥ 1, there is a unique $\epsilon \in \{0,1\}$ such that $\ell(s_\epsilon w) = \ell(w) - 1$. We let $\psi_w := \tau_{s_\epsilon} \cdot \phi_w = \phi_{s_\epsilon w} - e_1 \cdot \phi_w$. From the formulas above we get $\zeta \cdot \psi_w = \psi_{s_{1-\epsilon} s_\epsilon w}$ if $\ell(w) \geq 3$ and $\zeta \cdot \psi_w = 0$ if $\ell(w) = 1, 2$. So the subspace Ψ generated by the ψ_w is of ζ-torsion and contained in $\ker(\mathcal{S}^d)$. We show that this subspace is in fact equal to $\ker(\mathcal{S}^d)$. First of all we recall from the proof of [OS3] Prop. 8.6 that $E^d = \ker(\mathcal{S}^d) \oplus k e_1 \cdot \phi_1$. Then we notice that Ψ is stable under the left action of τ_ω for $\omega \in \Omega$. So $\Psi = e_1 \cdot \Psi \oplus (1 - e_1) \cdot \Psi$, and it is enough to show that $(1 - e_1) \cdot \Psi = (1 - e_1) \cdot E^d$ and $e_1 \cdot \Psi \oplus k e_1 \cdot \phi_1 = e_1 \cdot E^d$. The first identity is true because, for $w \in \widetilde{W}$, there exists $\eta \in \{0,1\}$ such that $\ell(s_\eta w) = \ell(w) + 1$ and $(1 - e_1) \cdot \phi_w = (1 - e_1) \cdot \psi_{s_\eta^{-1} w}$. To prove the second identity, we let $w \in \widetilde{W}$. If $\ell(w) = 0$, then $e_1 \cdot \phi_w = e_1 \cdot \phi_1$. If $\ell(w) > 1$, let $\epsilon \in \{0,1\}$ such that $\ell(s_\epsilon w) = \ell(w) - 1$. Then $e_1 \cdot \phi_w = e_1 \cdot \phi_{s_\epsilon w} - e_1 \cdot \psi_w$ lies in $e_1 \cdot \Psi \oplus k e_1 \cdot \phi_1$ by induction on $\ell(w)$.

Let $m \geq 1$. The restriction map $H^{\vee,f} \to (F^m H)^{\vee,f}$ is a homomorphism of H-bimodules and makes the finite dual $(F^m H)^{\vee,f}$ of $F^m H$ a quotient of the H-bimodule $H^{\vee,f}$. Furthermore, $(F^m H / F^{m+1} H)^\vee$ identifies with the sub-H-bimodule of $(F^m H)^{\vee,f}$ of the linear forms which are trivial on $F^{m+1} H$. We consider the linear map defined by

$$\begin{aligned} F^m H / F^{m+1} H &\longrightarrow {}^{\jmath}((F^m H / F^{m+1} H)^\vee)^{\jmath} \\ \tau_w &\longmapsto \tau_w^\vee|_{F^m H} \text{ for } w \in \widetilde{W} \text{ such that } \ell(w) = m \end{aligned} \tag{47}$$

By the above formulas, it is an isomorphism of H-bimodules.

2.4.10 The equivalence of categories

When $G = \mathrm{SL}_2(\mathbb{Q}_p)$, the functors $H^0(I, \_)$ and $\mathbf{X} \otimes_H \_$ are quasi-inverse equivalences between the category $\mathrm{Mod}^I(G)$ of all smooth representations generated by their I-fixed vectors and the category of left H-modules. In particular, $H^0(I, \_)$ is exact in $\mathrm{Mod}^I(G)$. (See [OS2] Prop. 3.25).

2.5 On some values of the functor $H^d(I, \_)$ when $\mathbf{G} = \mathrm{SL}_2$

We assume that $\mathbf{G} = \mathrm{SL}_2$ and that I is torsion free and therefore a Poincaré group of dimension d. It follows, in particular, that $p \geq 5$. By (22) and Proposition 2.4 we have

$$E^d \cong \ker(\mathcal{S}^d) \oplus \chi_{triv}$$

as H-bimodules where $\ker(\mathcal{S}^d) \cong \bigcup_{n \geq 1} (H / \zeta^n H)^\vee$. As a left or right H-module, $\ker(\mathcal{S}^d)$ is an injective envelope of $(H / \zeta H)^\vee$. Being injective, this is a ξ-divisible module on the left, resp. right, for any $\xi \in H$ which is a non-zero-divisor. For example, we know that H is free over $k[\zeta]$ (Lemma 2.7) so $Q(\zeta)$ is a non-zero-divisor for any nonzero polynomial $Q(X) \in k[X]$. If furthermore $\chi_{triv}(\xi) \neq 0$, then the whole space E^d is ξ-divisible. Recall that $\chi_{triv}(\zeta) = 1$.

Remark 2.17. χ_{triv} is the only nontrivial finite dimensional quotient of E^d as a left or right H-module.

Proof. Since $\ker(\mathcal{S}^d)$ is left and right ζ-torsion, a finite dimensional quotient of $\ker(\mathcal{S}^d)$ as a left, resp. right, module is annihilated by a power ζ^m of ζ from the left, resp. right. But $\ker(\mathcal{S}^d) \cdot \zeta^m = \zeta^m \cdot \ker(\mathcal{S}^d) = \ker(\mathcal{S}^d)$ since $\ker(\mathcal{S}^d)$ is ζ-divisible. Therefore any finite dimensional module quotient of $\ker(\mathcal{S}^d)$ is zero. ☐

Recall that $H^d(I, -)$ is a right exact functor which commutes with arbitrary direct sums. By choosing a free presentation of an arbitrary left H-module M this easily implies the formula

$$H^d(I, \mathbf{X} \otimes_H M) \cong E^d \otimes_H M.$$

This is an isomorphism of left H-modules.

Proposition 2.18. *Let $G = \mathrm{SL}_2(\mathfrak{F})$. For any non-zero-divisor $\xi \in H$ such that ξ is central in H and $\chi_{triv}(\xi) \neq 0$, we have $H^d(I, \mathbf{X}/\mathbf{X}\xi) = 0$.*

Proof. Using the equality $\mathbf{X}/\mathbf{X}\xi = \mathbf{X} \otimes_H H/H\xi$ we compute

$$\begin{aligned} H^d(I, \mathbf{X}/\mathbf{X}\xi) = E^d \otimes_H H/H\xi &= \chi_{triv} \otimes_H H/H\xi \oplus \ker(\mathcal{S}^d) \otimes_H H/H\xi \\ &= k/\chi_{triv}(\xi)k \oplus \ker(\mathcal{S}^d)/\ker(\mathcal{S}^d)\xi = 0 \ . \end{aligned}$$

□

Corollary 2.19. *Let $Q(X) \in k[X]$ be a nonzero polynomial. Then $H^d(I, \mathbf{X}/\mathbf{X}Q(\zeta)) = 0$, resp. $\cong \chi_{triv}$ as an H-bimodule, if $Q(1) \neq 0$, resp. $Q(1) = 0$.*

Proof. For the second part of the result, we simply notice that $\chi_{triv} \otimes_H H/HQ(\zeta) \cong \chi_{triv}$ as a left H-module. Therefore, proceeding as above, we obtain an isomorphism of left H-modules $H^d(I, \mathbf{X}/\mathbf{X}Q(\zeta)) \cong \chi_{triv}$. By Remark 2.17 this is an isomorphism of H-bimodules because $H^d(I, \mathbf{X}/\mathbf{X}Q(\zeta))$ is a one-dimensional quotient of E^d. □

Proposition 2.20. *We have $H^d(I, V) = 0$ for any irreducible admissible representation of $G := \mathrm{SL}_2(\mathfrak{F})$ except when $V = k_{triv}$ is the trivial representation in which case:*

$$H^d(I, k_{triv}) \cong \chi_{triv} \qquad \text{as an } H\text{-bimodule.}$$

Proof. The case when $V = k_{triv}$ is the trivial representation of G is a particular case of [OS3] Prop. 8.4.i. For the rest of the proof we therefore assume that $V \not\cong k_{triv}$. We first make we the following observations. Let $\bar{k}/k$ denote an algebraic closure of k. Then the scalar extension $V_{\bar{k}} := \bar{k} \otimes_k V$ is a smooth G-representation over $\bar{k}$.

- Since $H^d(I, -)$ commutes with arbitrary direct sums we have $H^d(I, V_{\bar{k}}) = H^d(I, V) \otimes_k \bar{k}$.
- Since V is admissible $\mathrm{End}_{\mathrm{Mod}(G)}(V)$ is finite dimensional over k.
- The G-representation $V_{\bar{k}}$ is of finite length with each irreducible constituent being admissible and not isomorphic to k_{triv} ([HV] Thm. III.4.1)-2), which needs the previous point as input).

By an argument with the exact cohomology sequence these observations reduce us to proving our assertion over $\bar{k}$. In fact, all we need in the following is that $\mathbb{F}_q \subseteq k$.

Given an irreducible admissible representation V of G, the space V^I is finite dimensional. Let $Q \in k[X]$ denote the minimum polynomial of ζ on V^I, so that $Q(\zeta)V^I = 0$. We claim that V is a quotient representation of $\mathbf{X}/\mathbf{X}Q(\zeta)$. For this we choose a nonzero vector $v_0 \in V^I$, which gives rise to the surjective G-equivariant map $\mathbf{X} \twoheadrightarrow V$ sending gI to gv_0. It restricts to the map $H \to V^I$ sending char_I to v_0. But $(gI)Q(\zeta) = gQ(\zeta) \mapsto gQ(\zeta)v_0 = 0$. It follows that the initial map factors over $\mathbf{X}/\mathbf{X}Q(\zeta)$.

If $Q(1) \neq 0$, then we have $H^d(I, \mathbf{X}/\mathbf{X}Q(\zeta)) = 0$, but $H^d(I, V)$ being a quotient of that space it is also zero. It remains to treat the case $Q(1) = 0$. Then we can choose the above vector v_0 so that $(\zeta - 1)v_0 = 0$ and $M := Hv_0$ is a simple H-submodule of V^I. Since ζ is the identity on M, it follows from [OS2] Thm. 3.33 that $\mathbf{X} \otimes_H M$ is an irreducible G-representation with $(\mathbf{X} \otimes_H M)^I = M$. The inclusion $M \subseteq V^I$ induces a nonzero map $\mathbf{X} \otimes_H M \to V$ which by irreducibility must be an isomorphism. It follows that $V = \mathbf{X} \otimes_H V^I$ and that V^I is a simple H-module. Hence there is a unique $\gamma \in \Gamma$ such that $V^I = e_\gamma V^I$ (notation in §2.4.6). It further follows that $H^d(I, V) = H^d(I, \mathbf{X} \otimes_H V^I) = E^d \otimes_H V^I \cong \chi_{triv} \otimes_H V^I \oplus \ker(\mathcal{S}^d) \otimes_H V^I = \chi_{triv} \otimes_H V^I$, the latter equality since $\ker(\mathcal{S}^d)$ is divisible by $\zeta - 1$.

- If $\gamma \neq \{1\}$, then the idempotent e_γ satisfies $\chi_{triv}(e_\gamma) = 0$, so $H^d(I, V) = \{0\}$.

- If $\gamma = \{1\}$, then we use Fact 2.8 to deduce that $V^I \cong \chi_{triv}$ or $V^I \cong \chi_{sign}$. If $V^I \cong \chi_{sign}$, then $\chi_{triv} \otimes_H V^I = \{0\}$ because $\chi_{triv}(\tau_{s_0}) = 0$ and $\chi_{sign}(\tau_{s_0}) = -1$. If $V^I = \chi_{triv}$, then by [OV] Lemma 2.25 we know that $\mathbf{X} \otimes_H V^I \cong k_{triv}$ so $V = k_{triv}$.

□

Remark 2.21. Let $z \in H$ be a central element H. Then $\mathcal{J}(z)$ is also a central element and from the isomorphism $\Delta^d : \quad E^d \xrightarrow{\cong} ({}^{\mathcal{J}}E^{0\,\mathcal{J}})^{\vee,f}$ (see (14)) we deduce that z centralizes the elements of the H-bimodule E^d, namely $z \cdot \phi = \phi \cdot z$ for any $\phi \in E^d$. In particular the left and the right actions on E^d of the central element $\zeta \in H$ coincide.

Lemma 2.22. *The kernel of the (left or right) action of ζ on E^d is isomorphic to $\iota(H/\zeta H)$ as an H-bimodule.*

Proof. By (22) and Proposition 2.4, we have $E^d \cong \bigcup_{n\geq 1}(H/\zeta^n H)^\vee \oplus \chi_{triv}$ as H-bimodules. Recall that $\chi_{triv}(\zeta) = 1$. The kernel of the action of ζ on $\bigcup_{n\geq 1}(H/\zeta^n H)^\vee \oplus \chi_{triv}$ is isomorphic to the H-bimodule $(H/\zeta H)^\vee$ which, by (44), is isomorphic to $\iota(H/\zeta H)$. □

3 Formulas for the left action of H on E^1 when $\mathbf{G} = \mathrm{SL}_2(\mathbb{Q}_p)$, $p \neq 2, 3$

There is no hypothesis on $\mathfrak{F}$ and $G = \mathrm{SL}_2(\mathfrak{F})$ in §3.1 –§3.5 with the exception that we assume $p \neq 2$ from §3.2 on.

3.1 Conjugation by ϖ

Recall the matrix $\varpi := \left(\begin{smallmatrix} 0 & 1 \\ \pi & 0 \end{smallmatrix}\right)$ (Remark 2.6) which normalizes the Iwahori subgroup J and its pro-p Sylow I as well as the torus T. We apply Section 2.2.6 to the following automorphism of the pair $(G, \mathbf{X})$:

$$\xi : G \longrightarrow G, \quad g \longmapsto \varpi^{-1} g \varpi \quad \text{and} \quad \mathcal{X} : \mathbf{X} \longrightarrow \mathbf{X}, \quad f \longmapsto f \circ \xi \ (\text{resp. } gI \mapsto \varpi g \varpi^{-1} I). \tag{48}$$

It gives rise to the involutive automorphism

$$\Gamma_\varpi := \Gamma_\xi : E^* = H^*(I, \mathbf{X}) \longrightarrow E^* = H^*(I, \mathbf{X}) \tag{49}$$

which is multiplicative for the Yoneda product as well as the cup product. It has all the properties listed in Section 2.2.6. In the following we sometimes abbreviate ${}^{\varpi}w := \varpi w \varpi^{-1}$ for any $w \in \widetilde{W}$. We need the following additional fact. Recall that $\phi_w \in H^d(I, \mathbf{X}(w))$ was defined in (15).

Lemma 3.1. *Assume I is a Poincaré group of dimension d. For $w \in \widetilde{W}$ we have*

$$\Gamma_\varpi(\phi_w) = \phi_{\varpi w \varpi^{-1}} \ . \tag{50}$$

Proof. We recall from (18) that we have the commutative diagram

$$\begin{array}{ccccc} H^d(I, \mathbf{X}(w)) & \xrightarrow{\mathrm{Sh}_w} & H^d(I_w, k) & \xrightarrow{\mathrm{cores}} & H^d(I, k) \\ \downarrow{\scriptstyle \Gamma_\varpi} & & \downarrow{\scriptstyle \varpi_*} & & \downarrow{\scriptstyle \varpi_*} \\ H^d(I, \mathbf{X}({}^{\varpi}w)) & \xrightarrow{\mathrm{Sh}_{\varpi w}} & H^d(I_{\varpi w}, k) & \xrightarrow{\mathrm{cores}} & H^d(I, k) \end{array}$$

where $\varpi_* = (\varpi^{-1})^*$ is the conjugation operator given on cocycles by sending c to $c(\varpi^{-1}\_\varpi)$. We will prove that the operator ϖ_* on $H^d(I, k)$ is the identity. For this we follow the same idea as in [OS3] §7.2.3 and [Koz2] Thm. 7.1.

For any $m \geq 1$ we have the open subgroup $K_{C,m} := \begin{pmatrix} 1+\mathfrak{M} & \mathfrak{M}^m \\ \mathfrak{M}^{m+1} & 1+\mathfrak{M} \end{pmatrix}$ of I. It is normalized by ϖ. Since $\mathrm{cores}_I^{K_{C,m}} : H^d(K_{C,m}, k) \xrightarrow{\cong} H^d(I, k)$ is an isomorphism ([OS3] Rmk. 7.3) and commutes with corestriction we are reduced to showing that the operator ϖ_* on $H^d(K_{C,m}, k)$ is the identity. But for m large enough the pro-p group $K_{C,m}$ is uniform by [OS3] Cor. 7.8 and Rmk. 7.10. So by [Laz] V.2.2.6.3 and V.2.2.7.2, the one dimensional k-vector space $H^d(K_{C,m}, k)$ is the maximal exterior power (via the cup product) of the d-dimensional k-vector space $H^1(K_{C,m}, k)$. Conjugation commuting with the cup product, the action of ϖ_* on $H^d(K_{C,m}, k)$ is the determinant of ϖ_* on $H^1(K_{C,m}, k)$. The latter is the dual of the Frattini quotient $(K_{C,m})_\Phi$. This reduces us further to showing that the determinant of ϖ_* on $(K_{C,m})_\Phi$ is equal to 1. For this we consider the subgroups $\mathfrak{U}^-_{m+1} = \begin{pmatrix} 1 & 0 \\ \mathfrak{M}^{m+1} & t \end{pmatrix}$, $\mathfrak{U}^+_m = \begin{pmatrix} 1 & \mathfrak{M}^m \\ 0 & 1 \end{pmatrix}$, and $T^m := \begin{pmatrix} 1+\mathfrak{M}^m & 0 \\ 0 & 1+\mathfrak{M}^m \end{pmatrix}$ of $K_{C,m}$. According to [OS3] Cor. 7.9 multiplication gives an isomorphism

$$\mathfrak{U}^-_{m+1}/(\mathfrak{U}^-_{m+1})^p \times T^m/(T^m)^p \times \mathfrak{U}^+_m/(\mathfrak{U}^+_m)^p \xrightarrow{\cong} (K_{C,m})_\Phi \ .$$

One easily checks that ϖ_* restricts to an involutive isomorphism $\mathfrak{U}^-_{m+1}/(\mathfrak{U}^-_{m+1})^p \cong \mathfrak{U}^+_m/(\mathfrak{U}^+_m)^p$. These are $\mathbb{F}_p$-vector spaces of dimension equal to $[\mathfrak{F} : \mathbb{Q}_p]$. Hence the determinant of ϖ_* on $\mathfrak{U}^-_{m+1}/(\mathfrak{U}^-_{m+1})^p \times \mathfrak{U}^+_m/(\mathfrak{U}^+_m)^p$ is equal to $(-1)^{[\mathfrak{F}:\mathbb{Q}_p]}$. On the other hand, for m large enough, the logarithm induces an isomorphism $T^m/(T^m)^p \cong 1 + \pi^m\mathfrak{O}/(1 + \pi^m\mathfrak{O})^p \cong \pi^m\mathfrak{O}/p\pi^m\mathfrak{O} \cong \mathfrak{O}/p\mathfrak{O}$ with respect to which ϖ_* corresponds to multiplication by -1. Hence its determinant on this factor is again equal to $(-1)^{[\mathfrak{F}:\mathbb{Q}_p]}$. $\square$

3.2 Elements of E^1 as triples

From now on we assume $p \neq 2$ unless it is specifically stated otherwise.

3.2.1 Definition

We refer to the notation introduced in §2.4.1. We introduce the following subsets of $\widetilde{W}$:

$$\widetilde{W}^0 := \{w \in \widetilde{W},\ \ell(s_0 w) = \ell(w) + 1\} \text{ and}$$
$$\widetilde{W}^1 := \{w \in \widetilde{W},\ \ell(s_1 w) = \ell(w) + 1\}.$$

Note that the intersection of these two subsets coincides with the set $\Omega = T^0/T^1$ of all elements in $\widetilde{W}$ with length 0. Recall as in [OS2, 3.3], we define for $m \geq 0$ the subgroups

$$I^+_m := \begin{pmatrix} 1+\mathfrak{M} & \mathfrak{O} \\ \mathfrak{M}^{m+1} & 1+\mathfrak{M} \end{pmatrix} \qquad \text{and} \qquad I^-_m = \varpi I^+_m \varpi^{-1} = \varpi^{-1} I^+_m \varpi = \begin{pmatrix} 1+\mathfrak{M} & \mathfrak{M}^m \\ \mathfrak{M} & 1+\mathfrak{M} \end{pmatrix} \tag{51}$$

of I and recall that

$$I_w = I \cap wIw^{-1} = \begin{cases} I^+_{\ell(w)} & \text{if } w \in \widetilde{W}^0, \\ I^-_{\ell(w)} & \text{if } w \in \widetilde{W}^1. \end{cases} \tag{52}$$

We abbreviate $h^1 := H^1(I, \mathbf{X})$ and $h^1(w) := H^1(I, \mathbf{X}(w))$ for $w \in \widetilde{W}$. Recall the Shapiro isomorphism $h^1(w) \cong H^1(I_w, k) = \mathrm{Hom}((I_w)_\Phi, k)$ (§2.2) where $(I_w)_\Phi$ denotes the Frattini quotient of I_w ([OS2] §3.8). By [OS2] Prop. 3.62 we have isomorphisms

$$(I_w)_\Phi \xrightarrow{\cong} \mathfrak{O}/\mathfrak{M} \times (1 + \mathfrak{M})/(1 + \mathfrak{M}^{\ell(w)+1})(1 + \mathfrak{M})^p \times \mathfrak{O}/\mathfrak{M}$$

for any $w \in \widetilde{W}$ (depending on a choice of a prime element in $\mathfrak{M}$). More precisely, when $w \in \widetilde{W}^0$:

$$(I^+_{\ell(w)})_\Phi \xrightarrow{\cong} \mathfrak{O}/\mathfrak{M} \times (1 + \mathfrak{M})/(1 + \mathfrak{M}^{\ell(w)+1})(1 + \mathfrak{M})^p \times \mathfrak{O}/\mathfrak{M}$$
$$\begin{pmatrix} 1+\pi x & y \\ \pi^{\ell(w)+1} z & 1+\pi t \end{pmatrix} \bmod \Phi(I_w) \longmapsto (z \bmod \mathfrak{M},\ 1 + \pi x \bmod (1 + \mathfrak{M}^{\ell(w)+1})(1 + \mathfrak{M})^p,\ y \bmod \mathfrak{M}) \tag{53}$$

and when $w \in \widetilde{W}^1$:

$$(54)\qquad \begin{aligned} (I^-_{\ell(w)})_\Phi &\xrightarrow{\cong} \mathfrak{O}/\mathfrak{M} \times (1+\mathfrak{M})/(1+\mathfrak{M}^{\ell(w)+1})(1+\mathfrak{M})^p \times \mathfrak{O}/\mathfrak{M} \\ \left(\begin{smallmatrix} 1+\pi x & \pi^{\ell(w)} y \\ \pi z & 1+\pi t \end{smallmatrix}\right) \bmod \Phi(I_w) &\longmapsto (z \bmod \mathfrak{M},\ 1+\pi x \bmod (1+\mathfrak{M}^{\ell(w)+1})(1+\mathfrak{M})^p,\ y \bmod \mathfrak{M}). \end{aligned}$$

By applying $\mathrm{Hom}(\_, k)$ and using the Shapiro isomorphism we deduce, for any $w \in \widetilde{W}$, a decomposition

$$h^1(w) = h^1_-(w) \oplus h^1_0(w) \oplus h^1_+(w)$$

such that

$$\begin{matrix} h^1_-(w) \\ h^1_0(w) \\ h^1_+(w) \end{matrix} \cong \mathrm{Hom}\begin{pmatrix} \text{left factor} & \\ \text{middle factor}, & k \\ \text{right factor} & \end{pmatrix}.$$

For any element $c \in h^1(w)$ we write this decomposition as

$$(55)\qquad \mathrm{Sh}_w(c) = (c^-, c^0, c^+) \text{ with } \\ c^\pm \in \mathrm{Hom}(\mathfrak{O}/\mathfrak{M}, k) \text{ and } c^0 \in \mathrm{Hom}((1+\mathfrak{M})/(1+\mathfrak{M}^{\ell(w)+1})(1+\mathfrak{M})^p, k)\ .$$

We will often denote by

$$(56)\qquad (c^-, c^0, c^+)_w$$

the element in $h^1(w)$ which has image the triple $(c^-, c^0, c^+) \in H^1(I_w, k)$ via the Shapiro isomorphism (with c^0 implicitly equal to 0 when $\ell(w) = 0$).

Remark 3.2. When $\mathfrak{F} = \mathbb{Q}_p$ and $p \neq 2$, we have $1 + p^2\mathbb{Z}_p = (1 + p\mathbb{Z}_p)^p$ since $\log : 1 + p\mathbb{Z}_p \xrightarrow{\cong} p\mathbb{Z}_p$. Therefore, when $\ell(w) \geq 1$, the identifications (53) and (54) become:

$$(57)\qquad \begin{aligned} (I_w)_\Phi = (I^+_{\ell(w)})_\Phi &\xrightarrow{\cong} \mathbb{Z}_p/p\mathbb{Z}_p \times (1+p\mathbb{Z}_p)/(1+p^2\mathbb{Z}_p) \times \mathbb{Z}_p/p\mathbb{Z}_p \\ \left(\begin{smallmatrix} 1+px & y \\ p^{\ell(w)+1} z & 1+pt \end{smallmatrix}\right) \bmod \Phi(I_w) &\longmapsto (z \bmod p\mathbb{Z}_p, 1+px \bmod 1+p^2\mathbb{Z}_p, y \bmod p\mathbb{Z}_p) \text{ when } w \in \widetilde{W}^0 \end{aligned}$$

(in particular, for $w \in \widetilde{W}^0$, $\ell(w) \geq 1$ we have $\mathrm{res}^{I_{s_1}}_{I_w}(\mathrm{Sh}_{s_1}(0, c^0, c^+)_{s_1}) = \mathrm{Sh}_w((0, c^0, c^+)_w)$ and

$$(58)\qquad \begin{aligned} (I_w)_\Phi = (I^-_{\ell(w)})_\Phi &\xrightarrow{\cong} \mathbb{Z}_p/p\mathbb{Z}_p \times (1+p\mathbb{Z}_p)/(1+p^2\mathbb{Z}_p) \times \mathbb{Z}_p/p\mathbb{Z}_p \\ \left(\begin{smallmatrix} 1+px & p^{\ell(w)} y \\ pz & 1+pt \end{smallmatrix}\right) \bmod \Phi(I_w) &\longmapsto (z \bmod p\mathbb{Z}_p,\ 1+px \bmod 1+p^2\mathbb{Z}_p, y \bmod p\mathbb{Z}_p) \text{ when } w \in \widetilde{W}^1. \end{aligned}$$

(in particular, for $w \in \widetilde{W}^1$, $\ell(w) \geq 1$ we have $\mathrm{res}^{I_{s_0}}_{I_w}(\mathrm{Sh}_{s_0}(c^-, c^0, 0)_{s_0}) = \mathrm{Sh}_w(c^-, c^0, 0)_w)$. When $\ell(w) = 0$, we have $(I_w)_\Phi = I_\Phi \xrightarrow{\cong} \mathbb{Z}_p/p\mathbb{Z}_p \times \mathbb{Z}_p/p\mathbb{Z}_p$.

Notation 3.3. For any subset $U \subseteq \widetilde{W}$ we have the k-subspaces

$$h^1_-(U) := \oplus_{w \in U} h^1_-(w), \quad h^1_0(U) := \oplus_{w \in U} h^1_0(w), \quad \text{and } h^1_+(U) := \oplus_{w \in U} h^1_+(w)$$

of h^1. We also let $h^1_\pm(U) := h^1_-(U) \oplus h^1_+(U)$ and $h^1(U) := h^1_0(U) \oplus h^1_\pm(U)$. The subsets of most interest to us are:

$$\begin{aligned} \widetilde{W}^\epsilon &:= \{w \in \widetilde{W} : \ell(s_\epsilon w) = \ell(w) + 1\} \quad \text{for } \epsilon \in \{0, 1\} \text{ as defined above, and,} \\ \widetilde{W}^{\epsilon, odd} &:= \{w \in \widetilde{W}^\epsilon : \ell(w) \text{ is odd}\}, \\ \widetilde{W}^{\epsilon, even} &:= \{w \in \widetilde{W}^\epsilon : \ell(w) \text{ is even}\}, \\ \widetilde{W}^{\epsilon, +even} &:= \widetilde{W}^{\epsilon, even} \setminus \Omega. \end{aligned}$$

We also define, for $k \geq 0$ and $\epsilon \in \{0,1\}$:

$$\widetilde{W}^{\ell \geq k} := \{w \in \widetilde{W} : \ell(w) \geq k\}$$
$$\widetilde{W}^{\epsilon, \ell \geq k} := \{w \in \widetilde{W} : \ell(s_\epsilon w) = \ell(w) + 1 \text{ and } \ell(w) \geq k\} \quad \text{for } \epsilon \in \{0,1\}.$$

3.2.2 Triples and conjugation by ϖ

Lemma 3.4. *Let $w \in \widetilde{W}$ and $(c^-, c^0, c^+)_w \in h^1(w)$. Its image by the map Γ_ϖ of conjugation by ϖ defined in (49) is*

$$(c^+, -c^0, c^-)_{\varpi w \varpi^{-1}} \in h^1(\varpi w \varpi^{-1})$$

and if $w \in \widetilde{W}^\epsilon$, then $\varpi w \varpi^{-1} \in \widetilde{W}^{1-\epsilon}$.

Proof. See Remark (2.6) for the second claim. By definition of the triples and by commutativity of diagram (18), the first claim follows directly from the observation that the matrices

$$\begin{pmatrix} 1+\pi x & \pi^{\ell(w)} y \\ \pi z & 1+\pi t \end{pmatrix} \in I^-_{\ell(w)} \quad \text{and} \quad \begin{pmatrix} 1+\pi t & z \\ \pi^{\ell(w)+1} y & 1+\pi x \end{pmatrix} \in I^+_{\ell(w)}$$

are conjugate to each other via ϖ.

□

3.2.3 Triples and cup product

Suppose $\mathfrak{F} = \mathbb{Q}_p$, $p \neq 2, 3$. We introduce the isomorphism

(59) $$\iota : 1 + p\mathbb{Z}_p / 1 + p^2\mathbb{Z}_p \xrightarrow{\simeq} \mathbb{Z}_p/p\mathbb{Z}_p, \quad 1 + px \mapsto x \bmod p\mathbb{Z}_p .$$

We choose and fix elements with the following constraints

(60) $$\alpha \in \mathbb{Z}_p/p\mathbb{Z}_p \setminus \{0\}, \quad \alpha^0 = \iota^{-1}(\alpha), \quad \mathbf{c} \in \mathrm{Hom}(\mathbb{Z}_p/p\mathbb{Z}_p, k) \text{ such that } \mathbf{c}(\alpha) = 1, \quad \mathbf{c}^0 := \mathbf{c}\iota$$

When $\ell(w) > 0$, the dimension of the Frattini quotient of I_w is 3, namely the dimension of I_w as a p-adic manifold. By [KS] Cor. 1.8 this means that I_w is uniform. Therefore, the algebra $H^*(I_w, k)$ is the exterior power (via the cup product) of the 3-dimensional k-vector space $H^1(I_w, k)$. In particular, $(\mathbf{c}, 0, 0)_{s_0} \cup (0, \mathbf{c}^0, 0)_{s_0} \cup (0, 0, \mathbf{c})_{s_0}$ is a nonzero element of $H^3(I, \mathbf{X}(s_0))$ and its image via

$$H^3(I, \mathbf{X}(s_0)) \xrightarrow{\mathrm{Sh}_{s_0}} H^3(I_{s_0}, k) \xrightarrow{\mathrm{cores}_I^{I_{s_0}}} H^3(I, k)$$

is a nonzero element of the one dimensional vector space $H^3(I, k)$ (see [OS3] Rmk. 7.3). We choose the isomorphism $\eta : H^3(I, k) \xrightarrow{\simeq} k$ sending that element to 1. As in §2.2.5, this choice of η yields a choice of a basis $(\phi_w)_{w \in \widetilde{W}}$ of $H^d(I, \mathbf{X})$ which is dual to $(\tau_w)_{w \in \widetilde{W}}$ via (14). By definition, we have

$$(\mathbf{c}, 0, 0)_{s_0} \cup (0, \mathbf{c}^0, 0)_{s_0} \cup (0, 0, \mathbf{c})_{s_0} = \phi_{s_0}$$

Lemma 3.5. *For any $w \in \widetilde{W}$ with $\ell(w) \geq 1$, we have*

(61) $$(\mathbf{c}, 0, 0)_w \cup (0, \mathbf{c}^0, 0)_w \cup (0, 0, \mathbf{c})_w = \phi_w$$

Proof. By definition (15) of ϕ_w, it is enough to prove that

$$\mathrm{cores}_I^{I_w} \circ \mathrm{Sh}_w \left((\mathbf{c},0,0)_w \cup (0,\mathbf{c}^0,0)_w \cup (0,0,\mathbf{c})_w\right) = \mathrm{cores}_I^{I_{s_0}} \circ \mathrm{Sh}_{s_0} \left((\mathbf{c},0,0)_{s_0} \cup (0,\mathbf{c}^0,0)_{s_0} \cup (0,0,\mathbf{c})_{s_0}\right).$$

• First suppose that $w \in \widetilde{W}^1$. Recall (see [OS3] §3.3), that the Shapiro isomorphism commutes with the cup product. We compute that $\mathrm{cores}_{I_{s_0}}^{I_w} \circ \mathrm{Sh}_w \left((\mathbf{c},0,0)_w \cup (0,\mathbf{c}^0,0)_w \cup (0,0,\mathbf{c})_w\right)$ is equal to

$$\begin{aligned}
&\mathrm{cores}_{I_{s_0}}^{I_w} [\mathrm{Sh}_w \left((\mathbf{c},0,0)_w\right) \cup \mathrm{Sh}_w \left((0,\mathbf{c}^0,0)_w\right) \cup \mathrm{Sh}_w \left((0,0,\mathbf{c})_w\right)]\\
=&\mathrm{cores}_{I_{s_0}}^{I_w} [\mathrm{res}_{I_w}^{I_{s_0}} \left(\mathrm{Sh}_{s_0} \left((\mathbf{c},0,0)_{s_0}\right) \cup \mathrm{Sh}_{s_0} \left((0,\mathbf{c}^0,0)_{s_0}\right)\right) \cup \mathrm{Sh}_w \left((0,0,\mathbf{c})_w\right)] \text{ by Remark 3.2}\\
=&\mathrm{Sh}_{s_0} \left((\mathbf{c},0,0)_{s_0}\right) \cup \mathrm{Sh}_{s_0} \left((0,\mathbf{c}^0,0)_{s_0}\right) \cup \mathrm{cores}_{I_{s_0}}^{I_w} [\mathrm{Sh}_w \left((0,0,\mathbf{c})_w\right)] \text{ by the projection formula ([OS3] §4.6)}\\
=&\mathrm{Sh}_{s_0} \left((\mathbf{c},0,0)_{s_0}\right) \cup \mathrm{Sh}_{s_0} \left((0,\mathbf{c}^0,0)_{s_0}\right) \cup \mathrm{Sh}_{s_0} \left((0,0,\mathbf{c})_{s_0}\right) \text{ by [OS2] Lemma 3.68-iv.}\\
=&\mathrm{Sh}_{s_0} \left((\mathbf{c},0,0)_{s_0} \cup (0,\mathbf{c}^0,0)_{s_0} \cup (0,0,\mathbf{c})_{s_0}\right) = \mathrm{Sh}_{s_0} \left(\phi_{s_0}\right)
\end{aligned}$$

which proves the expected statement after applying $\mathrm{cores}_I^{I_{s_0}}$.

• If $w \in \widetilde{W}^0$, we conjugate by ϖ using Γ_ϖ (see (49)):

$$\begin{aligned}
&\Gamma_\varpi\left((\mathbf{c},0,0)_w \cup (0,\mathbf{c}^0,0)_w \cup (0,0,\mathbf{c})_w\right)\\
&\quad= -(0,0,\mathbf{c})_{\varpi w\varpi^{-1}} \cup (0,\mathbf{c}^0,0)_{\varpi w\varpi^{-1}} \cup (\mathbf{c},0,0)_{\varpi w\varpi^{-1}} \quad \text{by (20) and Lemma 3.4}\\
&\quad= (\mathbf{c},0,0)_{\varpi w\varpi^{-1}} \cup (0,\mathbf{c}^0,0)_{\varpi w\varpi^{-1}} \cup (0,0,\mathbf{c})_{\varpi w\varpi^{-1}} \quad \text{by anticommutativity of } \cup\\
&\quad= \phi_{\varpi w\varpi^{-1}} \quad \text{since } \varpi w\varpi^{-1} \in \widetilde{W}^1\\
&\quad= \Gamma_\varpi(\phi_w) \quad \text{by (50)}
\end{aligned}$$

which concludes the proof since Γ_ϖ is bijective. □

Example 3.6. The subalgebra $H^*(I,\mathbf{X}(1))$ of E^*:

- $H^0(I,\mathbf{X}(1))$ has dimension 1,
- $H^3(I,\mathbf{X}(1))$ has dimension 1 with basis ϕ_1 which satisfies $\eta(\phi_1) = 1$.
- $H^1(I,\mathbf{X}(1))$ has dimension 2 and basis $(\mathbf{c},0,0)_1$ and $(0,0,\mathbf{c})_1$,
- $H^2(I,\mathbf{X}(1))$ is dual to $H^1(I,\mathbf{X}(1))$ via the cup product. We denote by $(\alpha,0,0)_1$ and $(0,0,\alpha)_1$ the dual of the basis of $H^1(I,\mathbf{X}(1))$ given above, it satisfies by definition:
  – $(\alpha,0,0)_1 \cup (\mathbf{c},0,0)_1 = (\mathbf{c},0,0)_1 \cup (\alpha,0,0)_1 = \phi_1 = (0,0,\alpha)_1 \cup (0,0,\mathbf{c})_1 = (0,0,\mathbf{c})_1 \cup (0,0,\alpha)_1$, while
  – $(\mathbf{c},0,0)_1 \cup (0,0,\mathbf{c})_1 = (0,0,\mathbf{c})_1 \cup (\mathbf{c},0,0)_1 = 0$.

3.3 Image of a triple under the anti-involution $\mathcal{J}$

Let $c \in h^1(w)$ seen as a triplet $(c^-,c^0,c^+)_w$ as in (55). Its image by $\mathcal{J}$ is an element in $h^1(w^{-1})$ whose image by the Shapiro isomorphism is given by (see (12))

$$(\mathrm{Sh}_w\, c)(w\_w^{-1}) : I_{w^{-1}} \to k\,.$$

Lemma 3.7. *Let $w \in \widetilde{W}$ and $c = (c^-, c^0, c^+)_w \in h^1(w)$.*
If $\ell(w)$ is even then

$$\mathcal{J}(c) = (c^-(u^2\_), c^0, c^+(u^{-2}\_))_{w^{-1}}. \tag{62}$$

If $\ell(w)$ is odd then

$$\mathcal{J}(c) = (-c^+(u^{-2}\_), -c^0, -c^-(u^2\_))_{w^{-1}}. \tag{63}$$

where $u \in (\mathfrak{O}/\mathfrak{M})^\times$ is such that $\omega_u^{-1} w$ lies in the subgroup of $\widetilde{W}$ generated by s_0 and s_1.

Proof. Notice that the intersection of Ω and of the subgroup of $\widetilde{W}$ generated by s_0 and s_1 is equal to $\{\pm 1\}$, therefore u^2 is determined by w.

- If $w = \omega_u(s_0s_1)^n$, then $I_{w^{-1}} = I^+_{2n}$ and for $X = \begin{pmatrix} 1+\pi x & y \\ \pi^{1+2n} z & 1+\pi t \end{pmatrix} \in I_{w^{-1}}$ we have $wXw^{-1} = \omega_u \begin{pmatrix} 1+\pi x & \pi^{2n} y \\ \pi z & 1+\pi t \end{pmatrix} \omega_u^{-1} = \begin{pmatrix} 1+\pi x & [u]^{-2}\pi^{2n} y \\ [u]^2 \pi z & 1+\pi t \end{pmatrix}$ so
$$\mathrm{Sh}_{w^{-1}}(\mathcal{J}(c)) = (c^-(u^2\_), c^0, c^+(u^{-2}\_))_{w^{-1}}.$$

- If $w = \omega_u(s_1s_0)^n$, then $I_{w^{-1}} = I^-_{2n}$ and for $X = \begin{pmatrix} 1+\pi x & \pi^{2n} y \\ \pi z & 1+\pi t \end{pmatrix} \in I_{w^{-1}}$ we have $wXw^{-1} = \omega_u \begin{pmatrix} 1+\pi x & y \\ \pi^{1+2n} z & 1+\pi t \end{pmatrix} \omega_u^{-1} = \begin{pmatrix} 1+\pi x & [u]^{-2} y \\ [u]^2 \pi^{1+2n} z & 1+\pi t \end{pmatrix}$ so
$$\mathrm{Sh}_{w^{-1}}(\mathcal{J}(c)) = (c^-(u^2\_), c^0, c^+(u^{-2}\_))_{w^{-1}}.$$

- If $w = \omega_u(s_1s_0)^n s_1$, then $I_{w^{-1}} = I^+_{2n+1}$ and for $X = \begin{pmatrix} 1+\pi x & y \\ \pi^{2+2n} z & 1+\pi t \end{pmatrix} \in I_{w^{-1}}$ we have $wXw^{-1} = \omega_u \begin{pmatrix} 1+\pi t & -z \\ -\pi^{2+2n} y & 1+\pi x \end{pmatrix} \omega_u^{-1} = \begin{pmatrix} 1+\pi t & -[u]^{-2} z \\ -\pi^{2+2n}[u]^2 y & 1+\pi x \end{pmatrix}$ so
$$\mathrm{Sh}_{w^{-1}}(\mathcal{J}(c)) = (-c^+(u^{-2}\_), -c^0, -c^-(u^2\_))_{w^{-1}}.$$

- If $w = \omega_u(s_0s_1)^n s_0$, then $I_{w^{-1}} = I^-_{2n+1}$ and for $X = \begin{pmatrix} 1+\pi x & \pi^{2n+1} y \\ \pi z & 1+\pi t \end{pmatrix} \in I_{w^{-1}}$ we have $wXw^{-1} = \omega_u \begin{pmatrix} 1+\pi t & -\pi^{1+2n} z \\ -\pi y & 1+\pi z \end{pmatrix} \omega_u^{-1} = \begin{pmatrix} 1+\pi t & -\pi^{1+2n}[u]^{-2} z \\ -\pi [u]^2 y & 1+\pi x \end{pmatrix}$ so
$$\mathrm{Sh}_{w^{-1}}(\mathcal{J}(c)) = (-c^+(u^{-2}\_), -c^0, -c^-(u^2\_))_{w^{-1}}.$$

∎

3.4 Action of τ_ω on E^1 for $\omega \in \Omega$

Let $w \in \widetilde{W}$, $\omega \in T^0/T^1$ and $c \in h^i(w)$ for some $i \geq 0$. By [OS3] Prop. 5.6, the left action of τ_ω on c corresponds to the following transformation, where again we identify c with its image in $H^i(I_w, k)$ by the Shapiro isomorphism:

$$\begin{array}{ccc} h^i(w) & \xrightarrow[\cong]{\tau_\omega} & h^i(\omega w) \\ \big\downarrow {\scriptstyle \mathrm{Sh}_w} & & {\scriptstyle \mathrm{Sh}_{\omega w}} \big\downarrow \\ H^i(I_w, k) & \xrightarrow{\omega_*(c) = c(\omega^{-1}\_\omega)} & H^i(I_w, k) \end{array} \tag{64}$$

In other words, for $\omega \in \Omega$, we have $\tau_\omega \cdot c \in h^i(\omega w)$ and

$$\mathrm{Sh}_{\omega w}(\tau_\omega \cdot c) = \omega_* \,\mathrm{Sh}_w(c)\ . \tag{65}$$

Using $c \cdot \tau_\omega = \mathcal{J}(\tau_{\omega^{-1}} \cdot \mathcal{J}(c))$, we also obtain $c \cdot \tau_\omega \in h^i(w\omega)$ and

$$\mathrm{Sh}_{w\omega}(c \cdot \tau_\omega) = \mathrm{Sh}_w(c)\ . \tag{66}$$

Now we suppose $i = 1$. We identify $c \in h^1(w)$ with a triple $(c^-, c^0, c^+)_w$ as in (55). For $u \in (\mathfrak{O}/\mathfrak{M})^\times$ and $\left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right) \in I_w$ we have $\omega_u^{-1} \left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right) \omega_u = \begin{pmatrix} x & [u]^2 y \\ [u]^{-2} z & t \end{pmatrix}$ and therefore

$$\tau_{\omega_u} \cdot (c^-, c^0, c^+)_w = (c^-(u^{-2}\,\_), c^0, c^+(u^2\,\_))_{\omega_u w}\ \in h^1(\omega_u w)\ . \tag{67}$$

In particular,

$$\tau_{s^2} \cdot (c^-, c^0, c^+)_w = (c^-, c^0, c^+)_{s^2 w} \in h^1(s^2 w) \tag{68}$$

for $s \in \{s_0, s_1\}$ since $s^2 = \omega_{-1}$. For the right action, it follows from (66) that

$$(c^-, c^0, c^+)_w \cdot \tau_{\omega_u} = (c^-, c^0, c^+)_{w\omega_u}\ \in h^1(w\omega_u)\ . \tag{69}$$

3.5 Action of the idempotents e_λ

For $\lambda : \Omega \to k^\times$ and $w \in \widetilde{W}$, recall that we defined the idempotent $e_\lambda \in k[\Omega]$ (see (36)) and that, for any $\omega \in \Omega$ we have $e_\lambda \tau_\omega = \tau_\omega e_\lambda = \lambda(\omega) e_\lambda$.

Lemma 3.8. *Let $\lambda, \mu : \Omega \to k^\times$, $w \in \widetilde{W}$. We consider an element $c \in h^i(w)$ with image $c_w \in H^i(I_w, k)$ by the Shapiro isomorphism. We have*

$$e_\lambda \cdot c = c \cdot e_\mu \text{ if and only if } c_w = \mu(w^{-1}\omega w)\lambda(\omega^{-1})\, \omega_*(c_w) \quad \text{for any } \omega \in \Omega.$$

Proof. The element $e_\lambda \cdot c$ lies in $\oplus_{\omega\in\Omega} H^i(I, \mathbf{X}(\omega w))$ and its component in $H^i(I, \mathbf{X}(\omega w))$ is

$$-\lambda(\omega^{-1})\,\mathrm{Sh}^{-1}_{\omega w}\left(\omega_* c_w\right)$$

The element $c \cdot e_\mu$ lies in $\oplus_{t\in\Omega} H^i(I, \mathbf{X}(wt))$ and its component in $H^i(I, \mathbf{X}(wt)) = H^i(I, \mathbf{X}(wtw^{-1}w))$ is

$$-\mu(t^{-1})\,\mathrm{Sh}^{-1}_{wt}\left(c_w\right) = -\mu(w^{-1}(wt^{-1}w^{-1})w)\,\mathrm{Sh}^{-1}_{wtw^{-1}w}\left(c_w\right)$$

These two elements are equal if and only if for any $\omega \in \Omega$ we have $\lambda(\omega^{-1})\omega_* c_w = \mu(w^{-1}\omega^{-1}w)c_w$. ☐

In the same context as in the lemma, we suppose that $i = 1$. Then we may see the image in $H^1(I_w, k)$ by the Shapiro isomorphism of $c \in h^1(w)$ as a (c^-, c^0, c^+) as in (55). For $u \in (\mathfrak{O}/\mathfrak{M})^\times$, we know from the calculation that gave (67) that

$$\omega_{u*}(c^-, c^0, c^+) = (c^-(u^{-2}\,\_)\,, c^0,\, c^+(u^2\,\_)) \in H^1(I_w, k).$$

If $\ell(w)$ is even, then the conjugation of μ by w is equal to μ and therefore $e_\lambda \cdot c = c \cdot e_\mu$ if and only if $c = \mu\lambda^{-1}(\omega_u)\, \omega_{u*}(c)$ for any $u \in (\mathfrak{O}/\mathfrak{M})^\times$. So

$$(\ell(w) \text{ even}): \quad e_\lambda \cdot c = c \cdot e_\mu \text{ if and only if } \begin{cases} c^- = \mu\lambda^{-1}(\omega_u) c^-(u^{-2}\,\_) \\ c^0 = \mu\lambda^{-1}(\omega_u) c^0 \\ c^+ = \mu\lambda^{-1}(\omega_u) c^+(u^2\,\_). \end{cases} \quad \text{for any } u \in (\mathfrak{O}/\mathfrak{M})^\times. \tag{70}$$

If $\ell(w)$ is odd, then the conjugation of μ by w is equal to μ^{-1} and therefore $e_\lambda \cdot c = c \cdot e_\mu$ if and only if $c_w = (\mu\lambda)^{-1}(\omega_u)\,\omega_*(c)$ for any $u \in (\mathfrak{O}/\mathfrak{M})^\times$ which is equivalent to

$$(71)\quad (\ell(w) \text{ odd}): \quad e_\lambda \cdot c = c \cdot e_\mu \text{ if and only if } \begin{cases} c^- = (\mu\lambda)^{-1}(\omega_u)c^-(u^{-2}\,\_) \\ c^0 = (\mu\lambda)^{-1}(\omega_u)c^0 \\ c^+ = (\mu\lambda)^{-1}(\omega_u)c^+(u^2\,\_). \end{cases} \quad \text{for any } u \in (\mathfrak{O}/\mathfrak{M})^\times.$$

An important special case of the above is the following. Suppose that $q = p$; for any $m \in \mathbb{Z}$ and $w \in \widetilde{W}$ we then have

$$(72)\quad (c^-, c^0, c^+)_w \cdot e_{\mathrm{id}^m} = e_{\mathrm{id}^{m(-1)^{\ell(w)}-2}} \cdot (c^-, 0, 0)_w + e_{\mathrm{id}^{m(-1)^{\ell(w)}}} \cdot (0, c^0, 0)_w + e_{\mathrm{id}^{m(-1)^{\ell(w)}+2}} \cdot (0, 0, c^+)_w\ .$$

3.6 Action of H on E^1 when $G = \mathrm{SL}_2(\mathbb{Q}_p)$, $p \neq 2, 3$

In this whole subsection, $G = \mathrm{SL}_2(\mathbb{Q}_p)$ with $p \neq 2, 3$. **We also choose** $\pi = p$. This is required in the proof of Lemma 9.1 which is used in the proof of Proposition 3.9. The isomorphism ι was introduced in (59). The following proposition is proved in §9.3. Together with (67), it gives the explicit left action of H on E^1 when $G = \mathrm{SL}_2(\mathbb{Q}_p)$ with $p \neq 2, 3$.

Proposition 3.9. *Let $w \in \widetilde{W}$ and $(c^-, c^0, c^+)_w \in h^1(w)$.*

$$\tau_{s_0} \cdot (c^-, c^0, c^+)_w = \begin{cases} (0, -c^0, -c^-)_{s_0 w} & \text{if } w \in \widetilde{W}^0,\ \ell(w) \geq 1, \\ e_1 \cdot (-c^-, -c^0, -c^+)_w + e_{\mathrm{id}} \cdot (0, -2c^-\iota, 0)_w + (0, 0, -c^-)_{s_0 w} & \text{if } w \in \widetilde{W}^1 \text{ with } \ell(w) \geq 2, \\ e_1 \cdot (-c^-, -c^0, -c^+)_w + e_{\mathrm{id}} \cdot (0, -2c^-\iota, c^0\iota^{-1})_w & \\ \qquad + e_{\mathrm{id}^2} \cdot (0, 0, c^-)_w + (0, 0, -c^-)_{s_0 w} & \text{if } w \in \widetilde{W}^1 \text{ with } \ell(w) = 1. \end{cases}$$

$$\tau_{s_1} \cdot (c^-, c^0, c^+)_w = \begin{cases} (-c^+, -c^0, 0)_{s_1 w} & \text{if } w \in \widetilde{W}^1,\ \ell(w) \geq 1, \\ e_1 \cdot (-c^-, -c^0, -c^+)_w + e_{\mathrm{id}^{-1}} \cdot (0, 2c^+\iota, 0)_w + (-c^+, 0, 0)_{s_1 w} & \text{if } w \in \widetilde{W}^0 \text{ with } \ell(w) \geq 2, \\ e_1 \cdot (-c^-, -c^0, -c^+)_w + e_{\mathrm{id}^{-1}} \cdot (-c^0\iota^{-1}, 2c^+\iota, 0)_w & \\ \qquad + e_{\mathrm{id}^{-2}} \cdot (c^+, 0, 0)_w + (-c^+, 0, 0)_{s_1 w} & \text{if } w \in \widetilde{W}^0 \text{ with } \ell(w) = 1. \end{cases}$$

$$\tau_{s_0} \cdot (c^-, 0, c^+)_\omega = (0, 0, -c^-)_{s_0\omega} \quad \text{for } \omega \in \Omega.$$
$$\tau_{s_1} \cdot (c^-, 0, c^+)_\omega = (-c^+, 0, 0)_{s_1\omega} \quad \text{for } \omega \in \Omega.$$

In these formulas, we use the notation e_{id^m} as introduced in (38) for $m \in \mathbb{Z}$. Recall, using (67), that for $(d^-, d^0, d^+)_w \in h^1(w)$, the component in $h^1(\omega_u w)$ of $e_{\mathrm{id}^m} \cdot (d^-, d^0, d^+)_w \in \bigoplus_{u \in \mathbb{F}_p^\times} h^1(\omega_u w)$ is given by

$$(73)\qquad -\mathrm{id}^m(u^{-1})\,\tau_{\omega_u} \cdot (d^-, d^0, d^+)_w = -u^{-m}(d^-(u^{-2}\_), d^0(\_), d^+(u^2\_))_{\omega_u w}\ .$$

Corollary 3.10. *Let $w \in \widetilde{W}$, $\omega \in \Omega$, and $(c^-, c^0, c^+)_w \in h^1(w)$.*

$$\zeta \cdot (c^-, 0, c^+)_\omega = (c^-, 0, 0)_{s_1 s_0 \omega} + (0, 0, c^+)_{s_0 s_1 \omega} + e_1 \cdot (0, 0, -c^-)_{s_0\omega} + e_1 \cdot (-c^+, 0, 0)_{s_1\omega} + e_1 \cdot (c^-, 0, c^+)_\omega.$$

$$\zeta\cdot(c^-,c^0,c^+)_w =$$
$$\begin{cases} (c^-,c^0,0)_{s_1s_0w} + e_{\mathrm{id}}\cdot(0,-2c^+\iota,0)_{s_0w} & \\ \qquad + e_{\mathrm{id}}\cdot(0,2c^+\iota,0)_{s_1w} + (0,0,c^+)_{s_0s_1w} & \text{if } w\in\widetilde{W}^0,\ \ell(w)\geq 3,\\ (c^-,c^0,0)_{s_1s_0w} + e_{\mathrm{id}}\cdot(0,-2c^+\iota,0)_{s_0w} & \\ \qquad + e_{\mathrm{id}}\cdot(0,2c^+\iota,0)_{s_1w} + e_{\mathrm{id}^2}\cdot(0,0,-c^+)_{s_1w} + (0,0,c^+)_{s_0s_1w} & \text{if } w\in\widetilde{W}^0,\ \ell(w)=2,\\ (c^-,c^0,0)_{s_1s_0w} + e_{\mathrm{id}}\cdot(0,-2c^+\iota,c^0\iota^{-1})_{s_0w} & \\ \qquad + e_{\mathrm{id}^2}\cdot(0,0,-c^+)_{s_0w} + (0,0,c^+)_{s_0s_1w} + e_1\cdot(-c^+,0,0)_{s_1w} & \text{if } w\in s_1\Omega. \end{cases}$$

$$\zeta\cdot(c^-,c^0,c^+)_w =$$
$$\begin{cases} (0,c^0,c^+)_{s_0s_1w} + e_{\mathrm{id}^{-1}}\cdot(0,2c^-\iota,0)_{s_1w} & \\ \qquad + e_{\mathrm{id}^{-1}}\cdot(0,-2c^-\iota,0)_{s_0w} + (c^-,0,0)_{s_1s_0w} & \text{if } w\in\widetilde{W}^1,\ \ell(w)\geq 3,\\ (0,c^0,c^+)_{s_0s_1w} + e_{\mathrm{id}^{-1}}\cdot(0,2c^-\iota,0)_{s_1w} & \\ \qquad + e_{\mathrm{id}^{-1}}\cdot(0,-2c^-\iota,0)_{s_0w} + e_{\mathrm{id}^{-2}}\cdot(-c^-,0,0)_{s_0w} + (c^-,0,0)_{s_1s_0w} & \text{if } w\in\widetilde{W}^1,\ \ell(w)=2,\\ (0,c^0,c^+)_{s_0s_1w} + e_{\mathrm{id}^{-1}}\cdot(-c^0\iota^{-1},2c^-\iota,0)_{s_1w} & \\ \qquad + e_{\mathrm{id}^{-2}}\cdot(-c^-,0,0)_{s_1w} + (c^-,0,0)_{s_1s_0w} + e_1\cdot(0,0,-c^-)_{s_0w} & \text{if } w\in s_0\Omega. \end{cases}$$

The decreasing filtration $(F^mE^1)_{m\geq 1}$ was introduced in §2.2.4.

Corollary 3.11. *We have* $\zeta\cdot E^1\supseteq F^3E^1$

Proof. It is easy to see that $\zeta\cdot E^1$ contains $h^1_-(\widetilde{W}^{0,\ell\geq 3})$ and $h^1_+(\widetilde{W}^{1,\ell\geq 3})$. Noticing that it also contains $h^1_0(\widetilde{W}^{\ell\geq 4})$, we deduce that it contains $h^1_-(\widetilde{W}^{1,\ell\geq 3})$ and $h^1_+(\widetilde{W}^{0,\ell\geq 3})$. But for c^0 as above and $\omega\in\Omega$, we have $\zeta\cdot(0,c^0,0)_{s_0\omega} = (0,c^0,0)_{s_0s_1s_0\omega} + e_{\mathrm{id}^{-1}}\cdot(-c^0\iota^{-1},0,0)_{s_1s_0\omega} = (0,c^0,0)_{s_0s_1s_0\omega} + \zeta e_{\mathrm{id}^{-1}}\cdot(-c^0\iota^{-1},0,0)_\omega$ so $(0,c^0,0)_{s_0s_1s_0\omega}\in\zeta\cdot E^1$ and likewise we would obtain $(0,c^0,0)_{s_1s_0s_1\omega}\in\zeta\cdot E^1$. □

Using the anti-involution $\mathcal{J}$, we would obtain the explicit right action of H on E^1. For example, using $(c^-,0,c^+)_1\cdot\zeta = \mathcal{J}(\zeta\cdot\mathcal{J}((c^-,0,c^+)_1)) = \mathcal{J}(\zeta\cdot(c^-,0,c^+)_1)$ we can compute:

$$\begin{aligned}(c^-,0,c^+)_1\cdot\zeta =& (c^-,0,0)_{s_0s_1} + (0,0,c^+)_{s_1s_0}\\ &+ e_{\mathrm{id}^{-2}}(c^-,0,0)_{s_0} + e_{\mathrm{id}^2}(0,0,c^+)_{s_1} + e_{\mathrm{id}^{-2}}(c^-,0,0)_1 + e_{\mathrm{id}^2}(0,0,c^+)_1.\end{aligned}\tag{74}$$

We give now further partial results on the right action of H on E^1.

Lemma 3.12. *Let $v,w\in\widetilde{W}$ such that $\ell(w)\geq 1$ and $(c^-,c^0,c^+)_v\in h^1(v)$.*

i. Suppose $\ell(v)+\ell(w)=\ell(vw)$. Then $(c^-,c^0,c^+)_v\cdot\tau_w = \begin{cases}(c^-,c^0,0)_{vw} & \text{if } vw\in\widetilde{W}^1,\\ (0,c^0,c^+)_{vw} & \text{if } vw\in\widetilde{W}^0.\end{cases}$

ii. In the case when $v\in\{s_0,s_1\}$ and $\ell(vw)=\ell(w)-1$ we have:

$$(0,c^0,0)_{s_0}\cdot\tau_w = -e_1\cdot(0,c^0,0)_w - e_{\mathrm{id}^{-1}}\cdot(c^0\iota^{-1},0,0)_w$$

$$(0,c^0,0)_{s_1}\cdot\tau_w = -e_1\cdot(0,c^0,0)_w + e_{\mathrm{id}}\cdot(0,0,c^0\iota^{-1})_w$$

Proof. i. Using (69), we see that we may restrict the proof to the case when v belongs to the set $\{(s_i s_{1-i})^n, s_1(s_i s_{1-i})^n : i = 0, 1, n \geq 0\}$. We treat the case $v \in W^1$. First suppose $v = (s_0 s_1)^n$. Then, using Lemma 3.7, (68) and Proposition 3.9:

$$\begin{aligned}(c^-, c^0, c^+)_v \cdot \tau_{s_0} &= \mathcal{J}(\tau_{s_0^{-1}} \cdot (c^-, c^0, c^+)_{v^{-1}}) = \mathcal{J}(\tau_{s_0} \cdot (c^-, c^0, c^+)_{s_0^2 v^{-1}}) \\ &= \mathcal{J}((0, -c^0, -c^-)_{s_0^{-1} v^{-1}}) = (c^-, c^0, 0)_{v s_0} \ .\end{aligned}$$

Next suppose that $v = s_0(s_1 s_0)^n$. Then

$$\begin{aligned}(c^-, c^0, c^+)_v \cdot \tau_{s_1} = \mathcal{J}(\tau_{s_1^{-1}} \cdot (-c^+, -c^0, -c^-)_{v^{-1}}) = \\ \mathcal{J}(\tau_{s_1} \cdot (-c^+, -c^0, -c^-)_{s_1^2 v^{-1}}) = \mathcal{J}((c^-, c^0, 0)_{s_1^{-1} v^{-1}}) = (c^-, c^0, 0)_{v s_1}.\end{aligned}$$

This is enough to conclude the proof when $v \in \widetilde{W}^1$ by induction on $\ell(w)$.

ii. We treat the case $v = s_0$ and suppose first that $w = s_0$. Then, using (63), Proposition 3.9, (68) and (71)

$$\begin{aligned}(0, c^0, 0)_{s_0} \cdot \tau_{s_0} &= -\mathcal{J}(\tau_{s_0^{-1}} \cdot (0, c^0, 0)_{s_0^{-1}}) = -\mathcal{J}(\tau_{s_0} \cdot (0, c^0, 0)_{s_0}) \\ &= -\mathcal{J}((0, -c^0, 0)_{s_0} \cdot e_1 + (0, 0, c^0 \iota^{-1})_{s_0} \cdot e_{\mathrm{id}}) \\ &= -e_1 \cdot (0, c^0, 0)_{s_0^{-1}} - e_{\mathrm{id}^{-1}} \cdot (-c^0 \iota^{-1}, 0, 0)_{s_0}^{-1} \\ &= -e_1 \cdot (0, c^0, 0)_{s_0} - e_{\mathrm{id}^{-1}} \cdot (c^0 \iota^{-1}, 0, 0)_{s_0} \ .\end{aligned}$$

For $w = s_0 \omega$ with $\omega \in \Omega$, apply τ_ω on the right to the above formula and use (69). For general w such that $\ell(s_0 w) = \ell(w) - 1$, apply $\tau_{s_0^{-1} w}$ on the right to the above formula and use Point i.

□

The increasing filtration $(F_n E^1)_{n \geq 0}$ was defined in §2.2.4.

Lemma 3.13. *If $\omega \in \Omega$, we have*

$$\zeta \cdot (c^-, 0, c^+)_\omega - (c^-, 0, c^+)_\omega \cdot \zeta \equiv (0, 0, c^+)_{s_0 s_1 \omega} + (c^-, 0, 0)_{s_1 s_0 \omega} - (0, 0, c^+)_{s_1 s_0 \omega} - (c^-, 0, 0)_{s_0 s_1 \omega} \bmod F_1 E^1 \ .$$

If $w \in \widetilde{W}^1$ of length ≥ 1

$$\zeta \cdot (c^-, c^0, c^+)_w - (c^-, c^0, c^+)_w \cdot \zeta \equiv (0, 0, c^+)_{s_0 s_1 w} - (c^-, 0, 0)_{s_0 s_1 w} \bmod F_{\ell(w)+1} E^1 \ .$$

If $w \in \widetilde{W}^0$ of length ≥ 1

$$\zeta \cdot (c^-, c^0, c^+)_w - (c^-, c^0, c^+)_w \cdot \zeta \equiv (c^-, 0, 0)_{s_1 s_0 w} - (0, 0, c^+)_{s_1 s_0 w} \bmod F_{\ell(w)+1} E^1 \ .$$

Proof. We use Cor. 3.10. Recall from (69) that $(c^-, c^0, c^+)_{w\omega} = (c^-, c^0, c^+)_w \tau_\omega$ for $\omega \in \Omega$. So it is enough to prove the lemma for $\omega = 1$ and for w of the form $(s_\epsilon s_{1-\epsilon})^n s_\epsilon$ or $(s_\epsilon s_{1-\epsilon})^n$ where $\epsilon \in \{0, 1\}$. By (74) we have

$$\zeta \cdot (c^-, 0, c^+)_1 - (c^-, 0, c^+)_1 \cdot \zeta \equiv (0, 0, c^+)_{s_0 s_1} + (c^-, 0, 0)_{s_1 s_0} - (c^-, 0, 0)_{s_0 s_1} - (0, 0, c^+)_{s_1 s_0} \bmod F_1 E^1.$$

Now for $w = (s_0 s_1)^n$ with $n \geq 1$ we have

$$\begin{aligned}\zeta \cdot (c^-, c^0, c^+)_w &\equiv (0, c^0, c^+)_{s_0 s_1 w} \bmod F_{2n+1} E^1 \quad \text{and} \\ \zeta \cdot (c^-, c^0, c^+)_{w^{-1}} &\equiv (c^-, c^0, 0)_{s_1 s_0 w^{-1}} \bmod F_{2n+1} E^1.\end{aligned}$$

Since $\mathcal{J}$ preserves $F_{2n+1}E^1$ we have, using (62):

$$(c^-, c^0, c^+)_w \cdot \zeta = \mathcal{J}(\zeta \cdot (c^-, c^0, c^+)_{w^{-1}}) \equiv \\ \mathcal{J}((c^-, c^0, 0)_{s_1 s_0 w^{-1}}) \equiv (c^-, c^0, 0)_{w s_0 s_1} \equiv (c^-, c^0, 0)_{s_0 s_1 w} \mod F_{2n+1}E^1$$

which gives the expected formula. Using $\mathcal{J}$, we then obtain the expected result for $w = (s_1 s_0)^n$. Likewise we treat the case $w = (s_0 s_1)^n s_0$ with $n \geq 0$. We have

$$\zeta \cdot (c^-, c^0, c^+)_w \equiv (0, c^0, c^+)_{s_0 s_1 w} \bmod F_{2n+2}E^1$$

and

$$(c^-, c^0, c^+)_w \cdot \zeta = \mathcal{J}(\zeta \cdot (-c^+, -c^0, -c^-)_{w^{-1}}) \\ \equiv \mathcal{J}((0, -c^0, -c^-)_{s_0 s_1 w^{-1}}) \equiv (c^-, c^0, 0)_{w s_1 s_0} \equiv (c^-, c^0, 0)_{s_0 s_1 w} \mod F_{2n+2}E^1$$

which gives the expected result for $w = (s_0 s_1)^n s_0$ and similarly we would treat the case $w = (s_1 s_0)^n s_1$. $\square$

3.7 Sub-H-bimodules of E^1

3.7.1 The H-bimodule $F^1 H$

In this paragraph 3.7.1, there is no condition on $\mathfrak{F}$ (in fact we may even have $p = 2$).

The elements $x_i := \tau_{s_i} \in F^1 H$ satisfy the relations:

1) $\tau_{s_i} x_i = -e_1 x_i = x_i \tau_{s_i}$ for $i \in \{0, 1\}$;

2) $\tau_\omega x_i = x_i \tau_{\omega^{-1}}$ for $i \in \{0, 1\}$ and $\omega \in \Omega$;

3) $\tau_{s_0} x_1 = x_0 \tau_{s_1}$ and $\tau_{s_1} x_0 = x_1 \tau_{s_0}$.

Given any H-bimodule M, a pair of elements $x_0, x_1 \in M$ which satisfy the relations 1) - 3) will be called an $F^1 H$-pair in M. The $F^1 H$-pair in M form a k-vector subspace of $M \times M$.

Example 3.14. For any $\ell \geq 0$ the elements $\tau_{s_0}(\tau_{s_1}\tau_{s_0})^\ell$ and $\tau_{s_1}(\tau_{s_0}\tau_{s_1})^\ell$ form an $F^1 H$-pair in $F^1 H$.

Lemma 3.15. *i. Given an $F^1 H$-pair $(x_0, x_1) \in M \times M$, there is a unique H-bimodule homomorphism $f_{(x_0,x_1)} : F^1 H \to M$ satisfying*

$$f_{(x_0,x_1)}(\tau_{s_0}) = x_0, \quad \textit{and} \quad f_{(x_0,x_1)}(\tau_{s_1}) = x_1 \ .$$

ii. The map $f \mapsto (f(\tau_{s_0}), f(\tau_{s_1}))$ yields a bijection between the space of all H-bimodule homomorphism $F^1 H \to M$ and the space of all $F^1 H$-pairs in M. The inverse map is given by $(x_0, x_1) \mapsto f_{(x_0,x_1)}$.

Proof. As a right H-module, we have $F^1 H = \tau_{s_0} H \oplus \tau_{s_1} H$ and $\tau_{s_i} H \simeq H/(\tau_{s_i} + e_1)H$ for $i = 0, 1$. Let $(x_0, x_1) \in M \times M$ satisfying $x_i(\tau_{s_i} + e_1) = 0$ for $i = 0, 1$. There is a unique homomorphism of right H-modules

$$f : F^1 H \longrightarrow M \text{ such that } f(\tau_{s_0}) = x_0 \text{ and } f(\tau_{s_1}) = x_1 \ .$$

We prove that f is a homomorphism of H-bimodules if and only if x_0, x_1 is an $F^1 H$-pair in M. The direct implication is clear. Now suppose that $x_0, x_1 \in M$ satisfy the relations 1) - 3). Let $w \in \widetilde{W}$. We want to show that the maps $\tau \mapsto \tau_w \cdot f(\tau)$ and $\tau \mapsto f(\tau_w \tau)$ are equal. Since they are both homomorphisms of right H-modules, it is enough to show that they coincide at τ_{s_i} for $i = 0, 1$, namely that $\tau_w x_i = f(\tau_w \tau_{s_i})$. We proceed by induction on $\ell(w)$. Using relations 2), it is easy to check that this equality holds when w has length 0. Now let $w \in \widetilde{W}$ with length ≥ 1.

- If $u := ws_{1-i}^{-1}$ has length $< \ell(w)$ we have:

$$\begin{aligned}\tau_w x_i &= \tau_u \tau_{s_{1-i}} x_i = \tau_u x_{1-i} \tau_{s_i} \quad \text{by 3)} \\ &= f(\tau_u \tau_{s_{1-i}})\tau_{s_i} = f(\tau_u \tau_{s_{1-i}} \tau_{s_i}) = f(\tau_w \tau_{s_i}) \quad \text{by induction and then right } H\text{-equivariance.}\end{aligned}$$

- Otherwise, $v := ws_i^{-1}$ has length $< \ell(w)$ and we have

$$\begin{aligned}\tau_w x_i &= \tau_v \tau_{s_i} x_i = -\tau_v x_i e_1 \quad \text{by 1) and 2)} \\ &= -f(\tau_v \tau_{s_i}) e_1 \quad \text{by induction} \\ &= f(-\tau_v \tau_{s_i} e_1) = f(\tau_v \tau_{s_i}^2) = f(\tau_w \tau_{s_i}) \quad \text{by right } H\text{-equivariance.}\end{aligned}$$

The map $f_{(x_0,x_1)}$ of the Lemma is the map f studied above. ☐

Remark 3.16. For any F^1H-pair (x_0, x_1) in M we have $\operatorname{im}(f_{(x_0,x_1)}) \subseteq \{m \in M : \zeta m = m\zeta\}$.

3.7.2 F^1H-pairs in E^1

In this paragraph we assume that $\mathfrak{F} = \mathbb{Q}_p$ with $p \geq 5$ and that $\pi = p$.

Lemma 3.17. *The F^1H-pairs (x_0, x_1) in E^1 which are contained in $h^1(s_0) \oplus h^1(s_1) \oplus h^1(\Omega)$ are given by*

$$x_0 := -(0, c^0, 0)_{s_0} - e_{\mathrm{id}^{-1}} \cdot (c^0\iota^{-1}, 0, 0)_1 \quad \textit{and} \quad x_1 := (0, c^0, 0)_{s_1} - e_{\mathrm{id}} \cdot (0, 0, c^0\iota^{-1})_1$$

where c^0 runs over the 1-dimensional k-vector space $\operatorname{Hom}((1 + p\mathbb{Z}_p)/(1 + p^2\mathbb{Z}_p), k)$.

Proof. To check that the pairs (x_0, x_1) in the assertion are indeed F^1H-pairs is an explicit computation based on the formulas in Sections 3.4 and 3.6.

As noted in Remark 3.16, an element which satisfies the relations 1), 2) and 3) commutes with the action of ζ. We determine the elements in $h^1(s_0) \oplus h^1(s_1) \oplus h^1(\Omega)$ which commute with the action of ζ. Let x be such an element. Since the elements in the assertion of the lemma do commute with the action of ζ, we may assume that x is of the form

$$x = (c_0^-, 0, c_0^+)_{s_0} + (c_1^-, 0, c_1^+)_{s_1} + \sum_{\omega \in \Omega} (c_\omega^-, 0, c_\omega^+)_\omega \in h^1(s_0) \oplus h^1(s_1) \oplus h^1(\Omega)\ .$$

By Lemma 3.13, we know that

$$\zeta \cdot x - x \cdot \zeta \equiv (0, 0, c_0^+)_{s_0 s_1 s_0} - (c_0^-, 0, 0)_{s_0 s_1 s_0} + (c_1^-, 0, 0)_{s_1 s_0 s_1} - (0, 0, c_1^+)_{s_1 s_0 s_1} \bmod F_2 E^1\ .$$

Therefore we have $c_0^- = c_0^+ = c_1^+ = c_1^- = 0$ and $x = \sum_{\omega\in\Omega}(c_\omega^-, 0, c_\omega^+)_\omega \in h^1(\Omega)$. By Lemma 3.13 again,

$$\zeta \cdot x - x \cdot \zeta \equiv \sum_{\omega \in \Omega} \Big((0, 0, c_\omega^+)_{s_0 s_1 \omega} + (c_\omega^-, 0, 0)_{s_1 s_0 \omega} - (0, 0, c_\omega^+)_{s_1 s_0 \omega} - (c_\omega^-, 0, 0)_{s_0 s_1 \omega} \Big) \bmod F_1 E^1$$

and therefore $x = 0$. This proves that the only elements in E^1 which are contained in $h^1(s_0) \oplus h^1(s_1) \oplus h^1(\Omega)$ and commute with the action of ζ are given by the formulas announced in the lemma. Therefore, these are also the only F^1H-pairs (x_0, x_1) in $h^1(s_0) \oplus h^1(s_1) \oplus h^1(\Omega)$. ☐

In the following we choose $\mathbf{c}^0 \in \operatorname{Hom}((1 + p\mathbb{Z}_p)/(1 + p^2\mathbb{Z}_p), k)$ as in §3.2.3 and let $(\mathbf{x}_0, \mathbf{x}_1)$ be the corresponding F^1H-pair in E^1 of Lemma 3.17. Recall that the H-bimodule homomorphism $f_{(\mathbf{x}_0,\mathbf{x}_1)}$ was introduced in Lemma 3.15.

Proposition 3.18. *i. For $\tau_w \in F^1H$ we have*

$$f_{(\mathbf{x}_0,\mathbf{x}_1)}(\tau_w) = \begin{cases} (0,\mathbf{c}^0,0)_w & \text{if } w \in \widetilde{W}^0 \text{ and } \ell(w) \geq 2, \\ -(0,\mathbf{c}^0,0)_w & \text{if } w \in \widetilde{W}^1 \text{ and } \ell(w) \geq 2, \\ (0,\mathbf{c}^0,0)_{s_1\omega} - e_{\mathrm{id}} \cdot (0,0,\mathbf{c}^0\iota^{-1})_\omega & \text{if } w = s_1\omega \in s_1\Omega, \\ -(0,\mathbf{c}^0,0)_{s_0\omega} - e_{\mathrm{id}^{-1}} \cdot (\mathbf{c}^0\iota^{-1},0,0)_\omega & \text{if } w = s_0\omega \in s_0\Omega. \end{cases}$$

ii. The H-bimodule homomorphism $f_{(\mathbf{x}_0,\mathbf{x}_1)} : F^1H \longrightarrow E^1$ is injective.

iii. The image of $f_{(\mathbf{x}_0,\mathbf{x}_1)}$ is contained in the centralizer of ζ.

iv. $\mathcal{J} \circ f_{(\mathbf{x}_0,\mathbf{x}_1)} = -f_{(\mathbf{x}_0,\mathbf{x}_1)} \circ \mathcal{J}$.

v. $\Gamma_\varpi \circ f_{(\mathbf{x}_0,\mathbf{x}_1)}(\tau_w) = f_{(\mathbf{x}_0,\mathbf{x}_1)}(\tau_{\varpi w \varpi^{-1}})$ for any $\tau_w \in F^1H$.

Proof. i. For $\omega \in \Omega$ we have by definition that $f_{(\mathbf{x}_0,\mathbf{x}_1)}(\tau_{s_i\omega}) = \mathbf{x}_i\tau_\omega$. Hence the last two equalities follow directly from (69).

For the first two equalities we first consider the cases $w = s_0s_1$ and $w = s_1s_0$. By the left H-equivariance of $f_{(\mathbf{x}_0,\mathbf{x}_1)}$ we have

$$f_{(\mathbf{x}_0,\mathbf{x}_1)}(\tau_w) = \begin{cases} \tau_{s_0} \cdot \mathbf{x}_1 & \text{if } w = s_0s_1, \\ \tau_{s_1} \cdot \mathbf{x}_0 & \text{if } w = s_1s_0. \end{cases}$$

Using Prop. 3.9 one easily checks that $\tau_{s_0} \cdot \mathbf{x}_1 = -(0,\mathbf{c}^0,0)_w$ and $\tau_{s_1} \cdot \mathbf{x}_0 = (0,\mathbf{c}^0,0)_w$. The assertion for a general w follows from this by using again the left H-equivariance together with the following general observation. For any $v, w \in \widetilde{W}$ such that $\ell(v) + \ell(w) = \ell(vw)$ and $\ell(w) \geq 1$ we have, by (67) and Prop. 3.9:

$$\tau_v \cdot (0,c^0,0)_w = (0,(-1)^{\ell(v)}c^0,0)_{vw} \ .$$

ii. It is immediate from i. that the set $\{f_{(\mathbf{x}_0,\mathbf{x}_1)}(\tau_w)\}_{w \in F^1H}$ is a k-basis of $\mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)})$.

iii. This is obvious, as noted in Remark 3.16.

iv. We first check that $\mathcal{J}(\mathbf{x}_i) = -\tau_{s_i^2} \cdot \mathbf{x}_i$ holds true. The case $i = 1$ being analogous we only compute

$$\begin{aligned} \mathcal{J}(\mathbf{x}_0) &= -\mathcal{J}((0,\mathbf{c}^0,0)_{s_0}) - \mathcal{J}((\mathbf{c}^0\iota^{-1},0,0)_1)\mathcal{J}(e_{\mathrm{id}^{-1}}) \\ &= (0,\mathbf{c}^0,0)_{s_0^{-1}} - (\mathbf{c}^0\iota^{-1},0,0)_1 \cdot e_{\mathrm{id}} \quad \text{by Lemma 3.7} \\ &= (0,\mathbf{c}^0,0)_{s_0^2s_0} - e_{\mathrm{id}^{-1}} \cdot (\mathbf{c}^0\iota^{-1},0,0)_1 \quad \text{by (67) and (69)} \\ &= \tau_{s_0^2} \cdot (0,\mathbf{c}^0,0)_{s_0} - e_{\mathrm{id}^{-1}} \cdot (\mathbf{c}^0\iota^{-1},0,0)_1 \quad \text{by (68)} \\ &= \tau_{s_0^2} \cdot (0,\mathbf{c}^0,0)_{s_0} + \tau_{s_0^2}e_{\mathrm{id}^{-1}} \cdot (\mathbf{c}^0\iota^{-1},0,0)_1 \quad \text{by } -e_{\mathrm{id}^{-1}} = \tau_{s_0^2} \cdot e_{\mathrm{id}^{-1}} \\ &= -\tau_{s_0^2} \cdot \mathbf{x}_0 \ . \end{aligned}$$

For a general $w \in \widetilde{W}^{1-i,\ell\geq 1}$ we have $\tau_w = \tau_{s_i}\tau_{s_i^{-1}w}$ and we deduce that

$$\begin{aligned} \mathcal{J}(f_{(\mathbf{x}_0,\mathbf{x}_1)}(\tau_w)) = \mathcal{J}(\mathbf{x}_i \cdot \tau_{s_i^{-1}w}) = \mathcal{J}(\tau_{s_i^{-1}w}) \cdot \mathcal{J}(\mathbf{x}_i) = -\tau_{w^{-1}s_i}\tau_{s_i^2} \cdot \mathbf{x}_i = -\tau_{w^{-1}s_i^{-1}} \cdot \mathbf{x}_i \\ = -f_{(\mathbf{x}_0,\mathbf{x}_1)}(\tau_{w^{-1}}) = -f_{(\mathbf{x}_0,\mathbf{x}_1)}(\mathcal{J}(\tau_w)) \end{aligned}$$

using left H-equivariance in the fifth equality.

v. Lemma 3.4 easily implies that $\Gamma_\varpi(\mathbf{x}_i) = \mathbf{x}_{1-i}$. For a general $w \in \widetilde{W}^{1-i,\ell\geq 1}$ we have $\varpi w \varpi^{-1} \in \widetilde{W}^{i,\ell\geq 1}$ and we deduce that

$$\begin{aligned}\Gamma_\varpi(f_{(\mathbf{x}_0,\mathbf{x}_1)}(\tau_w)) &= \Gamma_\varpi(\mathbf{x}_i \cdot \tau_{s_i^{-1}w}) = \Gamma_\varpi(\mathbf{x}_i)\cdot\Gamma_\varpi(\tau_{s_i^{-1}w}) = \mathbf{x}_{1-i}\cdot\tau_{\varpi s_i^{-1}w\varpi^{-1}} \\ &= \mathbf{x}_{1-i}\cdot\tau_{s_{1-i}^{-1}\varpi w\varpi^{-1}} = f_{(\mathbf{x}_0,\mathbf{x}_1)}(\tau_{\varpi w\varpi^{-1}})\end{aligned}$$

using in the second equality that Γ_ϖ is multiplicative (cf. §2.2.6). □

In Prop. 6.3 we will see that the inclusion in part ii. of the above proposition, in fact, is an equality. This, in particular, shows that there are no nonzero F^1H-pairs in $E^1 \setminus \mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)})$.

Remark 3.19. Recalling that e_{γ_0} was introduced in (39) we have

$$(1-e_{\gamma_0})\cdot\mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)}) = \mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)})\cdot(1-e_{\gamma_0}) = (1-e_{\gamma_0})\cdot\mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)})\cdot(1-e_{\gamma_0}) = (1-e_{\gamma_0})\cdot h^1_0(\widetilde{W})\ .$$

Proof. Since $1-e_{\gamma_0}$ is central in H it also must centralize $\mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)})$. Recall that $e_{\gamma_0} = e_{\mathrm{id}} + e_{\mathrm{id}^{-1}}$. The last equality then is immediate from Prop. 3.18-i. □

3.7.3 An H_ζ-bimodule inside E^1

3.7.3.1 A left H_ζ-bimodule inside E^1 Let M be any H-bimodule. To give a homomorphism of left (or right) H-modules $f : H \longrightarrow M$ simply means to give any element $x \in M$ as the image $x = f(1)$. We state a simple sufficient condition on x such that the corresponding f extends to the localization H_ζ.

Lemma 3.20. *Let $x \in M$ be such that $\zeta\cdot x\cdot\zeta = x$. Then*

$$\begin{aligned} H_\zeta &\longrightarrow M \\ \zeta^{-i}\tau &\longmapsto f_x(\zeta^{-i}\tau) := \tau\cdot x\cdot\zeta^i,\ \textit{resp.}\ {}_xf(\zeta^{-i}\tau) := \zeta^i\cdot x\cdot\tau,\ \textit{for } i\geq 0 \textit{ and } \tau\in H,\end{aligned}$$

is a well defined homomorphism of left, resp. right, H-modules; its image is contained in the space $\{y \in M : \zeta\cdot y\cdot\zeta = y\}$.

Proof. Easy exercise. □

Assume that $\mathfrak{F} = \mathbb{Q}_p$ with $p \geq 5$ and that $\pi = p$. We will apply the above lemma to the bimodule E^1.

Lemma 3.21. *The elements $x \in E^1$ which satisfy $\zeta\cdot x\cdot\zeta = x$ and lie in $h^1(1)\oplus e_{\mathrm{id}}h^1(s_0)\oplus e_{\mathrm{id}}h^1(s_1s_0)$ with $\tau_{s_0}\cdot x = 0$, resp. in $h^1(1)\oplus e_{\mathrm{id}^{-1}}h^1(s_1)\oplus e_{\mathrm{id}^{-1}}h^1(s_0s_1)$ with $\tau_{s_1}\cdot x = 0$, are*

$$\begin{aligned} x^+ &:= (0,0,c^+)_1 - e_{\mathrm{id}}\cdot(0,2c^+\iota,0)_{s_0} - e_{\mathrm{id}}\cdot(0,0,c^+)_{s_1s_0},\ \textit{resp.} \\ x^- &:= (c^-,0,0)_1 + e_{\mathrm{id}^{-1}}\cdot(0,2c^-\iota,0)_{s_1} - e_{\mathrm{id}^{-1}}\cdot(c^-,0,0)_{s_0s_1},\end{aligned}$$

where c^+ and c^- run over the 1-dimensional k-vector space $\mathrm{Hom}(\mathbb{Z}_p/p\mathbb{Z}_p, k)$.

Proof. We treat the first case, the other one being analogous. Consider any

$$x = (c^-,0,c^+)_1 + e_{\mathrm{id}}\cdot(b^-,b^0,b^+)_{s_0} + e_{\mathrm{id}}\cdot(d^-,d^0,d^+)_{s_1s_0}$$

such that $\tau_{s_0}\cdot x = 0$. Using Prop. 3.9 we compute

$$\begin{aligned} 0 = \tau_{s_0}\cdot x &= \tau_{s_0}\cdot(c^-,0,c^+)_1 + e_{\mathrm{id}^{-1}}\tau_{s_0}\cdot(b^-,b^0,b^+)_{s_0} + e_{\mathrm{id}^{-1}}\tau_{s_0}\cdot(d^-,d^0,d^+)_{s_1s_0} \\ &= (0,0,-c^-)_{s_0} + e_{\mathrm{id}^{-1}}\cdot\big(-e_1\cdot(b^-,b^0,b^+)_{s_0} + e_{\mathrm{id}}\cdot(0,-2b^-\iota,b^0\iota^{-1})_{s_0} + e_{\mathrm{id}^2}\cdot(0,0,b^-)_{s_0} - (0,0,b^-)_{s_0^2}\big) \\ &\qquad - e_{\mathrm{id}^{-1}}\cdot(0,d^0,d^-)_{s_0s_1s_0} \\ &= (0,0,-c^-)_{s_0} - e_{\mathrm{id}^{-1}}\cdot(0,0,b^-)_{s_0^2} - e_{\mathrm{id}^{-1}}\cdot(0,d^0,d^-)_{s_0s_1s_0}\ .\end{aligned}$$

It follows that $c^- = b^- = d^0 = d^- = 0$ and hence that

(75) $$x = (0,0,c^+)_1 + e_{\mathrm{id}} \cdot (0, b^0, b^+)_{s_0} + e_{\mathrm{id}} \cdot (0,0,d^+)_{s_1 s_0} \ .$$

Now we assume in addition that $\zeta \cdot x \cdot \zeta = x$. From Cor. 3.10 we deduce that

$$\begin{aligned}
\zeta \cdot x &= \zeta \cdot (0,0,c^+)_1 + e_{\mathrm{id}}\zeta \cdot (0, b^0, b^+)_{s_0} + e_{\mathrm{id}}\zeta \cdot (0,0,d^+)_{s_1 s_0} \\
&= (0,0,c^+)_{s_0 s_1} - e_1 \cdot (c^+,0,0)_{s_1} + e_1 \cdot (0,0,c^+)_1 + e_{\mathrm{id}} \cdot (0,b^0,b^+)_{s_0 s_1 s_0} \\
&\quad - e_{\mathrm{id}} \cdot (0, 2d^+\iota, 0)_{s_0 s_1 s_0} + e_{\mathrm{id}} \cdot (0, 2d^+\iota, 0)_{s_1^2 s_0} + e_{\mathrm{id}} \cdot (0,0,d^+)_1 \ .
\end{aligned}$$

Using Lemma 3.7, Cor. 3.10, Section 3.4, and (72) we compute

(76) $$\begin{aligned}
(0,0,c^+)_{s_0 s_1} \cdot \zeta &= -e_{\mathrm{id}} \cdot (0,2c^+\iota,0)_{s_0 s_1 s_0} - e_{\mathrm{id}} \cdot (0,2c^+\iota,0)_{s_0} + e_1 \cdot (c^+,0,0)_{s_0} + (0,0,c^+)_1 \\
-e_1 \cdot (c^+,0,0)_{s_1} \cdot \zeta &= -e_1 \cdot (0,0,c^+)_{s_1 s_0} - e_1 \cdot (c^+,0,0)_{s_0} \\
e_{\mathrm{id}} \cdot (0,2d^+\iota,0)_{s_1^2 s_0} \cdot \zeta &= -e_{\mathrm{id}} \cdot (0,2d^+\iota,0)_{s_0 s_1 s_0} \\
e_1 \cdot (0,0,c^+)_1 \cdot \zeta &= e_1 \cdot (0,0,c^+)_{s_1 s_0} \\
e_{\mathrm{id}} \cdot (0,0,d^+)_1 \cdot \zeta &= e_{\mathrm{id}} \cdot (0,0,d^+)_{s_1 s_0} \\
e_{\mathrm{id}} \cdot (0,b^0,b^+)_{s_0 s_1 s_0} \cdot \zeta &= -e_{\mathrm{id}} \cdot (0,b^0,0)_{(s_1 s_0)^2 s_0^2} + e_{\mathrm{id}} \cdot (0,2b^+\iota,0)_{(s_0 s_1)^2} + e_{\mathrm{id}} \cdot (0,0,b^+)_{s_0} \\
-e_{\mathrm{id}} \cdot (0,2d^+\iota,0)_{s_0 s_1 s_0} \cdot \zeta &= e_{\mathrm{id}} \cdot (0,2d^+\iota,0)_{(s_1 s_0)^2 s_0^2} \ .
\end{aligned}$$

Comparing the sum of these equations with (75) shows that $d^+ = -c^+$, $b^0 = -2c^+\iota$, and $b^+ = 0$. We conclude that $x = x^+$. □

We now choose $c^+ := c^- := \mathbf{c} \in \mathrm{Hom}(\mathbb{Z}_p/p\mathbb{Z}_p, k)$ as in §3.2.3 and let $(\mathbf{x}^+, \mathbf{x}^-)$ be the corresponding elements of Lemma 3.21. By Lemma 3.20 they give rise to the left H-module homomorphisms

(77) $$f_{\mathbf{x}^\pm} : H_\zeta \longrightarrow E^1 \ .$$

Remark 3.22. 1. We have $\Gamma_\varpi(\zeta) = \zeta$. Hence Γ_ϖ extends to an automorphism of H_ζ. The multiplicativity of Γ_ϖ, the formula $\Gamma_\varpi(e_\lambda) = e_{\lambda^{-1}}$, and Lemma 3.4 then imply that

$$\Gamma_\varpi \circ f_{\mathbf{x}^+} = f_{\mathbf{x}^-} \circ \Gamma_\varpi \quad \text{and} \quad \Gamma_\varpi \circ f_{\mathbf{x}^-} = f_{\mathbf{x}^+} \circ \Gamma_\varpi$$

and, in particular, $\Gamma_\varpi(\mathbf{x}^+) = \mathbf{x}^-$.

2. Here and in the subsequent points let x^- and x^+ be as in Lemma 3.21. We compute

$$\begin{aligned}
\mathcal{J}(x^+) &= \mathcal{J}((0,0,c^+)_1) - \mathcal{J}((0,2c^+\iota,0)_{s_0}) \cdot \mathcal{J}(e_{\mathrm{id}}) - \mathcal{J}((0,0,c^+)_{s_1 s_0}) \cdot \mathcal{J}(e_{\mathrm{id}}) \\
&= (0,0,c^+)_1 + (0,2c^+\iota,0)_{s_0^{-1}} \cdot e_{\mathrm{id}^{-1}} - (0,0,c^+)_{s_0 s_1} \cdot e_{\mathrm{id}^{-1}} \quad \text{by Lemma 3.7} \\
&= (0,0,c^+)_1 + e_{\mathrm{id}} \cdot (0,2c^+\iota,0)_{s_0^{-1}} - e_{\mathrm{id}} \cdot (0,0,c^+)_{s_0 s_1} \quad \text{by (67) and (69)} \\
&= (0,0,c^+)_1 + e_{\mathrm{id}}\tau_{s_0^2} \cdot (0,2c^+\iota,0)_{s_0} - e_{\mathrm{id}} \cdot (0,0,c^+)_{s_0 s_1} \quad \text{by (68)} \\
&= (0,0,c^+)_1 - e_{\mathrm{id}} \cdot (0,2c^+\iota,0)_{s_0} - e_{\mathrm{id}} \cdot (0,0,c^+)_{s_0 s_1} \quad \text{by } -e_{\mathrm{id}}\tau_{s_0^2} = e_{\mathrm{id}} \\
&= x^+ + e_{\mathrm{id}} \cdot (0,0,c^+)_{s_1 s_0} - e_{\mathrm{id}} \cdot (0,0,c^+)_{s_0 s_1} \\
&= (1 - e_{\mathrm{id}} - e_{\mathrm{id}}\tau_{s_0 s_1}) \cdot x^+ \quad \text{by Prop. 3.9} \\
&= (1 - e_{\mathrm{id}} - e_{\mathrm{id}}\zeta) \cdot x^+ \ .
\end{aligned}$$

and similarly

$$\mathcal{J}(x^-) = (1 - e_{\mathrm{id}^{-1}} - e_{\mathrm{id}^{-1}}\zeta) \cdot x^- \ .$$

Lemma 3.23. *1. For any $u \in \mathbb{F}_p^\times$ we have $x^+ \cdot \tau_{\omega_u} = u^{-2}\tau_{\omega_u} \cdot x^+$ and $x^- \cdot \tau_{\omega_u} = u^2\tau_{\omega_u} \cdot x^-$.*

2. We have $x^+ \cdot \tau_{s_0} = \tau_{s_0} \cdot x^+ = 0$ and $x^- \cdot \tau_{s_1} = \tau_{s_1} \cdot x^- = 0$.

3. We have

$$\begin{aligned} x^- \cdot \iota(\tau_{s_1}) &= -e_{\mathrm{id}^{-2}} \cdot x^- \quad \textit{and} \\ x^+ \cdot \iota(\tau_{s_0}) &= -e_{\mathrm{id}^2} \cdot x^+ \ . \end{aligned}$$

while, for $\mathbf{x}^+$ and $\mathbf{x}^-$ as above,

$$\begin{aligned} \mathbf{x}^+ \cdot \iota(\tau_{s_1}) &= -\tau_{\omega_{-1}}\iota(\tau_{s_0}) \cdot \mathbf{x}^- \cdot \zeta \quad \textit{and} \\ \mathbf{x}^- \cdot \iota(\tau_{s_0}) &= -\tau_{\omega_{-1}}\iota(\tau_{s_1}) \cdot \mathbf{x}^+ \cdot \zeta \ . \end{aligned}$$

where we recall that the involution ι was introduced in (29).

Proof. 1. For any $u \in \mathbb{F}_p^\times$ we compute using (67) and (69)

$$\begin{aligned} x^+ \cdot \tau_{\omega_u} &= (0,0,c^+)_1 \cdot \tau_{\omega_u} - e_{\mathrm{id}} \cdot (0, 2c^+\iota, 0)_{s_0} \cdot \tau_{\omega_u} - e_{\mathrm{id}} \cdot (0,0,c^+)_{s_1 s_0} \cdot \tau_{\omega_u} \\ &= (0,0,c^+)_{\omega_u} - e_{\mathrm{id}} \cdot (0, 2c^+\iota, 0)_{\omega_u^{-1} s_0} - e_{\mathrm{id}} \cdot (0,0,c^+)_{\omega_u s_1 s_0} \\ &= u^{-2}\tau_{\omega_u} \cdot (0,0,c^+)_1 - e_{\mathrm{id}}\tau_{\omega_u^{-1}} \cdot (0, 2c^+\iota, 0)_{s_0} - u^{-2} e_{\mathrm{id}}\tau_{\omega_u} \cdot (0,0,c^+)_{s_1 s_0} \\ &= u^{-2}\tau_{\omega_u} \cdot (0,0,c^+)_1 - u^{-1} e_{\mathrm{id}} \cdot (0, 2c^+\iota, 0)_{s_0} - u^{-2} e_{\mathrm{id}}\tau_{\omega_u} \cdot (0,0,c^+)_{s_1 s_0} \\ &= u^{-2}\tau_{\omega_u} \cdot (0,0,c^+)_1 - u^{-2}\tau_{\omega_u} e_{\mathrm{id}} \cdot (0, 2c^+\iota, 0)_{s_0} - u^{-2} e_{\mathrm{id}}\tau_{\omega_u} \cdot (0,0,c^+)_{s_1 s_0} \\ &= u^{-2}\tau_{\omega_u} \cdot x^+ \end{aligned}$$

and, by an analogous computation (or by applying Remark 3.22-1), we obtain $x^- \cdot \tau_{\omega_u} = u^2\tau_{\omega_u} \cdot x^-$.
2. For the identity $\tau_{s_0} \cdot x^+ = 0$, see Lemma 3.21. Now we compute:

$$\begin{aligned} x^+ \cdot \tau_{s_0} &= \mathcal{J}(\mathcal{J}(\tau_{s_0}) \cdot \mathcal{J}(x^+)) = \mathcal{J}(\tau_{s_0^2}\tau_{s_0} \cdot (x^+ + e_{\mathrm{id}}(0,0,c^+)_{s_1 s_0} - e_{\mathrm{id}} \cdot (0,0,c^+)_{s_0 s_1})) \quad \text{by Remark 3.22-2} \\ &= \mathcal{J}(\tau_{s_0^2}(\tau_{s_0} \cdot x^+ + e_{\mathrm{id}^{-1}}\tau_{s_0} \cdot (0,0,c^+)_{s_1 s_0} - e_{\mathrm{id}^{-1}}\tau_{s_0} \cdot (0,0,c^+)_{s_0 s_1})) \\ &= 0 \quad \text{by Lemma 3.21 and Prop. 3.9} \end{aligned}$$

We obtain the analogous statements for x^- using Remark 3.22-1.
3. The first identities easily come from Points 1 and 2. We treat the second equation of the last statement. The first one can either be established by an analogous computation or by applying Remark 3.22-1 to the second equation. Both sides of the second equation lie in the sub-H-bimodule $\ker(\zeta \cdot \mathrm{id}_{E^1} \cdot \zeta - \mathrm{id}_{E^1})$ of E^1 on which left multiplication by ζ is injective. Hence we may instead check the equation

$$-\zeta \cdot \mathbf{x}^- \cdot (\tau_{s_0} + e_1) = \tau_{\omega_{-1}} \cdot (\tau_{s_1} + e_1) \cdot \mathbf{x}^+ \ .$$

For the left hand side we first have, using Lemma 3.12 and Point 1:

$$\begin{aligned} x^- \cdot (\tau_{s_0} + e_1) &= (c^-,0,0)_{s_0} + e_{\mathrm{id}^{-1}} \cdot (0, 2c^-\iota, 0)_{s_1 s_0} - e_{\mathrm{id}^{-1}} \cdot (c^-,0,0)_{s_0 s_1 s_0} + e_{\mathrm{id}^{-2}} \cdot x^- \\ &= (c^-,0,0)_{s_0} + e_{\mathrm{id}^{-1}} \cdot (0, 2c^-\iota, 0)_{s_1 s_0} - e_{\mathrm{id}^{-1}} \cdot (c^-,0,0)_{s_0 s_1 s_0} + e_{\mathrm{id}^{-2}} \cdot (c^-,0,0)_1 \end{aligned}$$

and then by Cor. 3.10

$$\begin{aligned} -\zeta \cdot x^- \cdot (\tau_{s_0} + e_1) &= e_{\mathrm{id}^{-2}} \cdot (c^-,0,0)_{s_1 s_0} - (c^-,0,0)_{s_1\omega_{-1}} \\ &\qquad + e_{\mathrm{id}^{-1}} \cdot (c^-,0,0)_{s_0} + e_1 \cdot (0,0,c^-)_1 - e_{\mathrm{id}^{-2}} \cdot (c^-,0,0)_{s_1 s_0} \\ &= -(c^-,0,0)_{s_1\omega_{-1}} + e_{\mathrm{id}^{-1}} \cdot (c^-,0,0)_{s_0} + e_1 \cdot (0,0,c^-)_1 \ . \end{aligned}$$

For the right hand side we first compute using Prop. 3.9

$$\tau_{s_1 s_0^2} \cdot x^+ = -(c^+, 0, 0)_{s_1\omega_{-1}} + e_{\mathrm{id}^{-1}} \cdot (c^+, 0, 0)_{s_0}$$
$$e_1 \cdot x^+ = e_1 \cdot (0, 0, c^+)_1$$

and then see, by adding up, that it coincides with the above computation for the left hand side when $c^+ = c^- = \mathbf{c}$. ☐

Lemma 3.24. *The maps $f_{\mathbf{x}^+}$ and $f_{\mathbf{x}^-}$ defined in (77) induce an injective homomorphism of left H-modules*

$$H_\zeta/H_\zeta\tau_{s_0} \oplus H_\zeta/H_\zeta\tau_{s_1} \xrightarrow{f^\pm := f_{\mathbf{x}^+} + f_{\mathbf{x}^-}} E^1$$

the image of which is contained in the kernel of the endomorphism $\zeta \cdot \mathrm{id}_{E^1} \cdot \zeta - \mathrm{id}_{E^1}$.

Proof. By Lemma 3.23-2, the map is well defined. By definition of $\mathbf{x}^+$and $\mathbf{x}^-$, the last statement of the lemma is clear. We prove that the map is the injective. We first observe that it suffices to check the injectivity of the restriction of $f^\pm$ to $H/H\tau_{s_0} \oplus H/H\tau_{s_1}$. The elements τ_w with $w \in \widetilde{W}$ such that $\ell(ws_0) = \ell(w) + 1$ form a k-basis of $H/H\tau_{s_0}$; they are of the form $w = \omega(s_0s_1)^m$ or $= \omega s_1(s_0s_1)^m$ with $m \geq 0$ and $\omega \in \Omega$. Using (67) and Prop. 3.9 we obtain

$$\tau_w \cdot (0, 0, c^+)_1 \in \begin{cases} \mathbb{F}_p^\times (0, 0, c^+)_w & \text{if } w = \omega(s_0s_1)^m, \\ \mathbb{F}_p^\times (c^+, 0, 0)_w & \text{if } w = \omega s_1(s_0s_1)^m, \end{cases}$$

and

$$\tau_w \cdot (0, c^0, 0)_{s_0} = (0, (-1)^{\ell(w)}c^0, 0)_{ws_0} \in h^1_0(\widetilde{W}) \ \text{ for any } w \text{ as above,}$$

and

$$\tau_w e_{\mathrm{id}} \cdot (0, 0, c^+)_{s_1s_0} \in \begin{cases} F_{\ell(w)-2}E^1 + h^1_0(\widetilde{W}) & \text{for any } w \text{ as above with } m \geq 1, \\ \mathbb{F}_p^\times e_{\mathrm{id}^{-1}} \cdot (c^+, 0, 0)_{s_0} + h^1_0(\widetilde{W}) & \text{if } w = \omega s_1, \\ \mathbb{F}_p^\times e_{\mathrm{id}} \cdot (0, 0, c^+)_{\omega s_1 s_0} & \text{if } w = \omega. \end{cases}$$

It follows that

$$\tau_w \cdot \mathbf{x}^+ \in \begin{cases} k^\times (0, 0, \mathbf{c})_w + F_{\ell(w)-2}E^1 + h^1_0(\widetilde{W}) & \text{if } w = \omega(s_0s_1)^m \text{ with } m \geq 1, \\ k^\times (\mathbf{c}, 0, 0)_w + F_{\ell(w)-2}E^1 + h^1_0(\widetilde{W}) & \text{if } w = \omega s_1(s_0s_1)^m \text{ with } m \geq 1, \\ k^\times (\mathbf{c}, 0, 0)_w + k^\times e_{\mathrm{id}^{-1}}(\mathbf{c}, 0, 0)_{s_0} + h^1_0(\widetilde{W}) & \text{if } w = \omega s_1, \\ k^\times (0, 0, \mathbf{c})_w + k^\times e_{\mathrm{id}}(0, 0, \mathbf{c})_{\omega s_1 s_0} + h^1_0(\widetilde{W}) & \text{if } w = \omega. \end{cases} \tag{78}$$

Similarly the elements τ_w with $w \in \widetilde{W}$ such that $\ell(ws_1) = \ell(w) + 1$ form a k-basis of $H/H\tau_{s_1}$; they are of the form $w = \omega(s_1s_0)^m$ or $= \omega s_0(s_1s_0)^m$ with $m \geq 0$ and $\omega \in \Omega$. In this case we obtain

$$\tau_w \cdot \mathbf{x}^- \in \begin{cases} k^\times (\mathbf{c}, 0, 0)_w + F_{\ell(w)-2}E^1 + h^1_0(\widetilde{W}) & \text{if } w = \omega(s_1s_0)^m \text{ with } m \geq 1, \\ k^\times (0, 0, \mathbf{c})_w + F_{\ell(w)-2}E^1 + h^1_0(\widetilde{W}) & \text{if } w = \omega s_0(s_1s_0)^m \text{ with } m \geq 1, \\ k^\times (0, 0, \mathbf{c})_w + k^\times e_{\mathrm{id}} \cdot (0, 0, \mathbf{c})_{s_1} + h^1_0(\widetilde{W}) & \text{if } w = \omega s_0, \\ k^\times (\mathbf{c}, 0, 0)_w + k^\times e_{\mathrm{id}^{-1}} \cdot (\mathbf{c}, 0, 0)_{\omega s_0 s_1} + h^1_0(\widetilde{W}) & \text{if } w = \omega. \end{cases} \tag{79}$$

By comparing the lists (78) and (79) we easily see that the elements

$$\{\tau_w \cdot \mathbf{x}^+ : \ell(ws_0) = \ell(w) + 1\} \cup \{\tau_w \cdot \mathbf{x}^- : \ell(ws_1) = \ell(w) + 1\}$$

in E^1 are k-linearly independent even in $E^1/h^1_0(\widetilde{W})$. ☐

3.7.3.2 Structure of H_ζ-bimodule on $H_\zeta/H_\zeta\tau_{s_0}\oplus H_\zeta/H_\zeta\tau_{s_1}$. In this paragraph, the only condition on $\mathfrak{F}$ is that it has residue field $\mathbb{F}_p$. Recall the involution ι of H defined in (29).

We consider the homomorphism of k-algebras $\kappa : H \to H_\zeta$ given by the composition of the involution $\iota : H \to H$ and the inclusion $H \to H_\zeta$, the element $-\tau_{\omega_{-1}}\zeta^{-1} \in Z(H_\zeta)$ in the center of H_ζ and the character $\mu : \Omega \to k^\times, \omega_u \mapsto u^2$. Recall that as in Remark 2.12, we may refer to the idempotent corresponding to the latter as e_{id^2} instead of e_μ. As in §2.4.7, this yields a homomorphism of k-algebras $\kappa_2 : H \to M_2(H_\zeta)$ and an (H_ζ, H)-bimodule structure on $H_\zeta \oplus H_\zeta$ denoted by $(H_\zeta \oplus H_\zeta)[\kappa, -\tau_{\omega_{-1}}\zeta^{-1}, \mu]$ where $h \in H$ acts on $(\sigma^+, \sigma^-) \in H_\zeta \oplus H_\zeta$ via

$$((\sigma^+, \sigma^-), h) \longmapsto (\sigma^+, \sigma^-)\kappa_2(h) .$$

We consider the composite map $\kappa_2 \circ \iota^{-1}$. Again, it is a homomorphism of algebras $H \to M_2(H_\zeta)$ and it yields an (H_ζ, H)-bimodule structure on $H_\zeta \oplus H_\zeta$ denoted by $(H_\zeta \oplus H_\zeta)^\pm$. We spell out below the action on $(\sigma^+, \sigma^-) \in H_\zeta \oplus H_\zeta$ of the generators $\iota(\tau_{s_0})$, $\iota(\tau_{s_1})$, τ_{ω_u} for $u \in \mathbb{F}_p^\times$ of H

$$\begin{aligned}(\sigma^+, \sigma^-)\iota(\tau_{s_0}) &:= (-\sigma^+ e_{\mathrm{id}^2} - \sigma^-\tau_{\omega_{-1}}\iota(\tau_{s_1})\zeta^{-1}, 0)\\ (\sigma^+, \sigma^-)\iota(\tau_{s_1}) &:= (0, -\sigma^- e_{\mathrm{id}^{-2}} - \sigma^+\tau_{\omega_{-1}}\iota(\tau_{s_0})\zeta^{-1})\\ (\sigma^+, \sigma^-)\tau_{\omega_u} &:= (u^{-2}\sigma^+\tau_{\omega_u}, u^2\sigma^-\tau_{\omega_u}) .\end{aligned} \tag{80}$$

One easily checks that
$(\tau_{s_0}, 0)\iota(\tau_{s_1}) = (0, \tau_{s_1})\iota(\tau_{s_0}) = 0$,
$(\tau_{s_0}, 0)\iota(\tau_{s_0}) = -e_{\mathrm{id}^{-2}}(\tau_{s_0}, 0)$ and $(0, \tau_{s_1})\iota(\tau_{s_1}) = -e_{\mathrm{id}^2}(0, \tau_{s_1})$, and lastly
$(\tau_{s_0}, 0)\tau_{\omega_u} = u^{-2}\tau_{\omega_u^{-1}}(\tau_{s_0}, 0)$ and $(0, \tau_{s_1})\tau_{\omega_u} = u^2\tau_{\omega_u^{-1}}(0, \tau_{s_1})$. Hence this bimodule structure passes to the quotient $(H_\zeta/H_\zeta\tau_{s_0} \oplus H_\zeta/H_\zeta\tau_{s_1})^\pm$.

Remark 3.25. In $(H_\zeta/H_\zeta\tau_{s_0} \oplus H_\zeta/H_\zeta\tau_{s_1})^\pm$, we have

$$\tau_{s_0}(1,0) = (1,0)\tau_{s_0} = 0 \quad \text{ and } \tau_{s_1}(1,0) = (0,0)\tau_{s_1} = 0 . \tag{81}$$

The only non obvious statement is for the right actions. We prove it in the first case (it is actually a computation in $(H_\zeta \oplus H_\zeta)^\pm$):

$$\begin{aligned}(1,0)\tau_{s_0} &= -(1,0)\iota(\tau_{s_0}) - (1,0)e_1 = (e_{\mathrm{id}^2}, 0) + \sum_u (1,0)\tau_{\omega_u}\\ &= (e_{\mathrm{id}^2}, 0) + \sum_u (u^{-2}\tau_{\omega_u}, 0) = (e_{\mathrm{id}^2}, 0) - (e_{\mathrm{id}^2}, 0) = 0 .\end{aligned}$$

Lemma 3.26. *For any $\sigma \in (H_\zeta/H_\zeta\tau_{s_0} \oplus H_\zeta/H_\zeta\tau_{s_1})^\pm$ we have $\zeta\sigma\zeta = \sigma$ In particular, $(H_\zeta/H_\zeta\tau_{s_0} \oplus H_\zeta/H_\zeta\tau_{s_1})^\pm$ is an (H_ζ, H_ζ)-bimodule.*

Proof. It suffices to show that $\zeta(1,0)\zeta \equiv (1,0)$ and $\zeta(0,1)\zeta \equiv (0,1)$. Here and in the following we write $\equiv$ and $=$, for greater clarity, if an equality holds in $\sigma \in (H_\zeta/H_\zeta\tau_{s_0} \oplus H_\zeta/H_\zeta\tau_{s_1})^\pm$ and $(H_\zeta \oplus H_\zeta)^\pm$, respectively. We give the computation in the first case:

$$\begin{aligned}\zeta(1,0)\zeta &= \zeta(1,0)\iota(\tau_{s_1})\iota(\tau_{s_0}) \text{ by (81)}\\ &= \zeta(1,0)(0, -\tau_{\omega_{-1}}\iota(\tau_{s_0})\zeta^{-1})\iota(\tau_{s_0})\\ &= \zeta(1,0)(\tau_{\omega_{-1}}\iota(\tau_{s_0})\zeta^{-1}\tau_{\omega_{-1}}\iota(\tau_{s_1})\zeta^{-1}, 0) = \zeta(\iota(\tau_{s_0})\iota(\tau_{s_1})\zeta^{-2}, 0)\\ &= \zeta(\zeta^{-2}(\zeta - \tau_{s_1}\tau_{s_0}), 0) \equiv \zeta(\zeta^{-1}, 0) = (1,0) .\end{aligned}$$

□

Lemma 3.27. *We have an isomorphism of right H_ζ-modules*

$$\beta : H_\zeta/\tau_{s_0}H_\zeta \oplus H_\zeta/\tau_{s_1}H_\zeta \xrightarrow{\cong} (H_\zeta/H_\zeta\tau_{s_0} \oplus H_\zeta/H_\zeta\tau_{s_1})^{\pm}$$

sending $(1,0)$ *and* $(0,1)$ *to* $(1,0)$ *and* $(0,1)$*, respectively. In particular,* $(H_\zeta/H_\zeta\tau_{s_0} \oplus H_\zeta/H_\zeta\tau_{s_1})^{\pm}$ *is a free* $k[\zeta^{\pm 1}]$*-module of rank* $4(p-1)$ *on the left and on the right.*

Proof. That the rule given to define β yields a well defined module homomorphism is immediate from the fact that $(1,0)\tau_{s_0} = (0,1)\tau_{s_1} = 0$ (see (81)). To check the bijectivity we start by observing that, as a consequence of Lemma 2.7, a k-basis of $H_\zeta/\tau_{s_i}H_\zeta$ as well as $H_\zeta/H_\zeta\tau_{s_i}$ is given by

$$\{\zeta^j\tau_{\omega_u} : j \in \mathbb{Z}, u \in \mathbb{F}_q^\times\} \cup \{\zeta^j\tau_{\omega_u}\iota(\tau_{s_{1-i}}) : j \in \mathbb{Z}, u \in \mathbb{F}_q^\times\}$$

where we use the involution (29) of H. It follows that $H_\zeta/\tau_{s_0}H_\zeta \oplus H_\zeta/\tau_{s_1}H_\zeta$ and $(H_\zeta/H_\zeta\tau_{s_0} \oplus H_\zeta/H_\zeta\tau_{s_1})^{\pm}$ both have the k-basis

$$\{(\zeta^j\tau_{\omega_u}, 0),\ (\zeta^j\tau_{\omega_u}\iota(\tau_{s_1}), 0),\ (0, \zeta^j\tau_{\omega_u}),\ (0, \zeta^j\tau_{\omega_u}\iota(\tau_{s_0})) : \quad j \in \mathbb{Z}, u \in \mathbb{F}_q^\times\}.$$

The image under β of this set is

$$\{u^{-2}(\zeta^{-j}\tau_{\omega_u}, 0),\ -u^{-2}(0, \zeta^{-j-1}\tau_{\omega_{-u}}\iota(\tau_{s_0})), u^2(0, \zeta^{-j}\tau_{\omega_u}), -u^2(\zeta^{-j-1}\tau_{\omega_{-u}}\iota(\tau_{s_1}), 0) \quad : j \in \mathbb{Z}, u \in \mathbb{F}_q^\times\}.$$

which is a basis for $(H_\zeta/H_\zeta\tau_{s_0} \oplus H_\zeta/H_\zeta\tau_{s_1})^{\pm}$.

The $k[\zeta^{\pm 1}]$-modules $H_\zeta/H_\zeta\tau_{s_0} \oplus H_\zeta/H_\zeta\tau_{s_1}$ and $H_\zeta/\tau_{s_0}H_\zeta \oplus H_\zeta/\tau_{s_1}H_\zeta$ are free $k[\zeta^{\pm 1}]$-modules of rank $4(p-1)$ (respectively on the left and on the right). The last statement follows. □

3.7.3.3 On $\mathrm{im}(f^{\pm})$.

In this paragraph we assume that $\mathfrak{F} = \mathbb{Q}_p$ with $p \geq 5$ and that $\pi = p$.

Proposition 3.28. *The map* $f^{\pm}$ *in Lemma 3.24 yields an injective homomorphism of* H*-bimodules*

$$(H_\zeta/H_\zeta\tau_{s_0} \oplus H_\zeta/H_\zeta\tau_{s_1})^{\pm} \longrightarrow E^1$$

which we still denote by $f^{\pm}$*. Its image* $\mathrm{im}(f^{\pm})$ *is contained in the kernel of the endomorphism* $\zeta \cdot \mathrm{id}_{E^1} \cdot \zeta - \mathrm{id}_{E^1}$ *and is a sub-*H*-bimodule of* E^1 *on which* ζ *acts invertibly from the left and the right. Furthermore,* $\mathrm{im}(f^{\pm})$ *is a free* $k[\zeta^{\pm 1}]$*-module of rank* $4(p-1)$ *on the left and on the right.*

Proof. From Lemma 3.24 we know that

$$H_\zeta/H_\zeta\tau_{s_0} \oplus H_\zeta/H_\zeta\tau_{s_1} \xrightarrow{f^{\pm} = f_{\mathbf{x}^+} + f_{\mathbf{x}^-}} E^1$$

is an injective homomorphism of left H-modules the image of which is contained in the kernel of the endomorphism $\zeta \cdot \mathrm{id}_{E^1} \cdot \zeta - \mathrm{id}_{E^1}$. The right H-equivariance of

$$(H_\zeta/H_\zeta\tau_{s_0} \oplus H_\zeta/H_\zeta\tau_{s_1})^{\pm} \xrightarrow{f^{\pm}} E^1$$

is immediately seen by comparing the definition (80) with Lemma 3.23. The last statement follows directly from Lemma 3.27 □

In Prop. 6.8 we will see that the image of $f^{\pm}$ coincides in fact with the kernel of $\zeta \cdot \mathrm{id}_{E^1} \cdot \zeta - \mathrm{id}_{E^1}$.

Remark 3.29. 1. It follows from Remark 3.22-1 that the diagram

$$\begin{array}{ccc} (H_\zeta/H_\zeta\tau_{s_0} \oplus H_\zeta/H_\zeta\tau_{s_1})^\pm & \xrightarrow{f^\pm} & E^1 \\ {\scriptstyle (\sigma^+,\sigma^-)\mapsto(\Gamma_\varpi(\sigma^-),\Gamma_\varpi(\sigma^+))}\downarrow & & \downarrow{\scriptstyle \Gamma_\varpi} \\ (H_\zeta/H_\zeta\tau_{s_0} \oplus H_\zeta/H_\zeta\tau_{s_1})^\pm & \xrightarrow{f^\pm} & E^1 \end{array}$$

is commutative.

2. The maps

$$\begin{aligned} \delta_0 &: H_\zeta/H_\zeta\tau_{s_0} \longrightarrow H_\zeta/H_\zeta\tau_{s_0}, h \longmapsto h(1-e_{\mathrm{id}}-e_{\mathrm{id}}\zeta^{-1}) \\ \delta_1 &: H_\zeta/H_\zeta\tau_{s_1} \longrightarrow H_\zeta/H_\zeta\tau_{s_1}, h \longmapsto h(1-e_{\mathrm{id}^{-1}}-e_{\mathrm{id}^{-1}}\zeta^{-1}) \end{aligned}$$

are well defined isomorphisms of left H_ζ-modules.

Note that on the component $H_\zeta(1-e_{\mathrm{id}})/H_\zeta\tau_{s_0}(1-e_{\mathrm{id}})$ (resp. $H_\zeta(1-e_{\mathrm{id}^{-1}})/H_\zeta\tau_{s_1}(1-e_{\mathrm{id}^{-1}})$), the map δ_0 (resp. δ_1) is the identity map. On $H_\zeta e_{\mathrm{id}}/H_\zeta\tau_{s_0}e_{\mathrm{id}}$ (resp. $H_\zeta e_{\mathrm{id}^{-1}}/H_\zeta\tau_{s_1}e_{\mathrm{id}^{-1}}$), the map δ_0 (resp. δ_1) is the multiplication by ζ^{-1}.

Consider

(82) $$(H_\zeta/H_\zeta\tau_{s_0} \oplus H_\zeta/H_\zeta\tau_{s_1})^\pm \xrightarrow{\mathcal{J}\oplus\mathcal{J}} H_\zeta/\tau_{s_0}H_\zeta \oplus H_\zeta/\tau_{s_1}H_\zeta \xrightarrow{\beta} (H_\zeta/H_\zeta\tau_{s_0} \oplus H_\zeta/H_\zeta\tau_{s_1})^\pm$$

We have

$$\begin{aligned} f^\pm \circ \beta \circ (\mathcal{J}\oplus\mathcal{J})(\sigma^+,\sigma^-) &= f^\pm \circ \beta(\mathcal{J}(\sigma^+),\mathcal{J}(\sigma^-)) \\ &= \mathbf{x}^+ \cdot \mathcal{J}(\sigma^+) + \mathbf{x}^- \cdot \mathcal{J}(\sigma^-) \text{ since } f^\pm \circ \beta \text{ is right } H\text{-equivariant }. \end{aligned}$$

Let

(83) $$\mathcal{J}^\pm := \beta \circ (\mathcal{J}\oplus\mathcal{J}) \circ (\delta_0 \oplus \delta_1).$$

$$\begin{aligned} f^\pm \circ \mathcal{J}^\pm(\sigma^+,\sigma^-) &= \mathbf{x}^+ \cdot \mathcal{J}(\sigma^+(1-e_{\mathrm{id}}-e_{\mathrm{id}}\zeta^{-1})) + \mathbf{x}^- \cdot \mathcal{J}(\sigma^-(1-e_{\mathrm{id}^{-1}}-e_{\mathrm{id}^{-1}}\zeta^{-1})) \\ &= \mathbf{x}^+ \cdot (1-e_{\mathrm{id}^{-1}}-e_{\mathrm{id}^{-1}}\zeta^{-1})\mathcal{J}(\sigma^+) + \mathbf{x}^- \cdot (1-e_{\mathrm{id}}-e_{\mathrm{id}}\zeta^{-1})\mathcal{J}(\sigma^-) \\ &= (1-e_{\mathrm{id}}-e_{\mathrm{id}}\zeta)\cdot\mathbf{x}^+\cdot\mathcal{J}(\sigma^+) + (1-e_{\mathrm{id}^{-1}}-e_{\mathrm{id}^{-1}}\zeta)\cdot\mathbf{x}^-\cdot\mathcal{J}(\sigma^-) \text{by Lemma 3.23-1} \\ &= \mathcal{J}(\mathbf{x}^+)\cdot\mathcal{J}(\sigma^+) + \mathcal{J}(\mathbf{x}^-)\cdot\mathcal{J}(\sigma^-) \text{ by Remark 3.22-2} \\ &= \mathcal{J}(\sigma^+\cdot\mathbf{x}^+ + \sigma^-\cdot\mathbf{x}^-) = \mathcal{J}(f^\pm(\sigma^+,\sigma^-)) \ . \end{aligned}$$

It follows that the diagram

$$\begin{array}{ccc} (H_\zeta/H_\zeta\tau_{s_0} \oplus H_\zeta/H_\zeta\tau_{s_1})^\pm & \xrightarrow{f^\pm} & E^1 \\ {\scriptstyle \mathcal{J}^\pm}\downarrow & & \downarrow{\scriptstyle \mathcal{J}} \\ (H_\zeta/H_\zeta\tau_{s_0} \oplus H_\zeta/H_\zeta\tau_{s_1})^\pm & \xrightarrow{f^\pm} & E^1 \end{array}$$

is commutative.

4 Formulas for the left action of H on E^{d-1} when $G = \mathrm{SL}_2(\mathbb{Q}_p)$, $p \neq 2, 3$

For the moment, $G = \mathrm{SL}_2(\mathfrak{F})$ and I is a Poincaré group of dimension d (hence $p \geq 5$).

4.1 Elements of E^{d-1} as triples

Recall (see (14)) the isomorphism of H-bimodules $\Delta^{d-1} : E^{d-1} \to {}^{\mathcal{J}}((E^1)^{\vee,f})^{\mathcal{J}}$. The left action of $h \in H$ on $\alpha \in {}^{\mathcal{J}}((E^1)^{\vee,f})^{\mathcal{J}} \cong E^{d-1}$ is given by

$$(h, \alpha) \mapsto \alpha(\mathcal{J}(h)\_) \ . \tag{84}$$

The anti-involution $\mathcal{J}$ on E^{d-1} corresponds to the transformation

$$\begin{aligned} {}^{\mathcal{J}}((E^1)^{\vee,f})^{\mathcal{J}} &\longrightarrow {}^{\mathcal{J}}((E^1)^{\vee,f})^{\mathcal{J}} \\ \alpha &\longmapsto \alpha \circ \mathcal{J}. \end{aligned} \tag{85}$$

Proof. We prove that for $\alpha_0 \in E^{d-1}$ we have $\Delta^{d-1}(\mathcal{J}(\alpha_0)) = \Delta^{d-1}(\alpha_0) \circ \mathcal{J}$ in $(E^1)^{\vee,f}$. Let $\beta \in (E^1)^{\vee,f}$. By definition of Δ^{d-1} we have

$$\begin{aligned} [\Delta^{d-1}(\alpha_0) \circ \mathcal{J}](\beta) &= \eta \circ \mathcal{S}^d(\alpha_0 \cup \mathcal{J}(\beta)) = \eta \circ \mathcal{S}^d(\mathcal{J}(\mathcal{J}(\alpha_0) \cup \beta)) \text{ by [OS3] Rmk. 6.2} \\ &= \eta \circ \mathcal{S}^d(\mathcal{J}(\alpha_0) \cup \beta) \text{ by [OS3] Cor. 7.17} \\ &= [\Delta^{d-1}(\mathcal{J}(\alpha_0))](\beta) \ . \end{aligned}$$

□

We will abbreviate $h^{d-1}(w) := H^{d-1}(I, \mathbf{X}(w))$ for $w \in \widetilde{W}$ and will identify it with $h^1(w)^\vee \subseteq {}^{\mathcal{J}}((E^1)^{\vee,f})^{\mathcal{J}}$. Recall from (55) that an element c in $h^1(w) \subset E^1$ may be seen as a triple $(c^-, c^0, c^+)_w$ with

$$c^\pm \in \mathrm{Hom}(\mathfrak{O}/\mathfrak{M}, k) \text{ and } c^0 \in \mathrm{Hom}((1+\mathfrak{M})/(1+\mathfrak{M}^{\ell(w)+1})(1+\mathfrak{M})^p, k) \ .$$

For a given finite dimensional $\mathbb{F}_p$-vector space V, the k-dual of $\mathrm{Hom}_{\mathbb{F}_p}(V, k)$ identifies canonically with $V \otimes_{\mathbb{F}_p} k$ so we will see an element α of $(h^1(w))^\vee$ as a triple

$$(\alpha^-, \alpha^0, \alpha^+)_w \in \mathfrak{O}/\mathfrak{M} \otimes_{\mathbb{F}_p} k \times ((1+\mathfrak{M})/(1+\mathfrak{M}^{\ell(w)+1})(1+\mathfrak{M})^p) \otimes_{\mathbb{F}_p} k \times \mathfrak{O}/\mathfrak{M} \otimes_{\mathbb{F}_p} k \tag{86}$$

such that $\alpha(c) = c^-(\alpha^-) + c^0(\alpha^0) + c^+(\alpha^+)$. We still denote by $(\alpha^-, \alpha^0, \alpha^+)_w$ the image of this element in $h^{d-1}(w)$ via the inverse of Δ^{d-1} and then we have

$$(\alpha^-, \alpha^0, \alpha^+)_w \cup (c^-, c^0, c^+)_w = (c^-(\alpha^-) + c^0(\alpha^0) + c^+(\alpha^+))\, \phi_w \tag{87}$$

where $\phi_w \in h^d(w)$ was defined in §2.2.5. Since $\mathcal{J}$ respects the cup product and since $\mathcal{J}(\phi_w) = \phi_{w^{-1}}$ ([OS3] Rmk. 6.2 and (8.2)), we obtain from Lemma 3.7 the following result:

Lemma 4.1. *Let $w \in \widetilde{W}$ and $\alpha = (\alpha^-, \alpha^0, \alpha^+)_w \in h^{d-1}(w)$.*
If $\ell(w)$ is even then

$$\mathcal{J}(\alpha) = (u^{-2}\alpha^-, \alpha^0, u^2\alpha^+)_{w^{-1}}. \tag{88}$$

If $\ell(w)$ is odd then

$$\mathcal{J}(\alpha) = (-u^2\alpha^+, -\alpha^0, -u^{-2}\alpha^-)_{w^{-1}}. \tag{89}$$

where $u \in (\mathfrak{O}/\mathfrak{M})^\times$ is such that $\omega_u^{-1} w$ lies in the subgroup of $\widetilde{W}$ generated by s_0 and s_1.

From (20), (50) and Lemma 3.4 we obtain:

Lemma 4.2. *Let $w \in \widetilde{W}$ and $(\alpha^-, \alpha^0, \alpha^+)_w \in h^{d-1}(w)$. Its image by conjugation by ϖ defined in (49) is*

$$\Gamma_\varpi((\alpha^-, \alpha^0, \alpha^+)_w) = (\alpha^+, -\alpha^0, \alpha^-)_{\varpi w \varpi^{-1}} \in h^2(\varpi w \varpi^{-1}) \ .$$

In the next lemma we refer to the notation in §3.2.3.

Lemma 4.3. *Assume $G = \mathrm{SL}_2(\mathbb{Q}_p)$, $p \neq 2, 3$. For $w \in \widetilde{W}$, $\ell(w) \geq 1$ we have*

$$\begin{aligned} (0, \boldsymbol{\alpha}^0, 0)_w &= -(\mathbf{c}, 0, 0)_w \cup (0, 0, \mathbf{c})_w, \\ (\boldsymbol{\alpha}, 0, 0)_w &= (0, \mathbf{c}^0, 0)_w \cup (0, 0, \mathbf{c})_w, \\ (0, 0, \boldsymbol{\alpha})_w &= (\mathbf{c}, 0, 0)_w \cup (0, \mathbf{c}^0, 0)_w \ . \end{aligned} \tag{90}$$

Proof. By definition, $(0, \boldsymbol{\alpha}^0, 0)_w$ is the unique element in $h^2(w)$ such that

$$\begin{aligned} &\eta \circ \mathcal{S}^d\big((0, \boldsymbol{\alpha}^0, 0)_w \cup (0, \mathbf{c}^0, 0)_w\big) = \mathbf{c}^0(\boldsymbol{\alpha}^0) = 1, \\ &\eta \circ \mathcal{S}^d\big((0, \boldsymbol{\alpha}^0, 0)_w \cup (\mathbf{c}, 0, 0)_w\big) = 0, \\ &\eta \circ \mathcal{S}^d\big((0, \boldsymbol{\alpha}^0, 0)_w \cup (0, 0, \mathbf{c})_w\big) = 0, \end{aligned}$$

namely $(0, \boldsymbol{\alpha}^0, 0)_w \cup (0, \mathbf{c}^0, 0)_w = \phi_w$ while $(0, \boldsymbol{\alpha}^0, 0)_w \cup (\mathbf{c}, 0, 0)_w = (0, \boldsymbol{\alpha}^0, 0)_w \cup (0, 0, \mathbf{c})_w = 0$. By (61), we obtain the first formula of the lemma. The other formulas are obtained similarly.

□

For any subset $U \subseteq \widetilde{W}$ we define as in §3.2 the k-subspaces

$$h_-^{d-1}(U) := \oplus_{w \in U} h_-^{d-1}(w), \quad h_0^{d-1}(U) := \oplus_{w \in U} h_0^{d-1}(w), \quad \text{and } h_+^{d-1}(U) := \oplus_{w \in U} h_+^{d-1}(w)$$

of h^{d-1}. We also let $h_\pm^{d-1}(U) := h_-^{d-1}(U) \oplus h_+^{d-1}(U)$.

4.2 Left action of τ_ω on E^{d-1} for $\omega \in \Omega$

Let $w \in \widetilde{W}$. The action of τ_ω on the left on an element $\alpha \in h^{d-1}(w) \subseteq E^{d-1}$ was given at the beginning of §3.4. Here we make this action explicit when α is given by a triple

$$\alpha = (\alpha^-, \alpha^0, \alpha^+)_w \in (h^1(w))^\vee \subset {}^{\partial}((E^1)^{\vee, f})^{\partial} \cong E^{d-1}$$

as in (86). For $u \in (\mathfrak{O}/\mathfrak{M})^\times$, we compute $\tau_{\omega_u} \cdot \alpha \in h^1(\omega_u w)$. For $c = (c^-, c^0, c^+)_{\omega_u w} \in h^1(\omega_u w)$ we have $(\tau_{\omega_u} \cdot \alpha)(c) = c^-(u^2 \alpha^-) + c^0(\alpha^0) + c^+(u^{-2} \alpha^+)$ (see (67)) therefore

$$\tau_{\omega_u} \cdot (\alpha^-, \alpha^0, \alpha^+)_w = (u^2 \alpha^-, \alpha^0, u^{-2} \alpha^+)_{\omega_u w} \ . \tag{91}$$

In particular, for $s \in \{s_0, s_1\}$ we have (compare with (68))

$$\tau_{s^2} \cdot (\alpha^-, \alpha^0, \alpha^+)_w = (\alpha^-, \alpha^0, \alpha^+)_{s^2 w} \ . \tag{92}$$

Remark 4.4. Using (84) and the formulas in §3.5, we have for $w \in \widetilde{W}$ and $\alpha = (\alpha^-, \alpha^0, \alpha^+)_w \in h^{d-1}(w)$:

$$(93)\quad (\ell(w) \text{ even})): \quad e_\lambda \cdot \alpha = \alpha \cdot e_\mu \text{ if and only if } \begin{cases} \alpha^- = \mu^{-1}\lambda(\omega_u)\alpha^-(u^{-2}\,\_) \\ \alpha^0 = \mu^{-1}\lambda(\omega_u)\alpha^0 \\ \alpha^+ = \mu^{-1}\lambda(\omega_u)\alpha^+(u^2\,\_). \end{cases} \quad \text{for any } u \in (\mathfrak{O}/\mathfrak{M})^\times$$

and

$$(94)\quad (\ell(w) \text{ odd})): \quad e_\lambda \cdot \alpha = \alpha \cdot e_\mu \text{ if and only if } \begin{cases} \alpha^- = \mu\lambda(\omega_u)\alpha^-(u^{-2}\,\_) \\ \alpha^0 = \mu\lambda(\omega_u)\alpha^0 \\ \alpha^+ = \mu\lambda(\omega_u)\alpha^+(u^2\,\_). \end{cases} \quad \text{for any } u \in (\mathfrak{O}/\mathfrak{M})^\times.$$

4.3 Left action of H on E^2 when $G = \mathrm{SL}_2(\mathbb{Q}_p)$, $p \neq 2, 3$

Suppose that $G = \mathrm{SL}_2(\mathbb{Q}_p)$, $p \neq 2, 3$ and $\pi = p$. Then $d = 3$. The isomorphism ι was defined in (59). The following proposition is proved in §9.4. Together with (91), these formulas give the left action of H on E^2.

Proposition 4.5. *Let $w \in \widetilde{W}$, $\omega \in \Omega$ and $\alpha = (\alpha^-, \alpha^0, \alpha^+)_w \in (h^1(w))^\vee$ seen as an element of E^2. We have:*

$$\tau_{s_0} \cdot (\alpha^-, \alpha^0, \alpha^+)_w = \begin{cases} (-\alpha^+, 0, 0)_{s_0 w} & \text{if } w \in \widetilde{W}^0 \text{ with } \ell(w) \geq 1, \\ e_1 \cdot (-\alpha^-, -\alpha^0, -\alpha^+)_w + e_{\mathrm{id}} \cdot (2\iota(\alpha^0), 0, 0)_w + (-\alpha^+, -\alpha^0, 0)_{s_0 w} & \text{if } w \in \widetilde{W}^1 \text{ with } \ell(w) \geq 2, \\ e_1 \cdot (-\alpha^-, -\alpha^0, -\alpha^+)_w + e_{\mathrm{id}} \cdot (2\iota(\alpha^0), -\iota^{-1}(\alpha^+), 0)_w \\ \qquad + e_{\mathrm{id}^2} \cdot (\alpha^+, 0, 0)_w + (-\alpha^+, 0, 0)_{s_0 w} & \text{if } w \in \widetilde{W}^1 \text{ with } \ell(w) = 1. \end{cases}$$

$$\tau_{s_1} \cdot (\alpha^-, \alpha^0, \alpha^+)_w = \begin{cases} (0, 0, -\alpha^-)_{s_1 w} & \text{if } w \in \widetilde{W}^1 \text{ with } \ell(w) \geq 1, \\ -e_1 \cdot (\alpha^-, \alpha^0, \alpha^+)_w + e_{\mathrm{id}^{-1}} \cdot (0, 0, -2\iota(\alpha^0))_w + (0, -\alpha^0, -\alpha^-)_{s_1 w} & \text{if } w \in \widetilde{W}^0 \text{ with } \ell(w) \geq 2, \\ -e_1 \cdot (\alpha^-, \alpha^0, \alpha^+)_w + e_{\mathrm{id}^{-1}} \cdot (0, \iota^{-1}(\alpha^-), -2\iota(\alpha^0))_w \\ \qquad + e_{\mathrm{id}^{-2}} \cdot (0, 0, \alpha^-)_w + (0, 0, -\alpha^-)_{s_1 w} & \text{if } w \in \widetilde{W}^0 \text{ with } \ell(w) = 1. \end{cases}$$

$$\tau_{s_0} \cdot (\alpha^-, 0, \alpha^+)_\omega = (-\alpha^+, 0, 0)_{s_0 \omega}$$
$$\tau_{s_1} \cdot (\alpha^-, 0, \alpha^+)_\omega = (0, 0, -\alpha^-)_{s_1 \omega}.$$

Corollary 4.6. *Let $w \in \widetilde{W}$, $\omega \in \Omega$ and $\alpha = (\alpha^-, \alpha^0, \alpha^+)_w \in (h^1(w))^\vee$ seen as an element of E^2.*

$$\zeta \cdot (\alpha^-, 0, \alpha^+)_\omega = (\alpha^-, 0, 0)_{s_0 s_1 \omega} + (0, 0, \alpha^+)_{s_1 s_0 \omega} + e_1 \cdot (-\alpha^+, 0, 0)_{s_0 \omega} + e_1 \cdot (0, 0, -\alpha^-)_{s_1 \omega} + e_1 \cdot (\alpha^-, 0, \alpha^+)_\omega.$$

$$\zeta \cdot (\alpha^-, \alpha^0, \alpha^+)_w = \begin{cases} (\alpha^-, 0, 0)_{s_0 s_1 w} + (0, 0, \alpha^+)_{s_1 s_0 w} + e_1 \cdot (-\alpha^+, 0, 0)_{s_0 w} \\ \qquad + e_{\mathrm{id}^{-1}} \cdot (0, 0, -2\iota(\alpha^0))_{s_1 w} + e_{\mathrm{id}^{-2}} \cdot (0, 0, -\alpha^+)_{s_1 w} & \text{if } w \in s_0 \Omega, \\ (\alpha^-, 0, 0)_{s_0 s_1 w} + (0, 0, \alpha^+)_{s_1 s_0 w} + e_1 \cdot (0, 0, -\alpha^-)_{s_1 w} \\ \qquad + e_{\mathrm{id}} \cdot (2\iota(\alpha^0), 0, 0)_{s_0 w} + e_{\mathrm{id}^2} \cdot (-\alpha^-, 0, 0,)_{s_0 w} & \text{if } w \in s_1 \Omega \end{cases}$$

$$\zeta\cdot(\alpha^-,\alpha^0,\alpha^+)_w = \begin{cases} (\alpha^-,0,0)_{s_0s_1w} + (0,0,\alpha^+)_{s_1s_0w} + e_{\mathrm{id}^{-1}}\cdot(0,0,-2\iota(\alpha^0))_{s_1w} \\ \quad +e_{\mathrm{id}^{-1}}\cdot(0,-\iota^{-1}(\alpha^+),2\iota(\alpha^0))_{s_0w} + e_{\mathrm{id}^{-2}}\cdot(0,0,-\alpha^+)_{s_0w} & \text{if } w\in\widetilde{W}^1,\ \ell(w)=2, \\ (0,0,\alpha^+)_{s_1s_0w} + (\alpha^-,0,0)_{s_0s_1w} + e_{\mathrm{id}}\cdot(2\iota(\alpha^0),0,0)_{s_0w} \\ \quad +e_{\mathrm{id}}\cdot(-2\iota(\alpha^0),\iota^{-1}(\alpha^-),0)_{s_1w} + e_{\mathrm{id}^2}\cdot(-\alpha^-,0,0)_{s_1w} & \text{if } w\in\widetilde{W}^0,\ \ell(w)=2, \end{cases}$$

$$\zeta\cdot(\alpha^-,\alpha^0,\alpha^+)_w = \begin{cases} (\alpha^-,0,0)_{s_0s_1w} + (0,\alpha^0,\alpha^+)_{s_1s_0w} \\ \quad +e_{\mathrm{id}^{-1}}\cdot(0,0,-2\iota(\alpha^0))_{s_1w} + e_{\mathrm{id}^{-1}}\cdot(0,0,2\iota(\alpha^0))_{s_0w} & \text{if } w\in\widetilde{W}^1,\ \ell(w)\geq 3, \\ (0,0,\alpha^+)_{s_1s_0w} + (\alpha^-,\alpha_0,0)_{s_0s_1w} \\ \quad +e_{\mathrm{id}}\cdot(2\iota(\alpha^0),0,0)_{s_0w} + e_{\mathrm{id}}\cdot(-2\iota(\alpha^0),0,0)_{s_1w} & \text{if } w\in\widetilde{W}^0,\ \ell(w)\geq 3. \end{cases}$$

5 $k[\zeta]$-torsion in E^* when $G=\mathrm{SL}_2(\mathbb{Q}_p)$, $p\neq 2,3$

In this whole section $G=\mathrm{SL}_2(\mathfrak{F})$.

A) Without any assumption on $\mathfrak{F}$, we know that E^0 is a free left (resp. right) $k[\zeta]$-module (Lemma 2.7). Therefore it is $k[\zeta]$-torsion free on the left (resp. right).

B) Here we suppose that the group I is torsion free and its dimension as a Poincaré group is d. We study the $k[\zeta]$-torsion in E^d. We know by Remark 2.21 that the left and right actions of ζ on E^d coincide. Recall that we have the following isomorphism of H-bimodules

$$E^d \cong \ker(\mathcal{S}^d)\oplus\chi_{triv} \tag{95}$$

and by Proposition 2.4, we have $\ker(\mathcal{S}^d)\cong\bigcup_m(H/\zeta^mH)^\vee$ as H-bimodules. Therefore E^d is the direct sum of its one-dimensional subspace of $(\zeta-1)$-torsion and of its subspace $\ker(\mathcal{S}^d)$ of ζ-torsion. This applies in particular when $G=\mathrm{SL}_2(\mathbb{Q}_p)$, $p\neq 2,3$ and $d=3$.

C) We study the $k[\zeta]$-torsion in E^1.

Lemma 5.1. *Suppose that $G=\mathrm{SL}_2(\mathfrak{F})$.*

i Suppose that $p\neq 2$. For any $P\in k[X]$ such that $P(0)\neq 0$ there is no left (resp. right) $P(\zeta)$-torsion in E^1.

ii. If $\mathfrak{F}=\mathbb{Q}_p$, given any $0\neq P\in k[X]$, there is no left (resp. right) $P(\zeta)$-torsion in E^1.

Proof. Let $0\neq P\in k[X]$. Suppose that we know that $(\mathbf{X}/\mathbf{X}P(\zeta))^I\cong H/HP(\zeta)$. Then the exact sequence of (G,H)-bimodules $0\to\mathbf{X}\xrightarrow{\cdot P(\zeta)}\mathbf{X}\to\mathbf{X}/\mathbf{X}P(\zeta)\to 0$ induces the long exact sequence of H-bimodules

$$0\to E^1\xrightarrow{\cdot P(\zeta)}E^1\to H^1(I,\mathbf{X}/\mathbf{X}P(\zeta))\to E^2\to\ldots$$

In particular, there is no right $P(\zeta)$-torsion in E^1. Since $P(\zeta)\cdot c=\mathcal{J}(\mathcal{J}(c)\cdot P(\zeta))$ for any $c\in E^*$, there is no left $P(\zeta)$-torsion in E^1 either.

i. For any field extension k'/k and any $V\in\mathrm{Mod}(G)$ we have $(V\otimes_k k')^I=V^I\otimes_k k'$. Therefore we may assume that $\mathbb{F}_q\subseteq k$ (and that $p\neq 2$). Suppose that $P(0)\neq 0$. Then $H/HP(\zeta)$ is an H_ζ-module. Hence by [OS2] Thm. 3.33 we know that $(\mathbf{X}/\mathbf{X}P(\zeta))^I\cong H/HP(\zeta)$.

ii. Suppose $\mathfrak{F}=\mathbb{Q}_p$. Then $(\mathbf{X}/\mathbf{X}P(\zeta))^I\cong H/HP(\zeta)$ (see §2.4.10). □

D) Here we suppose that the group I is torsion free and its dimension as a Poincaré group is d. We study the $k[\zeta]$-torsion subspace in E^{d-1}.
Let $i \in \{0, \ldots, d\}$ and $\ell \geq 1$. Recall that the left action of ζ on ${}^{\jmath}((E^i)^{\vee,f})^{\jmath} \cong E^{d-i}$ is given by $(\zeta, \varphi) \mapsto \varphi(\zeta \cdot \_) : E^i \to k$. In particular, $\mathrm{coker}(\zeta^\ell \cdot : E^i \to E^i) = \{0\}$ implies $\ker(\zeta^\ell \cdot : E^{d-i} \to E^{d-i}) = \{0\}$. We explore the converse implication in the lemma below where we refer to the decreasing filtration $(F^m E^i)_{m\geq 0}$ introduced in §2.2.4.

Lemma 5.2. *Suppose that $G = \mathrm{SL}_2(\mathfrak{F})$ and I is a Poincaré group of dimension d. Let $i \in \{0, \ldots, d\}$. Suppose that there is $m \geq 0$ such that $\zeta^\ell \cdot E^i \supseteq F^m E^i$, then we have an isomorphism of H-bimodules:*

$$\ker(\zeta^\ell \cdot : E^{d-i} \to E^{d-i}) \cong {}^{\jmath}((E^i/\zeta^\ell \cdot E^i)^\vee)^{\jmath} \ .$$

In particular, $\ker(\zeta^\ell \cdot : E^{d-i} \to E^{d-i}) = 0$ if and only if $\mathrm{coker}(\zeta^\ell \cdot : E^i \to E^i) = 0$. The same statements are valid for the right action of ζ^ℓ.

Proof. The kernel of the left action of ζ^ℓ on ${}^{\jmath}((E^i)^{\vee,f})^{\jmath}$ is the space of all $\varphi \in (E^i)^{\vee,f}$ which are trivial on $\zeta^\ell \cdot E^i$. Suppose that there is $m \geq 0$ such that $\zeta^\ell \cdot E^i \supseteq F^m E^i$. Then any $\varphi \in (E^i)^\vee$ which is trivial on $\zeta^\ell \cdot E^i$ lies in $(E^i)^{\vee,f}$. Therefore, the kernel of the right action of ζ^ℓ on ${}^{\jmath}((E^i)^{\vee,f})^{\jmath}$ is the space of all $\varphi \in (E^i)^\vee$ which are trivial on $\zeta^\ell \cdot E^i$, namely $\ker(\cdot\zeta^\ell : {}^{\jmath}((E^i)^{\vee,f})^{\jmath} \to {}^{\jmath}((E^i)^{\vee,f})^{\jmath}) = {}^{\jmath}((E^i/\zeta^\ell \cdot E^i)^\vee)^{\jmath}$. ☐

Remark 5.3. It is easy to check that $\zeta^\ell \cdot E^0 \supset \zeta^\ell \cdot F^1 E^0 = F^{2\ell+1} E^0$. So we recover $\ker(\zeta^\ell \cdot : E^d \to E^d) \cong {}^{\jmath}((H/\zeta^\ell H)^\vee)^{\jmath}$ which is isomorphic to $(H/\zeta^\ell H)^\vee$. (compare with **B)** above).

Using Corollary 3.11 we obtain immediately:

Corollary 5.4. *Suppose that $G = \mathrm{SL}_2(\mathbb{Q}_p)$, $p \neq 2, 3$. We have an isomorphism of H-bimodules:*

$$\ker(\zeta \cdot : E^2 \to E^2) \cong {}^{\jmath}((E^1/\zeta \cdot E^1)^\vee)^{\jmath} \ .$$

Remark 5.5. We will see in Proposition 6.15 that this space is nontrivial.

Lemma 5.6. *Suppose that $G = \mathrm{SL}_2(\mathbb{Q}_p)$, $p \neq 2, 3$ and $\pi = p$. There is no left (resp. right) $P(\zeta)$-torsion in E^2 for any $P \in k[X]$ with $P(0) \neq 0$.*

Proof. We may prove the assertion after a base extension of k. Hence it suffices to consider the case $P(X) = X - a$ for some $a \in k^\times$. As in the proof of Lemma 5.1, it is enough to prove that there is no left $(\zeta - a)$-torsion in E^2 or equivalently that there is no right $(\zeta - a)$-torsion $(E^1)^{\vee,f}$ (see (14)). We prove that for a given $m \geq 1$, we have

$$(\zeta - a) \cdot E^1 + F^m E^1 = E^1.$$

By our assumption that $\pi = p$, we may use the formulas of Cor. 3.10.

- If $w \in \widetilde{W}^0$, $\ell(w) \geq 2$, we have $(\zeta - a) \cdot (c^-, c^0, 0)_w = (c^-, c^0, 0)_{s_1 s_0 w} - a \cdot (c^-, c^0, 0)_w$ and if $\ell(w) \geq 1$, we have $(\zeta - a) \cdot (c^-, 0, 0)_w = (c^-, 0, 0)_{s_1 s_0 w} - a \cdot (c^-, 0, 0)_w$.
  So by induction $h^1_-(\widetilde{W}^{0,\ell\geq 1}) + h^1_0(\widetilde{W}^{0,\ell\geq 2}) \subseteq (\zeta - a) \cdot E^1 + F^m E^1$. Using conjugation by ϖ, we have proved $h^1_-(\widetilde{W}^{0,\ell\geq 1}) + h^1_0(\widetilde{W}^{\ell\geq 2}) + h^1_+(\widetilde{W}^{1,\ell\geq 1}) \subseteq (\zeta - a) \cdot E^1 + F^m E^1$.
- If $\ell(w) \geq 3$ we have, $(\zeta - a) \cdot (0, 0, c^+)_w \in (0, 0, c^+)_{s_0 s_1 w} - a(0, 0, c^+)_w + h^1_0(\widetilde{W}^{\ell\geq 2})$ if $w \in \widetilde{W}^0$ therefore $h^1_-(\widetilde{W}^{1,\ell\geq 1}) + h^1_+(\widetilde{W}^{0,\ell\geq 1}) \subseteq (\zeta - a) \cdot E^1 + F^m E^1$ by induction and conjugation by ϖ.

So at this point we have $h^1_0(\widetilde{W}^{\ell\geq 2}) + h^1_\pm(\widetilde{W}^{\ell\geq 1}) \subseteq (\zeta - a) \cdot E^1 + F^m E^1$.

- But if $\ell(w) = 1$ we have $(\zeta - a)(0, c^0, 0)_w \in -a(0, c^0, 0)_w + h^1_0(\widetilde{W}^{\ell\geq 2}) + h^1_\pm(\widetilde{W}^{\ell\geq 1})$
  so $h^1_0(\widetilde{W}) + h^1_\pm(\widetilde{W}^{\ell\geq 1}) \subseteq (\zeta - a) \cdot E^1 + F^m E^1$.

- Lastly, $(c^-, 0, c^+)_\omega \in (\zeta - a) \cdot (c^-, 0, 0)_{s_0 s_1 \omega} + (\zeta - a) \cdot (0, 0, c^+)_{s_1 s_0 \omega} + h^1_0(\widetilde{W}) + h^1_\pm(\widetilde{W}^{\ell \geq 1})$ for $\omega \in \Omega$. So $h^1_0(\widetilde{W}) + h^1_\pm(\widetilde{W}) \subseteq (\zeta - a) \cdot E^1 + F^m E^1$.

□

6 Structure of E^1 and E^2 when $G = \mathrm{SL}_2(\mathbb{Q}_p)$, $p \neq 2, 3$

6.1 Preliminaries

We define the following endomorphisms of H-bimodules of E^*:

$$f := \zeta \cdot \mathrm{id}_{E^*} \cdot \zeta - \mathrm{id}_{E^*} : c \mapsto \zeta \cdot c \cdot \zeta - c$$

and

$$g := \zeta \cdot \mathrm{id}_{E^*} - \mathrm{id}_{E^*} \cdot \zeta : c \mapsto \zeta \cdot c - c \cdot \zeta.$$

We will restrict them to the graded pieces E^i and will then use the notation f_i and g_i. The following remarks are easy to check. Here $G = \mathrm{SL}_2(\mathfrak{F})$.

Remark 6.1. i. f and g commute. In fact,

$$f \circ g = (\zeta^2 + 1) \cdot \mathrm{id}_{E^*} \cdot \zeta - \zeta \cdot \mathrm{id}_{E^*} \cdot (\zeta^2 + 1) = g \circ f.$$

ii. It is clear that the left (resp. right) action of ζ on $\ker(f)$ induces a bijective map. Hence $\ker(f)$ is naturally a H_ζ-bimodule.

iii. We have the following inclusions of subalgebras of E^*:

$$\ker(g) \subseteq \ker(f) + \ker(g) \subseteq E^* .$$

We have indeed $\ker(f) \cdot \ker(f) \subseteq \ker(g)$ as well as $\ker(f) \cdot \ker(g) \subseteq \ker(f)$ and $\ker(g) \cdot \ker(f) \subseteq \ker(f)$.

iv. The spaces $\ker(f)$ and $\ker(g)$ are stable by conjugation by ϖ (see (49) and use that $\Gamma_\varpi(\zeta) = \zeta$).

v. The spaces $\ker(f)$ and $\ker(g)$ are stable by $\mathcal{J}$ (use that $\mathcal{J}(\zeta) = \zeta$).

Lemma 6.2. *Suppose $G = \mathrm{SL}_2(\mathfrak{F})$. We have*

i. $\ker(f_0) = \{0\}$ *and* $\ker(g_0) = E^0$.

ii. If I is a Poincaré group of dimension d, then $\ker(g_d) = E^d$ and $\ker(f_d) \cong \chi_{triv}$ as a left (resp. right) H-module.

iii. Suppose that $p \neq 2$ or $\mathfrak{F} = \mathbb{Q}_p$. Then $\ker(f_1) \cap \ker(g_1) = \{0\}$.

iv. Suppose that $\mathfrak{F} = \mathbb{Q}_p$ with $p \neq 2, 3$. Assume $\pi = p$. Then $\ker(f_2) \cap \ker(g_2) = \{0\}$.

Proof. The first point is clear, using in particular the freeness of H as a $k[\zeta]$-module. For the second point: we saw in §5**B)** that ζ centralizes the elements in E^d, therefore $\ker(g_d) = E^d$ and the kernel of f_d coincides with the kernel of the action of $\zeta^2 - 1$ on E^d. But E^d is the direct sum of its one-dimensional subspace of $(\zeta - 1)$-torsion and of its subspace of ζ-torsion. So $\ker(f_d)$ coincides with the subspace of $(\zeta - 1)$-torsion and is isomorphic to χ_{triv} as a left (resp. right) H-module.

The last two points come from the fact that for any i the space $\ker(f_i) \cap \ker(g_i)$ is contained in the $\zeta^2 - 1$ torsion space in E^i. But for $i = 1, 2$ and under the respective hypotheses, this torsion space is trivial by Lemmas 5.1 and 5.6. □

6.2 Structure of E^1

We suppose that $G = \mathrm{SL}_2(\mathbb{Q}_p)$ with $p \neq 2, 3$ and we choose $\pi = p$. Here we focus on the graded piece E^1 and work with the endomorphisms of H-bimodules

$$f_1 := \zeta \cdot \mathrm{id}_{E^1} \cdot \zeta - \mathrm{id}_{E^1} : c \mapsto \zeta \cdot c \cdot \zeta - c$$

and

$$g_1 := \zeta \cdot \mathrm{id}_{E^1} - \mathrm{id}_{E^1} \cdot \zeta : c \mapsto \zeta \cdot c - c \cdot \zeta .$$

6.2.1 On $\ker(g_1)$

In Prop. 3.18 we established the injectivity of the H-bimodule homomorphism

$$f_{(\mathbf{x}_0,\mathbf{x}_1)} : F^1 H \longrightarrow \ker(g_1) \ . \tag{96}$$

Proposition 6.3. *Assume $G = \mathrm{SL}_2(\mathbb{Q}_p)$ with $p \neq 2, 3$ and $\pi = p$. The map (96) is bijective, so $\ker(g_1)$ is isomorphic to $F^1 H$ as an H-bimodule. In particular, as a left (resp. right) $k[\zeta]$-module, $\ker(g_1)$ is free of rank $4(p-1)$.*

Proof. It is immediate from Prop. 3.18-i that $E^1 = \mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)}) \oplus h^1_{\pm}(\widetilde{W})$. Therefore we only need to check that g_1 is injective on $h^1_{\pm}(\widetilde{W})$. From §2.2.4 we know that, for $n \geq 0$, we have

$$\zeta \cdot F_n E^1 + F_n E^1 \cdot \zeta \subseteq F_{n+2} E^1 \quad \text{and hence} \quad g_1(F_n E^1) \subseteq F_{n+2} E^1 \ .$$

But Lemma 3.13 tells us that modulo $F_{\ell(w)+1} E^1$ we have

$$g_1((c^-, 0, c^+)_w) \equiv \begin{cases} (0,0,c^+)_{s_0 s_1 w} - (c^-,0,0)_{s_0 s_1 w} & \text{if } w \in W^{1,\ell \geq 1}, \\ (c^-,0,0)_{s_1 s_0 w} - (0,0,c^+)_{s_1 s_0 w} & \text{if } w \in W^{0,\ell \geq 1}, \\ (0,0,c^+)_{s_0 s_1 \omega} + (c^-,0,0)_{s_1 s_0 \omega} - (c^-,0,0)_{s_0 s_1 \omega} - (0,0,c^+)_{s_1 s_0 \omega} & \text{if } w = \omega \in \Omega. \end{cases}$$

This shows that g_1 is injective on $h^1_{\pm}(\widetilde{W})$. ☐

Remark 6.4. The above proposition implies in particular that $\ker(g_1)$ is the centralizer in E^1 of the full center Z of H.

6.2.2 On $\ker(f_1)$

In Prop. 3.28 we introduced and established the injectivity of the H_ζ-bimodule homomorphism

$$f^{\pm} : (H_\zeta / H_\zeta \tau_{s_0} \oplus H_\zeta / H_\zeta \tau_{s_1})^{\pm} \longrightarrow \ker(f_1) \ . \tag{97}$$

To show that this map is actually also surjective we need to introduce the vector subspace $\mathfrak{V} \subseteq E^1$ with basis

$$\begin{aligned} x &:= e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_1 \cdot e_{\mathrm{id}^{-1}}, \quad e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_{s_1} \cdot e_{\mathrm{id}} = x \cdot \tau_{s_1}, \\ y &:= e_{\mathrm{id}^{-1}} \cdot (\mathbf{c},0,0)_1 \cdot e_{\mathrm{id}}, \quad e_{\mathrm{id}^{-1}} \cdot (\mathbf{c},0,0)_{s_0} \cdot e_{\mathrm{id}^{-1}} = y \cdot \tau_{s_0}. \end{aligned} \tag{98}$$

Temporarily we put

$$\mathfrak{U} := \mathfrak{V} + \mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)}) + \mathrm{im}(f^{\pm}) \ .$$

But note that $\mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)}) + \mathrm{im}(f^{\pm}) = \mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)}) \oplus \mathrm{im}(f^{\pm})$ by Lemma 6.2-iii.

Lemma 6.5. *We have:*

a) $(x \cdot \tau_{s_1}) \cdot \tau_{s_1} = 0$ *and* $(y \cdot \tau_{s_0}) \cdot \tau_{s_0} = 0$;

b) $x \cdot \tau_{s_0} = 0$ *and* $y \cdot \tau_{s_1} = 0$;

c) $\tau_{s_0} \cdot x = 0 = \tau_{s_0} \cdot (x \cdot \tau_{s_1})$ *and* $\tau_{s_1} \cdot y = 0 = \tau_{s_1} \cdot (y \cdot \tau_{s_0})$;

d) $\tau_{s_1} \cdot x = y \cdot \tau_{s_0} + e_{\mathrm{id}^{-1}}\tau_{s_1} f_{\mathbf{x}^+}(1)$, $\tau_{s_0} \cdot y = x \cdot \tau_{s_1} + e_{\mathrm{id}}\tau_{s_0} \cdot f_{\mathbf{x}^-}(1)$;

e) $\zeta \cdot x - x = e_{\mathrm{id}}\tau_{s_0 s_1} \cdot f_{\mathbf{x}^+}(1) + 2e_{\mathrm{id}} \cdot f_{(\mathbf{x}_0,\mathbf{x}_1)}(\tau_{s_0})$;

f) $\zeta \cdot y - y = e_{\mathrm{id}^{-1}}\tau_{s_1 s_0} \cdot f_{\mathbf{x}^-}(1) + 2e_{\mathrm{id}^{-1}} \cdot f_{(\mathbf{x}_0,\mathbf{x}_1)}(\tau_{s_1})$;

g) $x \cdot \zeta - x, y \cdot \zeta - y \in \mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)}) \oplus \mathrm{im}(f^{\pm})$;

h) $(x \cdot \tau_{s_1}) \cdot \tau_{s_0} - x, (y \cdot \tau_{s_0}) \cdot \tau_{s_1} - y \in \mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)}) \oplus \mathrm{im}(f^{\pm})$;

i) $\tau_{s_1} \cdot (x \cdot \tau_{s_1}) - y \in \mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)}) \oplus \mathrm{im}(f^{\pm})$, $\tau_{s_0} \cdot (y \cdot \tau_{s_0}) - x \in \mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)}) \oplus \mathrm{im}(f^{\pm})$;

j) $\mathfrak{U}$ *is a sub-H-bimodule of* E^1.

Proof. a) is obvious. For the subsequent computations it is useful to note that we have

$$x = e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_1,\ x \cdot \tau_{s_1} = e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_{s_1},\ y = e_{\mathrm{id}^{-1}} \cdot (\mathbf{c},0,0)_1,\ y \cdot \tau_{s_0} = e_{\mathrm{id}^{-1}} \cdot (\mathbf{c},0,0)_{s_0}\ . \tag{99}$$

We also recall that $\mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)}) \oplus \mathrm{im}(f^{\pm})$ is a sub-H-bimodule of E^1.

Points b), c), d), e), and f) are a straightforward computation based on the formulas in Prop. 3.9. Point g) follows from e) and f) by applying $\mathcal{J}$. By b) we have $x \cdot \zeta = (x \cdot \tau_{s_1}) \cdot \tau_{s_0}$ and $y \cdot \zeta = (y \cdot \tau_{s_0}) \cdot \tau_{s_1}$; hence h) follows from g). i) follows from d) and h). j) follows from a) - d), h), and i). $\square$

Remark 6.6. By direct calculation, we have

$$\zeta \cdot (x \cdot \tau_{s_1}) \cdot \zeta - (x \cdot \tau_{s_1}) = -e_{\mathrm{id}} \cdot ((0,2\mathbf{c}\iota,0)_{(s_0 s_1)^2} + (0,2\mathbf{c}\iota,0)_{s_0 s_1}) \cdot e_{\mathrm{id}} = -(\zeta+1)e_{\mathrm{id}} \cdot (0,2\mathbf{c}\iota,0)_{s_0 s_1} \cdot e_{\mathrm{id}}.$$

Lemma 6.7. *We have* $E^1 = \mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)}) \oplus \mathrm{im}(f^{\pm}) \oplus \mathfrak{V} = \ker(g_1) \oplus \mathrm{im}(f^{\pm}) \oplus \mathfrak{V}$.

Proof. We remind the reader of the following consequences of (67) which we will silently use in the following:

$$e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_{\omega_u w} = u^{-1} e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_w \quad \text{and} \quad e_{\mathrm{id}^{-1}} \cdot (\mathbf{c},0,0)_{\omega_u w} = u e_{\mathrm{id}^{-1}} \cdot (\mathbf{c},0,0)_w$$

for any $w \in \widetilde{W}$ and $u \in \mathbb{F}_p^\times$. We also recall, using (70) and (71) that

$$x = e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_1,\ y = e_{\mathrm{id}^{-1}} \cdot (\mathbf{c},0,0)_1,\ x \cdot \tau_{s_1} = e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_{s_1},\ y \cdot \tau_{s_0} = e_{\mathrm{id}^{-1}} \cdot (\mathbf{c},0,0)_{s_0}\ .$$

Prop. 3.18-i tells us that

$$\begin{aligned} \mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)}) &= h^1_0(\widetilde{W}^{\ell \geq 2}) \oplus (\oplus_{u \in \mathbb{F}_p^\times} k((0,\mathbf{c}\iota,0)_{s_1\omega_u} - u^{-1} e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_1)) \\ &\qquad \oplus (\oplus_{u \in \mathbb{F}_p^\times} k((0,\mathbf{c}\iota,0)_{s_0\omega_u} + u e_{\mathrm{id}^{-1}} \cdot (\mathbf{c},0,0)_1)) \\ &= h^1_0(\widetilde{W}^{\ell \geq 2}) \oplus (\oplus_{u \in \mathbb{F}_p^\times} k((0,\mathbf{c}\iota,0)_{s_1\omega_u} - u^{-1} x)) \oplus (\oplus_{u \in \mathbb{F}_p^\times} k((0,\mathbf{c}\iota,0)_{s_0\omega_u} + u y)). \end{aligned} \tag{100}$$

This implies

$$\mathfrak{U} \supseteq \mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)}) \oplus kx \oplus ky = h^1_0(\widetilde{W}^{\ell \geq 1}) \oplus kx \oplus ky\ . \tag{101}$$

Next we observe that, by Lemma 3.12, we have

$$(0,0,\mathbf{c})_w = (0,0,\mathbf{c})_1 \cdot \tau_w \quad \text{and} \quad e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_w = x \cdot \tau_w \quad \text{for any } w \in \widetilde{W}^0, \text{ and}$$
$$(\mathbf{c},0,0)_w = (\mathbf{c},0,0)_1 \cdot \tau_w \quad \text{and} \quad e_{\mathrm{id}^{-1}} \cdot (\mathbf{c},0,0)_w = y \cdot \tau_w \quad \text{for any } w \in \widetilde{W}^1.$$

Furthermore, Prop. 3.9 implies

$$(0,0,\mathbf{c})_w = \begin{cases} \tau_w \cdot (0,0,\mathbf{c})_1 \\ -\tau_w \cdot (\mathbf{c},0,0)_1 \end{cases} \quad \text{and} \quad e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_w = \begin{cases} \tau_w \cdot x & \text{if } w = (s_0 s_1)^m \text{ with } m \geq 0, \\ -\tau_w \cdot y & \text{if } w = (s_0 s_1)^m s_0 \text{ with } m \geq 0, \end{cases}$$
$$(\mathbf{c},0,0)_w = \begin{cases} \tau_w \cdot (\mathbf{c},0,0)_1 \\ -\tau_w \cdot (0,0,\mathbf{c})_1 \end{cases} \quad \text{and} \quad e_{\mathrm{id}^{-1}} \cdot (\mathbf{c},0,0)_w = \begin{cases} \tau_w \cdot y & \text{if } w = (s_1 s_0)^m \text{ with } m \geq 0, \\ -\tau_w \cdot x & \text{if } w = (s_1 s_0)^m s_1 \text{ with } m \geq 0. \end{cases}$$

It follows, recalling that $\mathfrak{U}$ is a sub-H-bimodule of E^1 (Lemma 6.5-j), that

$$H \cdot \big(k(0,0,\mathbf{c})_1 \oplus k(\mathbf{c},0,0)_1\big) \cdot H \supseteq h^1_-(\widetilde{W}) \oplus h^1_+(\widetilde{W}) \qquad \text{and} \tag{102}$$
$$\mathfrak{U} \supseteq H \cdot \mathfrak{V} \cdot H \supseteq e_{\mathrm{id}^{-1}} h^1_-(\widetilde{W}) \oplus e_{\mathrm{id}} h^1_+(\widetilde{W}) \ . \tag{103}$$

By looking at the definition of $\mathbf{x}^\pm$ and using (101) and (103) we see that $(0,0,\mathbf{c})_1, (\mathbf{c},0,0)_1 \in \mathfrak{U}$. So (102) implies that $h^1_-(\widetilde{W}) \oplus h^1_+(\widetilde{W}) \subseteq \mathfrak{U}$, and together with (101) we obtain $\mathfrak{U} = E^1$.

It remains to check that

$$\mathfrak{V} \cap (\mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)}) \oplus \mathrm{im}(f^\pm)) = 0. \tag{104}$$

If $z = r_1 x + r_2 y + r_3 x\tau_{s_1} + r_4 y \tau_{s_0} \in \mathfrak{V}$ with $r_i \in k$ is an arbitrary element then $e_{\mathrm{id}} \cdot z \cdot e_{\mathrm{id}^{-1}} = r_1 x$, $e_{\mathrm{id}^{-1}} \cdot z \cdot e_{\mathrm{id}} = r_2 y$, $e_{\mathrm{id}} \cdot z \cdot e_{\mathrm{id}} = r_3 x \cdot \tau_{s_1}$, and $e_{\mathrm{id}^{-1}} \cdot z \cdot e_{\mathrm{id}^{-1}} = r_4 y \cdot \tau_{s_0}$. Hence it suffices to show that none of the elements x, y, $x \cdot \tau_{s_1}$, $y \cdot \tau_{s_0}$ is contained in $\mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)}) \oplus \mathrm{im}(f^\pm)$. Obviously we need to check this only for $x \cdot \tau_{s_1}$ and $y \cdot \tau_{s_0}$. First notice using (99) and (71) that

$$x \cdot \tau_{s_1} = e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_{s_1} \cdot e_{\mathrm{id}}, \ y \cdot \tau_{s_0} = e_{\mathrm{id}^{-1}} \cdot (\mathbf{c},0,0)_{s_0} \cdot e_{\mathrm{id}^{-1}} \ .$$

Therefore we only need to study

$$e_{\mathrm{id}} \cdot (\mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)}) + \mathrm{im}(f^\pm)) \cdot e_{\mathrm{id}} \ \oplus \ e_{\mathrm{id}^{-1}} \cdot (\mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)}) + \mathrm{im}(f^\pm)) \cdot e_{\mathrm{id}^{-1}}$$

and show that it does not contain $x \cdot \tau_{s_1}$ and $y \cdot \tau_{s_0}$. We focus on the case of $x \cdot \tau_{s_1}$, the case of $y \cdot \tau_{s_0}$ being analogous. It is immediate from Prop. 3.18 that $e_{\mathrm{id}} \cdot \mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)}) \cdot e_{\mathrm{id}} = e_{\mathrm{id}} \cdot h^1_0(\widetilde{W}^{\ell \geq 1}) \cdot e_{\mathrm{id}}$. Now assume that

$$x \cdot \tau_{s_1} = y_{\mathbf{x}_0,\mathbf{x}_1} + y^\pm \in e_{\mathrm{id}} \cdot (\mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)}) \oplus \mathrm{im}(f^\pm)) \cdot e_{\mathrm{id}} \ .$$

Applying the operator $\zeta \cdot \_ \cdot \zeta - 1$ on both sides and using Remark 6.6, we have

$$(\zeta + 1) \cdot z = (\zeta^2 - 1) \cdot y_{\mathbf{x}_0,\mathbf{x}_1}$$

where $z := -e_{\mathrm{id}} \cdot (0, 2\mathbf{c}\iota, 0)_{s_0 s_1} \cdot e_{\mathrm{id}} = -f_{\mathbf{x}_0,\mathbf{x}_1}(e_{\mathrm{id}} \tau_{s_0 s_1})$ by Prop. 3.18-i. So both z and $y_{\mathbf{x}_0,\mathbf{x}_1}$ lie in $\mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)})$. Recall that $f_{(\mathbf{x}_0,\mathbf{x}_1)}$ induces an isomorphism between $F^1 H$ and $\mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)})$ hence the latter is a free $k[\zeta]$-module. The identity above therefore implies that $z = (\zeta - 1) \cdot y_{\mathbf{x}_0,\mathbf{x}_1}$. This is impossible because $e_{\mathrm{id}} \tau_{s_0 s_1} \notin (\zeta - 1) F^1 H$.

This concludes the proof of the first equality of Lemma 6.7. The second equality then follows from Prop. 6.3. $\square$

Proposition 6.8. *Suppose $G = \mathrm{SL}_2(\mathbb{Q}_p)$ with $p \neq 2, 3$ and $\pi = p$. We have:*

i. The map $f^{\pm}$ described in (97) is bijective;

ii. $f_1 \circ g_1 = g_1 \circ f_1 = 0$ on E^1.

In particular (cf. Remark 3.29), as a left (resp. right) $k[\zeta^{\pm 1}]$-module, $\ker(f_1)$ is free of rank $4(p-1)$.

Proof. By [OS2] Remark 3.2.ii we have

$$\zeta\tau_w\zeta = \zeta^2\tau_w = \begin{cases} \tau_{(s_0s_1)^2w} & \text{if } w \in \widetilde{W}^{1,\ell\geq 1}, \\ \tau_{(s_1s_0)^2w} & \text{if } w \in \widetilde{W}^{0,\ell\geq 1}. \end{cases}$$

We deduce that

$$\begin{aligned} f_1(f_{(\mathbf{x}_0,\mathbf{x}_1)}(\tau_w)) &= f_{(\mathbf{x}_0,\mathbf{x}_1)}(\zeta^2\tau_w) - f_{(\mathbf{x}_0,\mathbf{x}_1)}(\tau_w) \\ &= \begin{cases} f_{(\mathbf{x}_0,\mathbf{x}_1)}(\tau_{(s_0s_1)^2w}) - f_{(\mathbf{x}_0,\mathbf{x}_1)}(\tau_w) & \text{if } w \in \widetilde{W}^{1,\ell\geq 1}, \\ f_{(\mathbf{x}_0,\mathbf{x}_1)}(\tau_{(s_1s_0)^2w}) - f_{(\mathbf{x}_0,\mathbf{x}_1)}(\tau_w) & \text{if } w \in \widetilde{W}^{0,\ell\geq 1} \end{cases} \end{aligned}$$

and, using Prop. 3.18-i, see that

$$f_1(f_{(\mathbf{x}_0,\mathbf{x}_1)}(\tau_w)) \in \begin{cases} k^\times(0, \mathbf{c}\iota, 0)_{(s_0s_1)^2w} + F_{\ell(w)+3}E^1 & \text{if } w \in \widetilde{W}^{1,\ell\geq 1}, \\ k^\times(0, \mathbf{c}\iota, 0)_{(s_1s_0)^2w} + F_{\ell(w)+3}E^1 & \text{if } w \in \widetilde{W}^{0,\ell\geq 1}. \end{cases} \tag{105}$$

On the other hand we observe that

$$\begin{aligned} f_1(x) = \zeta \cdot x \cdot \zeta - x &= e_{\mathrm{id}} \cdot (0, 0, \mathbf{c})_{s_0s_1} && \text{by Lemma 6.5-i and Prop. 3.9} \\ &= -e_{\mathrm{id}} \cdot (0, 2\mathbf{c}\iota, 0)_{s_0s_1s_0} - e_{\mathrm{id}} \cdot (0, 2\mathbf{c}\iota, 0)_{s_0} && \text{by (76)} \\ &\in F_3E^1 \cap \mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)}) && \text{by Prop. 3.18-i} \end{aligned}$$

and, by an analogous computation, $f_1(y) \in F_3E^1 \cap \mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)})$ as well. By Prop. 2.1 we conclude that $f_1(\mathfrak{V}) \subseteq F_4E^1 \cap \mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)}) = F_4E^1 \cap \ker(g_1)$ using Prop. 6.3. This together with (105) shows that f_1 is injective on $\mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)}) \oplus \mathfrak{V}$. Lemma 6.7 then implies that $\mathrm{im}(f^{\pm}) = \ker(f_1)$, which establishes Point i of the proposition. Furthermore, we have $f_1(\ker(g_1)) \subseteq \ker(g_1)$ since f_1 and g_1 commute (Remark 6.1-i). The fact that $f_1(\mathfrak{V}) \subseteq \ker(g_1)$ then shows, again invoking Lemma 6.7, that $f_1(E^1) \subseteq \ker(g_1)$ which amounts to our assertion ii. □

Remark 6.9. $(1 - e_{\gamma_0}) \cdot \ker(f_1) = (1 - e_{\gamma_0}) \cdot h^1_{\pm}(\widetilde{W})$.

Proof. We deduce from Cor. 3.10 that left multiplication by ζ preserves $(1 - e_{\gamma_0}) \cdot h^1_{\pm}(\widetilde{W})$ as well as $h^1_{\pm}(\widetilde{W}) \cdot (1 - e_{\gamma_0})$; for the latter use in addition that e_{γ_0} centralizes $h^1_0(\widetilde{W})$ by (72). Applying $\mathcal{J}$, which preserves $h^1_{\pm}(\widetilde{W})$ by Lemma 3.7, one sees that also right multiplication by ζ preserves $(1 - e_{\gamma_0}) \cdot h^1_{\pm}(\widetilde{W})$. We now compute

$$\begin{aligned} (1 - e_{\gamma_0}) \cdot \ker(f_1) &= (1 - e_{\gamma_0}) \cdot \mathrm{im}(f^{\pm}) \quad \text{by Prop. 6.8-i} \\ &= (1 - e_{\gamma_0})H \cdot \mathbf{x}^- \cdot \zeta^{\mathbb{N}} + (1 - e_{\gamma_0})H \cdot \mathbf{x}^+ \cdot \zeta^{\mathbb{N}} \\ &= H(1 - e_{\gamma_0}) \cdot \mathbf{x}^- \cdot \zeta^{\mathbb{N}} + H(1 - e_{\gamma_0}) \cdot \mathbf{x}^+ \cdot \zeta^{\mathbb{N}} \\ &= H(1 - e_{\gamma_0}) \cdot (\mathbf{c}, 0, 0)_1 \cdot \zeta^{\mathbb{N}} + H(1 - e_{\gamma_0}) \cdot (0, 0, \mathbf{c})_1 \cdot \zeta^{\mathbb{N}} \\ &= (1 - e_{\gamma_0})H \cdot (\mathbf{c}, 0, 0)_1 \cdot \zeta^{\mathbb{N}} + (1 - e_{\gamma_0})H \cdot (0, 0, \mathbf{c})_1 \cdot \zeta^{\mathbb{N}} \\ &= (1 - e_{\gamma_0}) \cdot h^1_-(\widetilde{W}) \cdot \zeta^{\mathbb{N}} + (1 - e_{\gamma_0}) \cdot h^1_+(\widetilde{W}) \cdot \zeta^{\mathbb{N}} \quad \text{by (67) and Prop. 3.9} \\ &\subseteq (1 - e_{\gamma_0}) \cdot h^1_{\pm}(\widetilde{W}) \quad \text{by the initial consideration.} \end{aligned}$$

Since $(1-e_{\gamma_0})\cdot\mathfrak{V}=0$ we conclude from Lemma 6.7 the right hand equality in

$$(1-e_{\gamma_0})\cdot h^1_0(\widetilde{W})\oplus(1-e_{\gamma_0})\cdot h^1_{\pm}(\widetilde{W})=(1-e_{\gamma_0})\cdot E^1=(1-e_{\gamma_0})\cdot\operatorname{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)})\oplus(1-e_{\gamma_0})\cdot\operatorname{im}(f^{\pm})\ .$$

The left hand summands are equal by Remark 3.19, of the right hand summands one contains the other by the above calculation since $\operatorname{im}(f^{\pm})=\ker(f_1)$. Hence the right hand summands must be equal as well. □

6.2.3 Structure of E^1 as an H-bimodule.

Recall the central idempotent $e_{\gamma_0}=e_{\mathrm{id}}+e_{\mathrm{id}^{-1}}$ in H.

Proposition 6.10. *Let $G=\mathrm{SL}_2(\mathbb{Q}_p)$ with $p\neq 2,3$ and assume $\pi=p$. We have the following.*

1. *As an H-bimodule, E^1 sits in an exact sequence of the form*
$$0\to\ker(f_1)\oplus\ker(g_1)\to E^1\to E^1/\ker(f_1)\oplus\ker(g_1)\to 0$$
where $E^1/\ker(f_1)\oplus\ker(g_1)$ is a 4-dimensional H-bimodule.
2. *As a left (resp. right) H-module, $E^1/\ker(f_1)\oplus\ker(g_1)$ is isomorphic to the direct sum of two copies of a simple 2-dimensional left (resp. right) H-module on which ζ and e_{γ_0} act by 1.*
3. *$E^1/\ker(g_1)$ is an H_ζ-bimodule.*

Proof. The first assertion follows from Lemma 6.7 and Prop. 6.8-i. As observed before we trivially have $f_1(\ker(g_1))\subseteq\ker(g_1)$. Hence f_1 induces a well defined endomorphism of $E^1/\ker(g_1)$. But Prop. 6.8-ii implies that this latter map is actually the zero map. It follows that $z\equiv\zeta\cdot z\cdot\zeta \bmod \ker(g_1)$ for any $z\in E^1$, which implies the third assertion.

It remains to determine the module structure of the 4-dimensional quotient $E^1/\ker(f_1)\oplus\ker(g_1)$ which has as a k-basis the cosets of x, y, $x\cdot\tau_{s_1}$, and $y\cdot\tau_{s_0}$. Obviously e_{γ_0} acts by 1 on these elements from the left and the right. It follows from Lemma 6.5 that ζ acts by 1 from the left and the right on this quotient. The same lemma also implies that x and $x\cdot\tau_{s_1}$ generate a 2-dimensional right H-submodule in $E^1/\ker(f_1)\oplus\ker(g_1)$. It is necessarily a simple module because the only one-dimensional modules on which e_{γ_0} acts by 1 are supersingular, namely annihilated by ζ (see (26)). Correspondingly one sees that y and $y\cdot\tau_{s_0}$ generate another 2-dimensional simple right H-submodule in $E^1/\ker(f_1)\oplus\ker(g_1)$. It is easy to check that these two simple right modules are isomorphic to each other via the map $x\mapsto y\cdot\tau_{s_1}, x\cdot\tau_{s_0}\mapsto y$. This proves in particular that $E^1/\ker(f_1)+\ker(g_1)$ is semisimple isotypic as a right H-module, and therefore also as a left H-module using $\mathcal{J}$. □

6.3 Structure of E^2

We still assume that $G=\mathrm{SL}_2(\mathbb{Q}_p)$ with $p\neq 2,3$ and that $\pi=p$. Here we focus on the graded piece E^2 and work with the endomorphisms of H-bimodules

$$f_2:=\zeta\cdot\mathrm{id}_{E^2}\cdot\zeta-\mathrm{id}_{E^2}:c\mapsto\zeta\cdot c\cdot\zeta-c\quad\text{ and }\quad g_2:=\zeta\cdot\mathrm{id}_{E^2}-\mathrm{id}_{E^2}\cdot\zeta:c\mapsto\zeta\cdot c-c\cdot\zeta$$

as introduced in §6.1. By Prop. 6.10 we have an exact sequence of H-bimodules

$$0\longrightarrow\ker(f_1)\oplus\ker(g_1)\longrightarrow E^1\longrightarrow E^1/(\ker(f_1)\oplus\ker(g_1))\longrightarrow 0$$

where $E^1/(\ker(f_1)\oplus\ker(g_1))$ is a 4-dimensional H-bimodule. Passing to duals, this gives an exact sequence of H-bimodules

$$0\longrightarrow(E^1/(\ker(f_1)\oplus\ker(g_1)))^\vee\longrightarrow(E^1)^\vee\longrightarrow(\ker(f_1)\oplus\ker(g_1))^\vee\longrightarrow 0\ .$$

We define the sub-H-bimodules

$$(E^1)^\vee_{f_1} := \{\xi \in (E^1)^\vee : \xi|\ker(g_1) = 0\} \text{ and } (E^1)^\vee_{g_1} := \{\xi \in (E^1)^\vee : \xi|\ker(f_1) = 0\}.$$

Then

$$(E^1)^\vee = (E^1)^\vee_{f_1} + (E^1)^\vee_{g_1} \quad \text{and} \quad (E^1)^\vee_{f_1} \cap (E^1)^\vee_{g_1} = (E^1/(\ker(f_1) \oplus \ker(g_1)))^\vee \ .$$

Lemma 6.11. *The composed map*

$$(E^1)^{\vee,f} \xrightarrow{\subseteq} (E^1)^\vee \longrightarrow (\ker(f_1) \oplus \ker(g_1))^\vee$$

is injective.

Proof. We have to prove, for $m \geq 1$, that

$$\ker(f_1) + \ker(g_1) + F^m E^1 = E^1 \ .$$

Because of Lemma 6.7 this boils down to proving that x, $x \cdot \tau_{s_1}$, y, $y \cdot \tau_{s_0}$ all lie in $\mathrm{im}(f_{(\mathbf{x}_0,\mathbf{x}_1)}) \oplus \mathrm{im}(f^\pm) \oplus F^m E^1$. Since $y = \Gamma_\varpi(x)$ it is enough to prove this for $x = e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_1$ and $x \cdot \tau_{s_1} = e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_{s_1}$. By Lemma 6.5-e) we know that $\zeta^m \cdot x - x$ and $\zeta^m \cdot x \cdot \tau_{s_1} - x \cdot \tau_{s_1}$ lie in $\ker(f_1) + \ker(g_1)$ for any $m \geq 1$. But, using Cor. 3.10, we have $\zeta^m \cdot x = e_{\mathrm{id}} \zeta^m \cdot (0,0,\mathbf{c})_1 = e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_{(s_0 s_1)^m} \in F^{2m} E^1$ and then $\zeta^m \cdot x \cdot \tau_{s_1} = e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_{(s_0 s_1)^m} \tau_{s_1} \in F^{2m-1} E^1$ by applying $\mathcal{J}$ and using Prop. 3.9. □

We put $K_{f_1} := (E^1)^{\vee,f} \cap (E^1)^\vee_{f_1}$ and $K_{g_1} := (E^1)^{\vee,f} \cap (E^1)^\vee_{g_1}$. Because of Lemma 6.11 we have $K_{f_1} \oplus K_{g_1} \subseteq (E^1)^{\vee,f}$. Since K_{f_1} and K_{g_1} inject into $\ker(f_1)^\vee$ and $\ker(g_1)^\vee$, respectively, we have $\zeta \cdot \eta \cdot \zeta = \eta$ for $\eta \in K_{f_1}$ and $\zeta \cdot \eta = \eta \cdot \zeta$ for $\eta \in K_{g_1}$.

Lemma 6.12. $(E^1)^{\vee,f} = K_{f_1} \oplus K_{g_1}$.

Proof. Let $\xi \in (E^1)^{\vee,f}$. We claim that there exists a linear map $\eta \in K_{g_1}$ such that $\eta|_{\ker(g_1)} = \xi|_{\ker(g_1)}$. This implies that $\xi - \eta \in K_{f_1}$.

- Suppose $\xi = \xi(1 - e_{\gamma_0})$. Then we can see ξ as an element in $((1 - e_{\gamma_0})E^1)^{\vee,f}$. Since $(1 - e_{\gamma_0})E^1 = (1 - e_{\gamma_0}) \ker(f_1) \oplus (1 - e_{\gamma_0}) \ker(g_1)$ where $(1 - e_{\gamma_0}) \ker(f_1) = (1 - e_{\gamma_0}) h^1_\pm(\widetilde{W})$ by Remark 6.9 and $(1 - e_{\gamma_0}) \ker(g_1) = (1 - e_{\gamma_0}) h^1_0(\widetilde{W})$ by Remark 3.19 and Prop. 6.3, we may define η to be zero on $(1 - e_{\gamma_0}) \ker(f_1)$ and $\eta|_{(1-e_{\gamma_0})\ker(g_1)} = \xi|_{(1-e_{\gamma_0})\ker(g_1)}$.

- Suppose $\xi = (1 - e_{\gamma_0}) \xi e_{\gamma_0}$. Then we can see ξ as an element in $(e_{\gamma_0} E^1 (1 - e_{\gamma_0}))^{\vee,f}$. Since $e_{\gamma_0} \ker(g_1)(1 - e_{\gamma_0}) = 0$ by Remark 3.19 and Prop. 6.3, the linear form ξ is already in K_{f_1}.

- Now suppose $\xi = e_{\gamma_0} \xi e_{\gamma_0}$. We may consider separately two cases, namely $\xi = e_{\mathrm{id}} \xi e_{\mathrm{id}}$ and $\xi = e_{\mathrm{id}} \xi e_{\mathrm{id}^{-1}}$ (the other cases following by conjugation by ϖ). We treat the first case, the second one being similar. If $\xi = e_{\mathrm{id}} \xi e_{\mathrm{id}}$, then we can see ξ as a linear map on $e_{\mathrm{id}} E^1 e_{\mathrm{id}}$ (recall that we are working in the H-bimodule $(E^1)^\vee$). By Lemma 6.7 and (98) we have

$$e_{\mathrm{id}} E^1 e_{\mathrm{id}} = e_{\mathrm{id}} (\ker(f_1) \oplus \ker(g_1)) e_{\mathrm{id}} \oplus k e_{\mathrm{id}} (0,0,\mathbf{c})_{s_1} \ .$$

  Define the linear map $\eta : E^1 \to k$ by

$$\eta|_{e_{\mathrm{id}} \ker(f_1) e_{\mathrm{id}}} := 0, \ \eta|_{e_{\mathrm{id}} \ker(g_1) e_{\mathrm{id}}} := \xi|_{e_{\mathrm{id}} \ker(g_1) e_{\mathrm{id}}}, \text{ and}$$
$$\eta(e_{\mathrm{id}}(0,0,\mathbf{c})_{s_1}) := \sum_{j=1}^{+\infty} \xi\big(e_{\mathrm{id}}(0, 2\mathbf{c}\iota, 0)_{(s_0 s_1)^j}\big) \ ,$$

which is well defined because $\xi \in (E^1)^{\vee,f}$. From (72) we have

$$e_{\mathrm{id}} E^1 e_{\mathrm{id}} = e_{\mathrm{id}} h^1_0(\widetilde{W}^{even}) + e_{\mathrm{id}} h^1_+(\widetilde{W}^{odd}) \ .$$

It remains to check that $\eta \in (E^1)^{\vee,f}$. Since $h^1_0(\widetilde{W}^{\ell\geq 2})$ is contained in $\ker(g_1)$ by Prop. 3.18, we only need to check that η is trivial on $e_{\mathrm{id}} \cdot h^1_+(\widetilde{W}^{odd,\ell\geq m})$ for m large enough. From Cor. 3.10 we deduce that $\zeta^{m+1} \cdot x \cdot \tau_{s_1} = \zeta^{m+1} e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_{s_1} = -e_{\mathrm{id}}(0, 2\mathbf{c}\iota, 0)_{(s_0 s_1)^{m+1}} - e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_{(s_0 s_1)^m s_0}$ for any $m \geq 0$. Hence

$$\begin{aligned}
& e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_{(s_0 s_1)^m s_0} \\
&\quad = -\zeta^{m+1} \cdot x \cdot \tau_{s_1} - e_{\mathrm{id}} \cdot (0, 2\mathbf{c}\iota, 0)_{(s_0 s_1)^{m+1}} \\
&\quad = -e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_{s_1} - (\zeta^{m+1} \cdot x \cdot \tau_{s_1} - x \cdot \tau_{s_1}) - e_{\mathrm{id}} \cdot (0, 2\mathbf{c}\iota, 0)_{(s_0 s_1)^{m+1}} \\
&\quad = -e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_{s_1} - (\sum_{j=0}^{m} \zeta^j)(\zeta \cdot x - x) \cdot \tau_{s_1} - e_{\mathrm{id}} \cdot (0, 2\mathbf{c}\iota, 0)_{(s_0 s_1)^{m+1}} \\
&\quad \in \ker(f_1) - e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_{s_1} + (\sum_{j=0}^{m} \zeta^j) e_{\mathrm{id}} \cdot (0, 2\mathbf{c}\iota, 0)_{s_0} \cdot \tau_{s_1} - e_{\mathrm{id}} \cdot (0, 2\mathbf{c}\iota, 0)_{(s_0 s_1)^{m+1}} \\
&\qquad\qquad \text{by Lemma 6.5-e)} \\
&\quad = \ker(f_1) - e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_{s_1} + (\sum_{j=0}^{m} \zeta^j) e_{\mathrm{id}} \cdot (0, 2\mathbf{c}\iota, 0)_{s_0 s_1} - e_{\mathrm{id}} \cdot (0, 2\mathbf{c}\iota, 0)_{(s_0 s_1)^{m+1}} \\
&\qquad\qquad \text{by Lemma 3.12-i} \\
&\quad = \ker(f_1) - e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_{s_1} + (\sum_{j=0}^{m} e_{\mathrm{id}} \cdot (0, 2\mathbf{c}\iota, 0)_{(s_0 s_1)^{j+1}} - e_{\mathrm{id}} \cdot (0, 2\mathbf{c}\iota, 0)_{(s_0 s_1)^{m+1}}) \\
&\qquad\qquad \text{by Cor. 3.10} \\
&\quad = \ker(f_1) - e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_{s_1} + (\sum_{j=1}^{m} e_{\mathrm{id}} \cdot (0, 2\mathbf{c}\iota, 0)_{(s_0 s_1)^j}) \ .
\end{aligned}$$

Since η is zero on $\ker(f_1)$ it follows that

$$\eta(e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_{(s_0 s_1)^m s_0}) = \eta(-e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_{s_1} + \sum_{j=1}^{m} e_{\mathrm{id}} \cdot (0, 2\mathbf{c}\iota, 0)_{(s_0 s_1)^j}) = -\xi(\sum_{j=m+1}^{\infty} e_{\mathrm{id}} \cdot (0, 2\mathbf{c}\iota, 0)_{(s_0 s_1)^j}).$$

An analogous computation gives

$$\eta(e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_{(s_1 s_0)^m s_1}) = \eta(e_{\mathrm{id}} \cdot (0,0,\mathbf{c})_{s_1} - \sum_{j=1}^{m} e_{\mathrm{id}} \cdot (0, 2\mathbf{c}\iota, 0)_{(s_0 s_1)^j}) = \xi(\sum_{j=m+1}^{\infty} e_{\mathrm{id}} \cdot (0, 2\mathbf{c}\iota, 0)_{(s_0 s_1)^j}).$$

Both are zero for m large enough.

□

Recall from (14) that we have an isomorphism of H-bimodules

$$E^2 \xrightarrow{\cong} {}^{\jmath}((E^1)^{\vee,f})^{\jmath}. \tag{106}$$

Proposition 6.13. *Suppose $G = \mathrm{SL}_2(\mathbb{Q}_p)$ with $p \neq 2,3$ and $\pi = p$. Via the isomorphism (106), we have $\ker(f_2) \cong K_{f_1}$ and $\ker(g_2) \cong K_{g_1}$ and as H-bimodules*

$$E^2 = \ker(f_2) \oplus \ker(g_2).$$

In particular, $f_2 \circ g_2 = g_2 \circ f_2 = 0$.

Proof. Let us denote the isomorphism (106) temporarily by $\mathfrak{j}$. We had observed already that $\zeta\eta\zeta = \eta$ for $\eta \in K_{f_1}$ and $\zeta\eta = \eta\zeta$ for $\eta \in K_{g_1}$. It follows that $\mathfrak{j}^{-1}(K_{f_1}) \subseteq \ker(f_2)$ and $\mathfrak{j}^{-1}(K_{g_1}) \subseteq \ker(g_2)$. We also know from Lemma 6.2-iv that $\ker(f_2) \cap \ker(g_2) = \{0\}$. Therefore, our assertion is a consequence of Lemma 6.12. $\square$

From Lemma 6.2-i-ii, Propositions 6.8-ii and 6.13 we get:

Corollary 6.14. *Under the same assumptions, we have $f \circ g = g \circ f = 0$ on E^*.*

In the following two sections we determine the H-bimodule structure of the two summands $\ker(g_2)$ and $\ker(f_2)$.

6.3.1 On $\ker(g_2)$

The surjective restriction map $(E^1)^\vee \longrightarrow \ker(g_1)^\vee$ induces the injective map of H-bimodules

$$\ker(g_2) \cong {}^{\jmath}(K_{g_1})^{\jmath} \longrightarrow {}^{\jmath}(\ker(g_1)^\vee)^{\jmath} .$$

We have to determine the image of this map. From Prop. 3.18-i we know that $h^1_0(\widetilde{W}^{\ell \geq 2}) \subseteq \ker(g_1) \subseteq h^1_0(\widetilde{W}) \oplus h^1_\pm(\Omega)$. Hence the decreasing filtration

$$F^n \ker(g_1) := \begin{cases} \ker(g_1) & \text{if } n = 1, \\ h^1_0(\widetilde{W}^{\ell \geq n}) & \text{if } n \geq 2 \end{cases}$$

is well defined as well as the corresponding finite dual

$$\ker(g_1)^{\vee,f} := \bigcup_{n \geq 1} (\ker(g_1)/F^n \ker(g_1))^\vee .$$

If $\xi \in (E^1)^{\vee,f}$ satisfies $\xi | F^n E^1 = 0$ for some $n \geq 2$ then obviously $\xi_{|\ker(g_1)} | F^n \ker(g_1) = 0$ and hence $\xi_{|\ker(g_1)} \in \ker(g_1)^{\vee,f}$. Vice versa, let $\eta \in \ker(g_1)^{\vee,f}$ such that $\eta | F^n \ker(g_1) = 0$ for some $n \geq 2$. We first choose an extension $\dot\eta$ of η to $h^1_0(\widetilde{W}) \oplus h^1_\pm(\Omega)$ and then extend $\dot\eta$ further to $\ddot\eta$ on E^1 by setting $\ddot\eta | h^1_\pm(\widetilde{W}^{\ell \geq 1}) := 0$. then clearly $\ddot\eta | F^n E^1 = 0$, i.e., $\ddot\eta \in (E^1)^{\vee,f}$. This shows that our η has an extension in $(E^1)^{\vee,f}$. By Prop. 6.13 it then must also have an extension $\xi \in (E^1)^{\vee,f}$ which satisfies $\xi_{|\ker(f_1)} = 0$, i.e., $\xi \in K_{g_1}$. We see that the above restriction map induces an isomorphism of H-bimodules

$$\ker(g_2) \cong {}^{\jmath}(K_{g_1})^{\jmath} \xrightarrow{\cong} {}^{\jmath}(\ker(g_1)^{\vee,f})^{\jmath} . \tag{107}$$

Proposition 6.15. *Suppose $G = \mathrm{SL}_2(\mathbb{Q}_p)$ with $p \neq 2,3$ and $\pi = p$. The space $\ker(g_2)$ is the subspace of ζ-torsion in E^2 on the left and on the right. We have an isomorphism of H-bimodules*

$$\ker(g_2) \cong (F^1 H)^{\vee,f} \cong \bigcup_{n \geq 1} (F^1 H/\zeta^n F^1 H)^\vee . \tag{108}$$

In particular, $\ker(g_2)$ is $k[\zeta]$-divisible.

Proof. Prop. 3.18-i makes it directly visible that the isomorphism of H-bimodules $F^1H \cong \ker(g_1)$ in Prop. 6.3 respects the filtrations on both sides. Combined with (107) we therefore obtain an isomorphism of H-bimodules

$$\ker(g_2) \cong {}^{\mathcal{J}}((F^1H)^{\vee,f})^{\mathcal{J}} \cong ({}^{\mathcal{J}}(F^1H)^{\mathcal{J}})^{\vee,f} \cong (F^1H)^{\vee,f}$$

where the last isomorphism is induced by $\mathcal{J} : H \to H$. Since $\zeta^n \cdot F^1H = F^{2n+1}H$ for $n \geq 1$ by [OS2] Remark 3.2.ii, we also have

$$\ker(g_2) \cong \bigcup_{n\geq 1} (F^1H/\zeta^n F^1H)^\vee \ .$$

In particular, this makes visible that $\ker(g_2)$ is ζ-torsion. On the other hand $\ker(f_2)$ does not contain any left or right ζ-torsion since it is an H_ζ-bimodule. It therefore follows from Prop. 6.13 that $\ker(g_2)$ is the full subspace of left (or right) ζ-torsion in E^2. By Lemma 2.7 F^1H is a finitely generated free $k[\zeta]$-module. Hence $\bigcup_{n\geq 1}(F^1H/\zeta^n F^1H)^\vee \cong k[\zeta^{\pm 1}]/k[\zeta] \otimes_{k[\zeta]} F^1H$ noncanonically as a $k[\zeta]$-module, which shows that $\ker(g_2)$ is $k[\zeta]$-divisible. □

Corollary 6.16. *Under the same assumptions, we have* $\ker(f_1) \cdot \ker(g_2) = 0 = \ker(g_2) \cdot \ker(f_1)$.

Proof. Let $a \in \ker(f_1)$ and $b \in \ker(g_2)$. By Prop. 6.15 we find an $m \geq 1$ such that $\zeta^m \cdot b = 0 = b \cdot \zeta^m$. Then $a \cdot b = \zeta^m \cdot a \cdot \zeta^m \cdot b = 0 = b \cdot \zeta^m \cdot a \cdot \zeta^m$. □

6.3.2 On $\ker(f_2)$

We proceed in a way which is entirely analogous to section §3.7.3. Consider the following elements of E^2:

$$\mathbf{a}^+ := (\alpha, 0, 0)_1 - e_{\mathrm{id}} \cdot (0, \iota^{-1}\alpha, 0)_{s_0} = (\alpha, 0, 0)_1 - (0, \iota^{-1}\alpha, 0)_{s_0} \cdot e_{\mathrm{id}^{-1}} \text{ and}$$
$$\mathbf{a}^- := (0, 0, \alpha)_1 + e_{\mathrm{id}^{-1}} \cdot (0, \iota^{-1}\alpha, 0)_{s_1} = (0, 0, \alpha)_1 + (0, \iota^{-1}\alpha, 0)_{s_1} \cdot e_{\mathrm{id}}$$

where α is chosen as in (60) (see also (94)). It is easy to verify that

$$\mathcal{J}(\mathbf{a}^+) = \mathbf{a}^+ \quad \text{and} \quad \mathcal{J}(\mathbf{a}^-) = \mathbf{a}^- \tag{109}$$

using Lemma 4.1 and (92). In order to check that $\mathbf{a}^+$ lies in $\ker(f_2)$ we compute

$$\begin{aligned}
\mathbf{a}^+ \cdot \zeta &= \mathcal{J}(\zeta \cdot \mathcal{J}(\mathbf{a}^+)) = \mathcal{J}(\zeta \cdot \mathbf{a}^+) \\
&= \mathcal{J}((\alpha,0,0)_{s_0 s_1} + e_1 \cdot (0,0,-\alpha)_{s_1} + e_1 \cdot (\alpha,0,0)_1) \quad \text{by Cor. 4.6} \\
&= (\alpha,0,0)_{s_1 s_0} + (\alpha,0,0)_{s_1^{-1}} \cdot e_1 + (\alpha,0,0)_1 \cdot e_1 \quad \text{by Lemma 4.1} \\
&= (\alpha,0,0)_{s_1 s_0} + \tau_{\omega_{-1}} \cdot (\alpha,0,0)_{s_1} \cdot e_1 + (\alpha,0,0)_1 \cdot e_1 \quad \text{by (92)} \\
&= (\alpha,0,0)_{s_1 s_0} + e_{\mathrm{id}^2} \cdot (\alpha,0,0)_{s_1} + e_{\mathrm{id}^2} \cdot (\alpha,0,0)_1 \quad \text{by (93) and (94).}
\end{aligned}$$

Hence

$$\begin{aligned}
\zeta \cdot \mathbf{a}^+ \cdot \zeta &= \zeta \cdot \big((\alpha,0,0)_{s_1 s_0} + e_{\mathrm{id}^2} \cdot (\alpha,0,0)_{s_1} + e_{\mathrm{id}^2} \cdot (\alpha,0,0)_1\big) \\
&= (\alpha,0,0)_1 + e_{\mathrm{id}} \cdot (0, \iota^{-1}(\alpha), 0)_{s_1^2 s_0} + e_{\mathrm{id}^2}(-\alpha,0,0)_{s_1^2 s_0} \\
&\qquad + e_{\mathrm{id}^2} \cdot (\alpha,0,0)_{s_0 s_1^2} + e_{\mathrm{id}^2} \cdot (-\alpha,0,0,)_{s_0 s_1} + e_{\mathrm{id}^2} \cdot (\alpha,0,0)_{s_0 s_1} \quad \text{by Cor. 4.6} \\
&= (\alpha,0,0)_1 + e_{\mathrm{id}} \cdot (0, \iota^{-1}(\alpha), 0)_{s_1^2 s_0} \\
&= (\alpha,0,0)_1 - e_{\mathrm{id}} \cdot (0, \iota^{-1}(\alpha), 0)_{s_0} \quad \text{by (92)} \\
&= \mathbf{a}^+ \ .
\end{aligned}$$

Using Lemma 4.2 we notice that $\Gamma_\varpi(\mathbf{a}^+) = \mathbf{a}^-$. Hence Remark 6.1-iv implies that also $\mathbf{a}^- \in \ker(f_2)$. As in Lemma 3.20 we therefore have the homomorphism of left H_ζ-modules

$$H_\zeta \oplus H_\zeta \xrightarrow{f_{\mathbf{a}^+} + f_{\mathbf{a}^-}} \ker(f_2) \tag{110}$$

sending $(1,0)$ and $(0,1)$ to $\mathbf{a}^+$ and $\mathbf{a}^-$, respectively.

Remark 6.17. Let $w \in \widetilde{W}$ with $\ell(w) \geq 1$. From Proposition 4.5 we obtain

$$\tau_w \cdot \mathbf{a}^+ = \begin{cases} 0 & \text{if } w^{-1} \in \widetilde{W}^1 \\ (0,0,-\alpha)_w & \text{if } w^{-1} \in \widetilde{W}^0,\ \ell(w) \text{ odd} \\ (\alpha,0,0)_w & \text{if } w^{-1} \in \widetilde{W}^0,\ \ell(w) \text{ even} \end{cases}$$
$$\text{and} \tag{111}$$
$$\tau_w \cdot \mathbf{a}^- = \begin{cases} 0 & \text{if } w^{-1} \in \widetilde{W}^0 \\ (-\alpha,0,0)_w & \text{if } w^{-1} \in \widetilde{W}^1,\ \ell(w) \text{ odd} \\ (0,0,\alpha)_w & \text{if } w^{-1} \in \widetilde{W}^1,\ \ell(w) \text{ even}\ . \end{cases}$$

Lemma 6.18. *1. For any $u \in \mathbb{F}_p^\times$ we have $\mathbf{a}^+ \cdot \tau_{\omega_u} = u^{-2}\tau_{\omega_u} \cdot \mathbf{a}^+$ and $\mathbf{a}^- \cdot \tau_{\omega_u} = u^2 \tau_{\omega_u} \cdot \mathbf{a}^-$.*

2. We have $\mathbf{a}^+ \cdot \tau_{s_0} = \tau_{s_0} \cdot \mathbf{a}^+ = 0$ and $\mathbf{a}^- \cdot \tau_{s_1} = \tau_{s_1} \cdot \mathbf{a}^- = 0$.

3. We have

$$\mathbf{a}^+ \cdot \iota(\tau_{s_1}) = -\tau_{\omega_{-1}} \iota(\tau_{s_0}) \cdot \mathbf{a}^- \cdot \zeta \quad \textit{and}$$
$$\mathbf{a}^- \cdot \iota(\tau_{s_0}) = -\tau_{\omega_{-1}} \iota(\tau_{s_1}) \cdot \mathbf{a}^+ \cdot \zeta\ .$$

Proof. 1. Using using (67), (69) we compute:

$$\begin{aligned} \mathbf{a}^+ \cdot \tau_{\omega_u} &= \mathcal{J}(\tau_{\omega_u^{-1}} \cdot ((\alpha,0,0)_1 + e_{\mathrm{id}} \cdot (0, \iota^{-1}\alpha, 0)_{s_0^{-1}})) \\ &= \mathcal{J}((u^{-2}\alpha,0,0)_{\omega_u^{-1}} + u^{-1} e_{\mathrm{id}} \cdot (0,\iota^{-1}\alpha,0)_{s_0^{-1}}) \\ &= (\alpha,0,0)_{\omega_u} - u^{-1}(0,\iota^{-1}\alpha,0)_{s_0} \cdot e_{\mathrm{id}^{-1}} \\ &= u^{-2}(\tau_{\omega_u} \cdot (\alpha,0,0)_1 - \tau_{\omega_u} e_{\mathrm{id}} \cdot (0,\iota^{-1}\alpha,0)_{s_0}) \\ &= u^{-2}\tau_{\omega_u} \cdot \mathbf{a}^+ \end{aligned}$$

and, by an analogous computation (or by conjugation by ϖ), we obtain the second claim of Point 1.
2. Point 2 follows from (111) and (109).
3. We check the first identity. Since $\mathbf{a}^-, \mathbf{a}^+ \in \ker(f_2)$, we may as well check the following

$$-\zeta \cdot \mathbf{a}^+ \cdot (\tau_{s_1} + e_1) = (\tau_{s_0^{-1}} + e_1) \cdot \mathbf{a}^- \tag{112}$$

For the left hand side, we have using Lemma 4.1, Prop. 4.5, (92) and (93),(94)

$$\begin{aligned} \mathbf{a}^+ \cdot (\tau_{s_1} + e_1) &= \mathcal{J}((\tau_{s_1} + e_1) \cdot ((\alpha,0,0)_{\omega_{-1}} - e_{\mathrm{id}} \cdot (0,\iota^{-1}\alpha,0)_{s_0^{-1}})) \\ &= \mathcal{J}((0,0,-\alpha)_{s_1^{-1}} + e_1 \cdot (\alpha,0,0)_1) = (\alpha,0,0)_{s_1} + e_{\mathrm{id}^2} \cdot (\alpha,0,0)_1 \end{aligned}$$

and then using Corollary 4.6:

$$\begin{aligned}-\zeta\cdot\mathbf{a}^+\cdot(\tau_{s_1}+e_1) &= -(\alpha,0,0)_{s_0s_1^2}+e_1\cdot(0,0,\alpha)_{s_1^2}+e_{\mathrm{id}^2}\cdot(\alpha,0,0,)_{s_0s_1}-e_{\mathrm{id}^2}\cdot(\alpha,0,0,)_{s_0s_1}\\ &= -(\alpha,0,0)_{s_0^{-1}}+e_1\cdot(0,0,\alpha)_1\end{aligned}$$

For the right hand side we have, using Remark 6.17:

$$\tau_{s_0^{-1}}\cdot\mathbf{a}^- = (-\alpha,0,0)_{s_0^{-1}},\quad e_1\cdot\mathbf{a}^- = e_1\cdot(0,0,\alpha)_1\ .$$

By adding up, we see that (112) holds.

□

By Lemma 6.18-2, the map (110) factors through a homomorphism of left H_ζ-modules

$$H_\zeta/H_\zeta\tau_{s_0}\oplus H_\zeta/H_\zeta\tau_{s_1}\xrightarrow{f_{\mathbf{a}^+}+f_{\mathbf{a}^-}}\ker(f_2)\ . \tag{113}$$

Proposition 6.19. *Suppose $G=\mathrm{SL}_2(\mathbb{Q}_p)$ with $p\neq 2,3$ and $\pi=p$. The map (113) induces an isomorphism of H_ζ-bimodules*

$$(H_\zeta/H_\zeta\tau_{s_0}\oplus H_\zeta/H_\zeta\tau_{s_1})^{\pm}\xrightarrow{\simeq}\ker(f_2) \tag{114}$$

Proof. We need to verify that the map is bijective and right H-equivariant. We may compare with the proof of Proposition 3.28. Just like in that proof, the right H-equivariance is seen by comparing the definition (80) with Lemma 6.18.

Concerning the injectivity we first observe that it suffices to check the injectivity of the restriction of the map to $H/H\tau_{s_0}\oplus H/H\tau_{s_1}$. The elements τ_w with $w\in\widetilde{W}$ such that $\ell(ws_0)=\ell(w)+1$ from a k-basis of $H/H\tau_{s_0}$; they are of the form $w=\omega(s_0s_1)^m$ or $=\omega s_1(s_0s_1)^m$ with $m\geq 0$ and $\omega\in\Omega$. Using (91) and (111), we see that

$$\tau_w\cdot\mathbf{a}^+\in\begin{cases}k^\times(\alpha,0,0)_w & \text{if } w=\omega(s_0s_1)^m \text{ with } m\geq 1,\\ k^\times(0,0,\alpha)_w & \text{if } w=\omega s_1(s_0s_1)^m \text{ with } m\geq 0,\\ k^\times(\alpha,0,0)_w+k^\times e_{\mathrm{id}}(0,\iota^{-1}\alpha,\mathbf{c},0)_{s_0} & \text{if } w=\omega.\end{cases} \tag{115}$$

Similarly the elements τ_w with $w\in\widetilde{W}$ such that $\ell(ws_1)=\ell(w)+1$ form a k-basis of $H/H\tau_{s_1}$; they are of the form $w=\omega(s_1s_0)^m$ or $=\omega s_0(s_1s_0)^m$ with $m\geq 0$ and $\omega\in\Omega$. In this case we obtain

$$\tau_w\cdot\mathbf{a}^-\in\begin{cases}k^\times(0,0,\alpha)_w & \text{if } w=\omega(s_1s_0)^m \text{ with } m\geq 1,\\ k^\times(\alpha,0,0)_w & \text{if } w=\omega s_0(s_1s_0)^m \text{ with } m\geq 0,\\ k^\times(0,0,\alpha)_w+k^\times e_{\mathrm{id}^{-1}}(0,\iota^{-1}\alpha,\mathbf{c},0)_{s_1} & \text{if } w=\omega.\end{cases} \tag{116}$$

By comparing the lists (115) and (116) we easily see that the elements

$$\{\tau_w\cdot\mathbf{a}^+:\ell(ws_0)=\ell(w)+1\}\cup\{\tau_w\cdot\mathbf{a}^-:\ell(ws_1)=\ell(w)+1\}$$

in E^2 are k-linearly independent. This concludes the proof of the injectivity. For the surjectivity, we gather the following arguments:

- A basis for $\ker(g_1)$ is given by the set of all $f_{(x_0,x_1)}(\tau_w)$, $w\in\widetilde{W}$, $\ell(w)\geq 1$. These elements are spelled out in Proposition 3.18. From these formulas, we see that an element in $\ker(f_2)$ lies necessarily in the space $h^2_\pm(\widetilde{W}^{\ell\geq 2})+h^2(s_1\Omega)+h^2(s_0\Omega)+h^2(\Omega)$.

– From (115) and (116), we deduce that $h^2_-(\widetilde{W}^{1,\ell\geq 1}) + h^2_+(\widetilde{W}^{0,\ell\geq 1}) = \sum_{w\in\widetilde{W},\ell(w)\geq 1} k\tau_w\cdot\mathbf{a}^- + k\tau_w\cdot\mathbf{a}^+$ is contained in the image of the map of the proposition.

  – So it is contained in $\ker(f_2)$ which is invariant under $\mathcal{J}$. Therefore by Lemma 4.1, the whole space $h^2_\pm(\widetilde{W}^{\ell\geq 1})$ is contained in $\ker(f_2)$.
  – But this map is also right H-equivariant. So for $w\in\widetilde{W}$ with length ≥ 1, the elements $\mathbf{a}^+\cdot\tau_{w^{-1}} = \mathcal{J}(\tau_w\cdot\mathbf{a}^+)$ and $\mathbf{a}^-\cdot\tau_{w^{-1}} = \mathcal{J}(\tau_w\cdot\mathbf{a}^-)$ also lie in this image (see (109)). Therefore the whole space $h^2_\pm(\widetilde{W}^{\ell\geq 1})$ is contained in the image of the map.

- The component in
$$h^2(\Omega) + h^2_0(s_1\Omega) + h^2(s_0\Omega)$$
of $\ker(f_2)$ is spanned by all $\tau_\omega\cdot\mathbf{a}^+$ and $\tau_\omega\cdot\mathbf{a}^-$ for $\omega\in\Omega$.

  To verify this statement we notice, using the third lines of (115) and (116), that it is equivalent to saying that the component in $h^2_0(s_1\Omega)+h^2(s_0\Omega)$ of $\ker(f_2)$ is zero. But the latter follows easily from the formulas for $f_{(x_0,x_1)}(s_\epsilon\tau_\omega)$, $\omega\in\Omega$, $\epsilon=0,1$ given in Proposition 3.18.

- We have proved that $\ker(f_2) = h^2_\pm(\widetilde{W}^{\ell\geq 1}) \oplus \bigoplus_{\omega\in\Omega} k\,\tau_\omega\cdot\mathbf{a}^- \oplus k\,\tau_\omega\cdot\mathbf{a}^+$ and this space is contained in the image of the map.

□

Corollary 6.20. *Suppose $G=\mathrm{SL}_2(\mathbb{Q}_p)$ with $p\neq 2,3$ and $\pi=p$.*

i. The H_ζ-bimodules $\ker(f_1)$ and $\ker(f_2)$ are isomorphic.

ii. $\ker(f_2)$ is a free $k[\zeta^{\pm 1}]$-module of rank $4(p-1)$ on the left and on the right.

Proof. Combine Propositions 6.8 and 6.19. □

Remark 6.21. 1. It follows from $\Gamma_\varpi(\mathbf{a}^+)=\mathbf{a}^-$ (see also Remark 6.1-iv) that the diagram

$$\begin{array}{ccc} (H_\zeta/H_\zeta\tau_{s_0}\oplus H_\zeta/H_\zeta\tau_{s_1})^\pm & \xrightarrow{(114)} & \ker(f_2) \\ \Big\downarrow {\scriptstyle(\sigma^+,\sigma^-)\mapsto(\Gamma_\varpi(\sigma^-),\Gamma_\varpi(\sigma^+))} & & \Big\downarrow {\scriptstyle\Gamma_\varpi} \\ (H_\zeta/H_\zeta\tau_{s_0}\oplus H_\zeta/H_\zeta\tau_{s_1})^\pm & \xrightarrow{(114)} & \ker(f_2) \end{array}$$

is commutative.

2. It follows from (109) (see also Remark 6.1-v) that the diagram

$$\begin{array}{ccc} (H_\zeta/H_\zeta\tau_{s_0}\oplus H_\zeta/H_\zeta\tau_{s_1})^\pm & \xrightarrow{(114)} & \ker(f_2) \\ \Big\downarrow {\scriptstyle\beta\circ(\mathcal{J}\oplus\mathcal{J})} & & \Big\downarrow {\scriptstyle\mathcal{J}} \\ (H_\zeta/H_\zeta\tau_{s_0}\oplus H_\zeta/H_\zeta\tau_{s_1})^\pm & \xrightarrow{(114)} & \ker(f_2) \end{array}$$

is commutative. Compare with Remark 3.29-2. The maps in the diagram are all bijective.

7 On the left H-module $H^*(I,V)$ when $G=\mathrm{SL}_2(\mathbb{Q}_p)$ with $p\neq 2,3$ and V is of finite length

We suppose that $G=\mathrm{SL}_2(\mathbb{Q}_p)$ with $p\neq 2,3$. The goal of this section is to investigate the cohomology $H^*(I,V)=\mathrm{Ext}^*_{\mathrm{Mod}(G)}(\mathrm{X},V)$ for any finite length representation V in $\mathrm{Mod}(G)$.

Remark 7.1. Recall that our assumption on G guarantees that the pro-p Iwahori subgroup I has cohomological dimension 3. We therefore have $H^i(I,V)=0$ for $i\geq 4$ and any V in $\mathrm{Mod}(G)$.

In a **first step** we fix a nonzero polynomial $Q\in k[X]$ and consider the smooth G-representation $\mathbf{X}/\mathbf{X}Q(\zeta)$. Since H is free over $k[\zeta]$ (Lemma 2.7), right multiplication by $Q(\zeta)$ induces an injective map on $\mathbf{X}^I$ and therefore on $\mathbf{X}$. So we have the short exact sequence of smooth G-representations

$$0\to\mathbf{X}\xrightarrow{\cdot Q(\zeta)}\mathbf{X}\longrightarrow\mathbf{X}/\mathbf{X}Q(\zeta)\to 0\ .$$

Hence we obtain the long exact cohomology sequence (of H-bimodules)

$$\begin{aligned}(117)\qquad & 0\longrightarrow E^0\xrightarrow{\cdot Q(\zeta)}E^0\longrightarrow(\mathbf{X}/\mathbf{X}Q(\zeta))^I\longrightarrow E^1\xrightarrow{\cdot Q(\zeta)}E^1\longrightarrow H^1(I,\mathbf{X}/\mathbf{X}Q(\zeta))\\ &\longrightarrow E^2\xrightarrow{\cdot Q(\zeta)}E^2\longrightarrow H^2(I,\mathbf{X}/\mathbf{X}Q(\zeta))\longrightarrow E^3\xrightarrow{\cdot Q(\zeta)}E^3\longrightarrow H^3(I,\mathbf{X}/\mathbf{X}Q(\zeta))\longrightarrow 0\end{aligned}$$

and therefore the short exact sequences

$$(118)\qquad 0\to E^i/E^iQ(\zeta)\longrightarrow H^i(I,\mathbf{X}/\mathbf{X}Q(\zeta))\longrightarrow\ker(E^{i+1}\xrightarrow{\cdot Q(\zeta)}E^{i+1})\to 0\ .$$

Note that all three terms in these short exact sequences are annihilated by $Q(\zeta)$ from the right. Next we collect in the following proposition what we have proved in the previous sections about E^* as a left or a right $k[\zeta]$-module.

Proposition 7.2. *As left or right $k[\zeta]$-modules we have the following isomorphisms (for 2. and 3. we need $\pi=p$):*

1. $H\cong k[\zeta]^{4(p-1)}$;
2. $E^1\cong k[\zeta^{\pm1}]^{4(p-1)}\oplus k[\zeta]^{4(p-1)}$;
3. $E^2\cong k[\zeta^{\pm1}]^{4(p-1)}\oplus\big(k[\zeta^{\pm1}]/k[\zeta]\big)^{4(p-1)}$;
4. $E^3\cong k\oplus\big(k[\zeta^{\pm1}]/k[\zeta]\big)^{4(p-1)}$ *with ζ acting by 1 on the summand k.*

Proof. 1. See Lemma 2.7.

4. According to (22) and Prop. 2.4 we have

$$E^3\cong k\oplus\bigcup_{m\geq1}(H/\zeta^mH)^\vee\qquad\text{as } H\text{-bimodules.}$$

Using 1. we obtain

$$\bigcup_{m\geq1}(H/\zeta^mH)^\vee\cong\Big(\bigcup_{m\geq1}(k[\zeta]/\zeta^mk[\zeta])^\vee\Big)^{4(p-1)}\cong\Big(\bigcup_{m\geq1}(\frac{1}{\zeta^m}k[\zeta]/k[\zeta])^\vee\Big)^{4(p-1)}\cong\big(k[\zeta^{\pm1}]/k[\zeta]\big)^{4(p-1)}\ .$$

3. By Propositions 6.13 and 6.15 and Corollary 6.20, we have $E^2=A\oplus B$ with $A\cong k[\zeta^{\pm1}]^{4(p-1)}$ and $B\cong\bigcup_{m\geq1}(F^1H/\zeta^mF^1H)^\vee$, the latter even as an H-bimodule. But F^1H is of finite codimension

in H. Hence the elementary divisor theorem implies that also $F^1H \cong k[\zeta]^{4(p-1)}$. Therefore the same computation as in the proof of 4. above shows that $B \cong \left(k[\zeta^{\pm 1}]/k[\zeta]\right)^{4(p-1)}$.

2. According to Propositions 6.10, 6.3, and 6.8 the H-bimodule E^1 has the two sub-H-bimodules $A := \ker(f_1)$ and $B := \ker(g_1)$ which have the following properties:

a. $A \oplus B \subseteq E^1$ with $E^1/(A \oplus B)$ being 4-dimensional;

b. $A \cong k[\zeta^{\pm 1}]^{4(p-1)}$ and $B \cong F^1H \cong k[\zeta]^{4(p-1)}$ as left or as right $k[\zeta]$-modules;

c. E^1/B is a $k[\zeta^{\pm 1}]$-module;

d. ζ acts on $E^1/A \oplus B$ from the left and from the right by 1.

We give the argument for the left $k[\zeta]$-action, the other case being entirely analogous. Again the elementary divisor theorem implies that E^1/A as a $k[\zeta]$-module is of the form $E^1/A = F \oplus \bar{D}$ with F being free of rank $4(p-1)$ and $\bar{D}$ being finite dimensional. Since the natural map $\bar{D} \hookrightarrow E^1/A \oplus B$ is injective ζ must act by 1 on $\bar{D}$. Suppose that $\bar{D} = 0$. Then we have the short exact sequence $0 \to A \to E^1 \to F \to 0$ which splits since F is free. We therefore assume in the following that $\bar{D} \neq 0$, and we let $D \subset E^1$ denote the preimage of $\bar{D}$ in E^1. Then ζ acts bijectively on D which therefore is a $k[\zeta^{\pm 1}]$-module, which contains the free $k[\zeta^{\pm 1}]$-module A with a finite dimensional quotient. Applying this time the elementary divisor theorem to the $k[\zeta^{\pm 1}]$-module D we see that it must be of the form $D = F' \oplus D'$ with $F' \cong k[\zeta^{\pm 1}]^{4(p-1)}$ and finite dimensional D'. This D' then is a $k[\zeta]$-submodule of E^1 on which ζ acts by 1 so that $(\zeta - 1)D' = 0$. It therefore follows from Lemma 5.1.ii that $D' = 0$. Hence we have a short exact sequence $0 \to F' \to E^1 \to F \to 0$, which also must split. □

Lemma 7.3. *The multiplication by $Q(\zeta)$ on $k[\zeta]$ and on $k[\zeta^{\pm 1}]$ has zero kernel and a finite dimensional cokernel whereas on $k[\zeta^{\pm 1}]/k[\zeta]$ it has a finite dimensional kernel and zero cokernel.*

Proof. The only part of the statement which might not be entirely obvious is the surjectivity of the multiplication on $k[\zeta^{\pm 1}]/k[\zeta]$. This is clear if $Q(\zeta)$ is a power of ζ. We therefore assume that $Q(\zeta)$ is prime to ζ. But then $k[\zeta]/Q(\zeta)k[\zeta] = k[\zeta^{\pm 1}]/Q(\zeta)k[\zeta^{\pm 1}]$. □

In the three next statements we assume in addition that $\pi = p$.

Corollary 7.4. *The multiplication by $Q(\zeta)$ from the right on E^* has finite dimensional kernel and cokernel.*

Using (118) we deduce the following result.

Corollary 7.5. *The k-vector space $H^*(I, \mathbf{X}/\mathbf{X}Q(\zeta))$ is finite dimensional.*

Next we consider the left $k[\zeta]$-action on $H^*(I, \mathbf{X}/\mathbf{X}Q(\zeta))$. For this we introduce the polynomial $P(X) := Q(X)Q(\frac{1}{X})X^{\deg(Q)}$.

Proposition 7.6. *$H^*(I, \mathbf{X}/\mathbf{X}Q(\zeta))$ is left $P(\zeta)$-torsion.*

Proof. We start with the following observation. By Corollary 6.14, we know that for any $x \in E^*$, we have $\zeta \cdot x \cdot \zeta - x \in \ker(g)$. We deduce, for any $m \geq 0$ and $0 \leq i \leq m$, that $\zeta^m \cdot x \cdot \zeta^i \equiv \zeta^{m-i} \cdot x \bmod \ker(g)$ We choose m to be $2\deg(Q)$ which is $\geq \deg(P)$. The coefficients of the polynomial $P = \sum_{i=0}^m a_i X^i$ satisfy $a_{m-i} = a_i$ for any i. For $x \in E^*$, have

$$\begin{aligned} P(\zeta) \cdot x - \zeta^m \cdot x \cdot P(\zeta) &= \sum_{i=0}^m a_i \zeta^i \cdot x - \sum_{i=0}^m a_i \zeta^m \cdot x \cdot \zeta^i \equiv \sum_{i=0}^m a_i \zeta^i \cdot x - \sum_{i=0}^m a_i \zeta^{m-i} \cdot x \bmod \ker(g) \\ &= \sum_{i=0}^m a_i \zeta^i \cdot x - \sum_{i=0}^m a_{m-i} \zeta^{m-i} \cdot x \equiv 0 \bmod \ker(g) \ . \end{aligned} \tag{119}$$

Now we prove the proposition. Because of the exact sequences (118) it suffices to show that $E^*/E^*Q(\zeta)$ and $\ker(E^* \xrightarrow{\cdot Q(\zeta)} E^*)$ are left $P(\zeta)$-torsion. Obviously both modules are annihilated by $P(\zeta)$ from the right. That $\ker(E^* \xrightarrow{\cdot Q(\zeta)} E^*)$ is of left $P(\zeta)$-torsion follows from the above observation: suppose $x \cdot Q(\zeta) = 0$, then $x \cdot P(\zeta) = 0$ and $P(\zeta) \cdot x \in \ker(g)$ so $P(\zeta)^2 \cdot x = P(\zeta) \cdot x \cdot P(\zeta) = 0$. Now let $x \in E^*$. From (119), we deduce that $P(\zeta)^2 \cdot x - \zeta^m P(\zeta) \cdot x \cdot P(\zeta) = P(\zeta) \cdot x \cdot P(\zeta) - \zeta^m \cdot x \cdot P(\zeta)^2$ so

$$P(\zeta)^2 \cdot x = (\zeta^m P(\zeta) \cdot x + P(\zeta) \cdot x - \zeta^m \cdot x \cdot P(\zeta)) \cdot P(\zeta) \in E^* \cdot Q(\zeta).$$

This shows that $E^*/E^*Q(\zeta)$ is left $P(\zeta)$-torsion. $\square$

Remark 7.7. The formula (119) actually holds true for any nonzero polynomial $P(X) \in k[X]$ with the property that $X^m P(\frac{1}{X}) = P(X)$ for some integer $m \geq \deg(P)$. It shows that, for any $x \in E^*$ and any $j \geq 1$, we have

$$P(\zeta)^j \cdot x \equiv \zeta^{mj} \cdot x \cdot P(\zeta)^j \bmod \ker(g)$$

and symmetrically

$$x \cdot P(\zeta)^j \equiv P(\zeta)^j \cdot x \cdot \zeta^{mj} \bmod \ker(g)\ .$$

This easily implies that the multiplicative subset $\{P(\zeta)^n : n \geq 0\}$ of $H = E^0$ satisfies the left and right Ore conditions inside the full algebra E^*. Therefore the corresponding classical ring of fractions $E^*_{P(\zeta)}$ exists. This applies in particular to $P(X) = X$ so that H_ζ is part of the larger ring E^*_ζ. We will come back to these localizations elsewhere.

Lemma 7.8. *i.* $\mathrm{Mod}^I(G)$ *is closed under the formation of subrepresentations and quotient representations.*

ii. The functor $V \longrightarrow V^I$ *is exact on* $\mathrm{Mod}^I(G)$.

Proof. i. For quotient representations the assertion is obvious. For a subrepresentation U of a representation V in $\mathrm{Mod}^I(G)$ we consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{X} \otimes_H U^I & \longrightarrow & \mathbf{X} \otimes_H V^I & \longrightarrow & \mathbf{X} \otimes_H (V/U)^I & & \\ & & \downarrow & & \cong\downarrow & & \cong\downarrow & & \\ 0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & V/U & \longrightarrow & 0. \end{array}$$

The upper horizontal row is exact by the left exactness of the functor $(-)^I$ and the fact that $\mathbf{X}$ is projective as a (right) H-module (cf. the proof of [OS2] Prop. 3.25). By the equivalence of categories in §2.4.10 the middle and right perpendicular arrows are isomorphisms. Hence the left one is an isomorphism as well. This shows that U lies in $\mathrm{Mod}^I(G)$.
ii. This a consequence of the equivalence of categories in §2.4.10. $\square$

Lemma 7.9. *The* G*-representation* $\mathbf{X}/\mathbf{X}Q(\zeta)$ *is of finite length. Furthermore, the following sets of isomorphism classes of* G*-representations coincide:*

a. irreducible smooth G*-representations* V *such that* $Q(\zeta)V^I = 0$;

b. irreducible quotient representations of $\mathbf{X}/\mathbf{X}Q(\zeta)$;

c. irreducible subquotient representations of $\mathbf{X}/\mathbf{X}Q(\zeta)$.

Proof. First of all we have, by Lemma 7.8, that $(\mathbf{X}/\mathbf{X}Q(\zeta))^I = H/HQ(\zeta)$. This is finite dimensional over k by Prop. 7.2.1 and hence is an H-module of finite length. The equivalence of categories in §2.4.10 then implies that $\mathbf{X}/\mathbf{X}Q(\zeta)$ is of finite length.

Also by Lemma 7.8 the H-module V^I, for any irreducible subquotient V of $\mathbf{X}/\mathbf{X}Q(\zeta)$, is a subquotient of $H/HQ(\zeta)$ and hence satisfies $Q(\zeta)V^I = 0$. On the other hand consider any irreducible smooth G-representation V such that $Q(\zeta)V^I = 0$. By the equivalence of categories V^I is a simple H-module. We therefore have a surjection $H \twoheadrightarrow V^I$, which factors over $H/HQ(\zeta)$ and then gives rise to a surjection $\mathbf{X}/\mathbf{X}Q(\zeta) = \mathbf{X} \otimes_H H/HQ(\zeta) \twoheadrightarrow \mathbf{X} \otimes_H V^I = V$. ☐

Remark 7.10. As pointed out in the proof of the previous lemma the H-module V^I is finite dimensional for any irreducible smooth G-representation V. Hence there always is a nonzero polynomial $Q \in k[X]$ such that $Q(\zeta)V^I = 0$.

Combining all of the above we may now establish in a **second step** our main result.

Theorem 7.11. *Let $G = \mathrm{SL}_2(\mathbb{Q}_p)$ with $p \neq 2, 3$. For any representation V of finite length in $\mathrm{Mod}(G)$ we have:*

i. The k-vector space $H^*(I, V)$ is finite dimensional;

ii. Assume $\pi = p$. If V lies in $\mathrm{Mod}^I(G)$ and $Q(\zeta)V^I = 0$ for some nonzero polynomial $Q \in k[X]$, then the left H-module $H^*(I, V)$ is $P(\zeta)$-torsion for the polynomial $P(X) := Q(X)Q(\frac{1}{X})X^{\deg(Q)}$.

Proof. Let $Q(\zeta) \in k[\zeta] \setminus \{0\}$ and U a subquotient representation of $\mathbf{X}/\mathbf{X}Q(\zeta)$. We show by downwards induction w.r.t. the cohomology degree $i = 3, \ldots, 0$ that $H^i(I, U)$ is a finite dimensional left H-module which, when $\pi = p$, is of $P(\zeta)$-torsion.

• Here $i = 3$. First assume that U is irreducible. According to Lemma 7.9 we have a surjection $\mathbf{X}/\mathbf{X}Q(\zeta) \twoheadrightarrow U$. Because of the bound 3 for the cohomological dimension of I this surjection induces a surjection $H^3(I, \mathbf{X}/\mathbf{X}Q(\zeta)) \twoheadrightarrow H^3(I, U)$. By Cor. 7.5 and Prop. 7.6, the left H-module $H^3(I, U)$ is finite dimensional and, when $\pi = p$, of $P(\zeta)$-torsion. By another induction it is easy to see that the result still holds when U is a subquotient representation of $\mathbf{X}/\mathbf{X}Q(\zeta)$.

• Assume the statement is true at rank i for $1 \leq i \leq 3$.

Consider again first the case of an irreducible subquotient of $\mathbf{X}/\mathbf{X}Q(\zeta)$. We call it V and write it as part of an exact sequence $0 \to U \to \mathbf{X}/\mathbf{X}Q(\zeta) \to V \to 0$, which gives rise to an exact sequence of H-modules

$$H^{i-1}(I, \mathbf{X}/\mathbf{X}Q(\zeta)) \to H^{i-1}(I, V) \to H^i(I, U) \ .$$

By induction and by Cor. 7.5 and Prop. 7.6, it follows that $H^{i-1}(I, V)$ is finite dimensional and $P(\zeta)$-torsion when $\pi = p$. As above it is then easy to see that the result still holds when V is any subquotient representation of $\mathbf{X}/\mathbf{X}Q(\zeta)$.

Now we turn to the proof of the assertion of the Theorem. By a straightforward induction using the long exact cohomology sequence as well as Lemma 7.8 (for ii.) we may assume that V is irreducible. According to Remark 7.10 and Lemma 7.9, there is a nonzero polynomial $Q(\zeta)$ and a surjection $\mathbf{X}/\mathbf{X}Q(\zeta) \twoheadrightarrow V$. So V is a quotient of $\mathbf{X}/\mathbf{X}Q(\zeta)$ and the above result applies.

☐

Over an algebraically closed field k we refer to [OV] §5 for the notion of an irreducible admissible supercuspidal representation. Note that for our group G every irreducible representation is admissible as a consequence of the equivalence of categories in §2.4.10. We extend this notion as follows to arbitrary k. Let V be an irreducible representation in $\mathrm{Mod}(G)$. By this equivalence of categories V^I is a finite dimensional H-module. Hence, if $\bar{k}$ denotes an algebraic closure of k, the base extension $\bar{k} \otimes_k V$ is still generated by its I-fixed vectors and $(\bar{k} \otimes_k V)^I = \bar{k} \otimes_K V^I$ is a finite dimensional $\bar{k} \otimes_k H$-module. The equivalence of categories over $\bar{k}$ therefore implies that $\bar{k} \otimes_k V$ is a representation of finite length of G over $\bar{k}$. We will call V *supersingular* if all irreducible constituents of $\bar{k} \otimes_k V$ are supersingular in the sense of [OV] §5.

Corollary 7.12. *Let $G = \mathrm{SL}_2(\mathbb{Q}_p)$ with $p \neq 2, 3$. An irreducible representation V in* $\mathrm{Mod}(G)$ *is supersingular if and only if the left H-module $H^*(I, V)$ is supersingular.*

Proof. It is shown in [OV] Thm. 5.3 that, when k is algebraically closed, an irreducible (admissible) representation V_0 in $\mathrm{Mod}(G)$ is supersingular if and only V_0^I is ζ-torsion, namely if and only V_0^I is supersingular. Hence V is supersingular if and only if V_0^I is ζ-torsion for all irreducible constituents V_0 of $\bar{k} \otimes_k V$. By Lemma 7.8 the latter is equivalent to $(\bar{k} \otimes_k V)^I$ being ζ-torsion hence to V^I being ζ-torsion, i.e., being supersingular (see §2.4.5). But by the equivalence of categories in §2.4.10 the H-module V^I is simple. If it is ζ-torsion it must satisfy $\zeta V^I = 0$. So we apply Thm. 7.11.ii with $Q := X$ to see that then all of $H^*(I, V)$ is ζ-torsion and hence supersingular. □

We remind the reader that in Prop. 2.20 we had determined for which irreducible representations V the top cohomology $H^d(I, V)$ vanishes.

8 The commutator in E^* of the center of H when $G = \mathrm{SL}_2(\mathbb{Q}_p)$, $p \neq 2, 3$

We assume in this section that $G = \mathrm{SL}_2(\mathbb{Q}_p)$, $p \neq 2, 3$ and $\pi = p$. Recall that we denote by Z the center of H. In this section we consider the subalgebra

$$\mathcal{C}_{E^*}(Z) = \{\mathscr{E} \in E^*,\ z \cdot \mathscr{E} = \mathscr{E} \cdot z \quad \forall z \in Z\}$$

of E^*. We are going to describe the product in this algebra. We denote by $\mathcal{C}_{E^i}(Z)$ its i^{th} graded piece.

Proposition 8.1. *$\mathcal{C}_{E^*}(Z)$ coincides with the commutator of ζ in E^*, namely with* $\ker(g)$*:*

$$\mathcal{C}_{E^*}(Z) = \{\mathscr{E} \in E^*,\ \zeta \cdot \mathscr{E} = \mathscr{E} \cdot \zeta\}.$$

Proof. As H-bimodules, we have

$$\ker(g_0) \cong H,\ \ker(g_1) \cong F^1H,\ \text{and}\ \ker(g_2) \cong (F^1H)^{\vee,f} \cong \bigcup_{n \geq 1} (F^1H/\zeta^n F^1H)^\vee$$

(see Propositions 6.3 and 6.15). So these spaces are contained in $\mathcal{C}_{E^*}(Z)$. Lastly we explained in Remark 2.21 (see also §5**B)**) that the elements of Z centralize the elements of $E^3 = \ker(g_3)$. □

We recall some notations and results from §2.4.9, §6.2.1 and §6.3.1:

- $\mathcal{C}_{E^0}(Z) = H$,
- We have an isomorphism of H-bimodules $f_{(\mathbf{x}_0,\mathbf{x}_1)} : F^1H \longrightarrow \mathcal{C}_{E^1}(Z)$. We keep track of its inverse

(120)
$$f^{-1}_{(\mathbf{x}_0,\mathbf{x}_1)} : \mathcal{C}_{E^1}(Z) \xrightarrow{\simeq} F^1H\ ,\ .$$

- We have an isomorphism of H-bimodules (see (107))

(121)
$$\mathcal{C}_{E^2}(Z) \xrightarrow{\simeq} {}^{\jmath}((F^1H)^{\vee,f})^{\jmath}$$

and we denote by $\alpha^\star_w$ the preimage of $\tau^\vee_w|_{F^1H}$ by this map for $w \in \widetilde{W}$, $\ell(w) \geq 1$. The set of all these $\alpha^\star_w s$ forms a basis of $\mathcal{C}_{E^2}(Z)$.

- $\mathcal{C}_{E^3}(Z) \cong {}^{\jmath}(H^{\vee,f})^{\jmath}$ as H-bimodules. As in §2.4.9, the element in E^3 corresponding to $\tau_w^\vee$ is denoted by ϕ_w.

Remark 8.2. Let $w \in \widetilde{W}$ with $\ell(w) \geq 1$, $\omega \in \Omega$. Using formulas (46), we obtain immediately

$$\tau_\omega \cdot \alpha_w^\star = \alpha_{\omega w}^\star$$

$$\tau_{s_\epsilon} \cdot \alpha_w^\star = \begin{cases} 0 & \text{if } w \in \widetilde{W}^\epsilon \text{ with } \ell(w) \geq 1, \\ -e_1 \cdot \alpha_w^\star + \alpha_{s_0 w}^\star & \text{if } w \in \widetilde{W}^{1-\epsilon} \text{ with } \ell(w) \geq 2, \\ -e_1 \cdot \alpha_w^\star & \text{if } w \in \widetilde{W}^{1-\epsilon} \text{ with } \ell(w) = 1. \end{cases}$$

$$\zeta \cdot \alpha_w^\star = \begin{cases} 0 & \text{if } \ell(w) \leq 2, \\ \alpha_{s_\epsilon s_{1-\epsilon} w}^\star & \text{if } w \in \widetilde{W}^\epsilon \text{ with } \ell(w) \geq 3. \end{cases}$$

Remark 8.3. i. We have $\alpha_w^\star \cup f_{(\mathbf{x}_0,\mathbf{x}_1)}^{-1}(\tau_w) = \delta_{v,w}\phi_w$ for all $v, w \in \widetilde{W}$ with $\ell(v), \ell(w) \geq 1$.

- In particular, using (50) and Proposition 3.18-v we see that the image of $\alpha_w^\star$ by conjugation by ϖ is $\alpha_{\varpi w \varpi^{-1}}^\star$.
- Using Proposition 3.18-iv and recalling by [OS3] (89) (8.2) that $\jmath(\phi_w) = \phi_{w^{-1}}$, we deduce (see also [OS3] Rmk. 6.2) that $\jmath(\alpha_w^\star) = -\alpha_{w^{-1}}^\star$.

ii. Recall that the element $\alpha^0 \in 1 + p\mathbb{Z}_p/1 + p^2\mathbb{Z}_p$ was chosen in (60). For $w \in \widetilde{W}$ with $\ell(w) \geq 1$, there is a unique element in $\ker(g_2)$ which, when seen as a linear form in $(E^1)^{\vee,f}$, coincides with $(0, \alpha^0, 0)_w$ if $w \in \widetilde{W}^0$ (resp. $-(0, \alpha^0, 0)_w$ if $w \in \widetilde{W}^1$) on $\ker(g_1)$ (see Lemma 6.12 and Proposition 6.13). By Proposition 3.18-i, this element is $\alpha_w^\star$. By definition, it is zero on $\ker(f_1)$.

When $w \in \widetilde{W}^0$, the element $\alpha_w^\star - (0, \alpha^0, 0)_w$ is an element of $\ker(f_2)$ which coincides with $-(0, \alpha^0, 0)_w$ on $\ker(f_1)$. But Remark 6.9 implies that $(1 - e_{\gamma_0}) \cdot (0, \alpha^0, 0)_w$ is trivial on $\ker(f_1)$. Therefore, and by conjugation by ϖ (Lemma 4.2),

$$\begin{aligned} \alpha_w^\star - (0, \alpha^0, 0)_w &\in e_{\gamma_0} \cdot \ker(f_2) \text{ if } w \in \widetilde{W}^0 \\ \alpha_w^\star + (0, \alpha^0, 0)_w &\in e_{\gamma_0} \cdot \ker(f_2) \text{ if } w \in \widetilde{W}^1 \ . \end{aligned} \tag{122}$$

8.1 The product $(\mathcal{C}_{E^1}(Z), \mathcal{C}_{E^1}(Z)) \to \mathcal{C}_{E^2}(Z)$

Recall using (47) that we have a homomorphism of H-bimodules

$$\begin{aligned} F^1H &\longrightarrow {}^{\jmath}((F^1H/F^2H)^\vee)^{\jmath} \\ \tau_w &\longmapsto \begin{cases} -\tau_w^\vee|_{F^1H} & \text{if } \ell(w) = 1, \\ 0 & \text{if } \ell(w) \geq 2 \end{cases} \end{aligned} \tag{123}$$

which is trivial on F^2H. Identifying $(F^1H/F^2H)^\vee$ with the sub-H-bimodule of the linear forms in $(F^1H)^{\vee,f}$ which are trivial on F^2H, we obtain a homomorphism of H-bimodules

$$\begin{aligned} F^1H \otimes_H F^1H &\longrightarrow {}^{\jmath}((F^1H/F^2H)^\vee)^{\jmath} \hookrightarrow {}^{\jmath}((F^1H)^{\vee,f})^{\jmath} \\ \tau_v \otimes \tau_w &\longrightarrow \begin{cases} -\tau_v \cdot \tau_w^\vee|_{F^1H} & \text{if } \ell(w) = 1 \\ 0 & \text{if } \ell(w) \geq 2 \ . \end{cases} \end{aligned} \tag{124}$$

Remark 8.4. Let $v, w \in \widetilde{W}$ with length ≥ 1, $\omega, \omega' \in \Omega$ and $\epsilon \in \{0,1\}$. Using (46), we see that the map above has the following outputs:

$$\tau_{\omega s_\epsilon} \otimes \tau_{\omega' s_\epsilon} \longmapsto e_1 \cdot \tau_{s_\epsilon}^\vee|_{F^1 H} = \tau_{s_\epsilon}^\vee|_{F^1 H} \cdot e_1$$
$$\tau_{\omega s_\epsilon} \otimes \tau_{\omega' s_{1-\epsilon}} \longmapsto 0$$
and
$$\tau_v \otimes \tau_w \longmapsto 0 \text{ if } \ell(v) \geq 2 \text{ or } \ell(w) \geq 2.$$

We see that (124) is a symmetric bilinear map onto a 2-dimensional k-vector space.

Proposition 8.5. *Assume that $G = \mathrm{SL}_2(\mathbb{Q}_p)$, $p \neq 2, 3$ and $\pi = p$. We have a commutative diagram of H-bimodules*

(125)
$$\begin{array}{ccc} \mathcal{C}_{E^1}(Z) \otimes_H \mathcal{C}_{E^1}(Z) & \xrightarrow{\textit{Yoneda product}} & \mathcal{C}_{E^2}(Z) \\ {\scriptstyle (120)\otimes(120)}\downarrow \cong & & {\scriptstyle (121)}\downarrow \cong \\ F^1H \otimes_H F^1H & \xrightarrow{(124)} & {}^{\jmath}((F^1H)^{\vee,f})^{\jmath} \end{array}$$

Proof. Because of the isomorphism (120), the H-bimodule $\mathcal{C}_{E^1}(Z) \otimes_H \mathcal{C}_{E^1}(Z)$ is generated by the elements of the form $f^{-1}_{(\mathbf{x}_0,\mathbf{x}_1)}(\tau_{s_\epsilon}) \otimes f^{-1}_{(\mathbf{x}_0,\mathbf{x}_1)}(\tau_{s'_\epsilon}) = \mathbf{x}_\epsilon \otimes \mathbf{x}_{\epsilon'}$ for $\epsilon, \epsilon' \in \{0,1\}$. Therefore, using Remark 8.2, it is enough to prove that

$$\mathbf{x}_\epsilon \cdot \mathbf{x}_{1-\epsilon} = 0 \qquad \text{and} \qquad \mathbf{x}_\epsilon \cdot \mathbf{x}_\epsilon = e_1 \cdot \alpha^\star_{s_\epsilon} \ .$$

We verify these identities now. In the calculations below, we use formulas (67), (69), (70) the definition of the idempotents (36), Proposition 3.9, Lemma 3.12-i and Proposition 2.1.

- First we check that

$$\begin{aligned}
\mathbf{x}_0 \cdot \mathbf{x}_1 &= -\left((0,\mathbf{c}^0,0)_{s_0} + e_{\mathrm{id}^{-1}} \cdot (\mathbf{c}^0\iota^{-1},0,0)_1\right) \cdot \left((0,\mathbf{c}^0,0)_{s_1} - (0,0,\mathbf{c}^0\iota^{-1})_1 \cdot e_{\mathrm{id}^{-1}}\right) \\
&= -(0,\mathbf{c}^0,0)_{s_0} \cdot (0,\mathbf{c}^0,0)_{s_1} + (0,\mathbf{c}^0,0)_{s_0} \cdot (0,0,\mathbf{c}^0\iota^{-1})_1 \cdot e_{\mathrm{id}^{-1}} \\
&\quad - e_{\mathrm{id}^{-1}} \cdot (\mathbf{c}^0\iota^{-1},0,0)_1 \cdot (0,\mathbf{c}^0,0)_{s_1} + e_{\mathrm{id}^{-1}} \cdot (\mathbf{c}^0\iota^{-1},0,0)_1 \cdot (0,0,\mathbf{c}^0\iota^{-1})_1 \cdot e_{\mathrm{id}^{-1}} \\
&= -\left((0,\mathbf{c}^0,0)_{s_0} \cdot \tau_{s_1} \cup \tau_{s_0} \cdot (0,\mathbf{c}^0,0)_{s_1}\right) + \left((0,\mathbf{c}^0,0)_{s_0} \cup \tau_{s_0} \cdot (0,0,\mathbf{c}^0\iota^{-1})_1\right) \cdot e_{\mathrm{id}^{-1}} \\
&\quad - e_{\mathrm{id}^{-1}} \cdot \left((\mathbf{c}^0\iota^{-1},0,0)_1 \cdot \tau_{s_1} \cup (0,\mathbf{c}^0,0)_{s_1}\right) + e_{\mathrm{id}^{-1}} \cdot \left((\mathbf{c}^0\iota^{-1},0,0)_1 \cup (0,0,\mathbf{c}^0\iota^{-1})_1\right) \cdot e_{\mathrm{id}^{-1}} \\
&= (0,\mathbf{c}^0,0)_{s_0 s_1} \cup (0,\mathbf{c}^0,0)_{s_0 s_1} + e_{\mathrm{id}^{-1}} \cdot \left((\mathbf{c}^0\iota^{-1},0,0)_1 \cup (0,0,\mathbf{c}^0\iota^{-1})_1\right) \cdot e_{\mathrm{id}^{-1}} \\
&= 0 \text{ by Example 3.6.}
\end{aligned}$$

  Likewise, by conjugation by ϖ (see Proposition 3.18-v) we have $\mathbf{x}_1 \cdot \mathbf{x}_0 = 0$.

- Next we compute

$$\begin{aligned}
\mathbf{x}_0 \cdot \mathbf{x}_0 &= [(0,\mathbf{c}^0,0)_{s_0} + e_{\mathrm{id}^{-1}} \cdot (\mathbf{c}^0\iota^{-1},0,0)_1] \cdot [(0,\mathbf{c}^0,0)_{s_0} + e_{\mathrm{id}^{-1}} \cdot (\mathbf{c}^0\iota^{-1},0,0)_1] \\
&= (0,\mathbf{c}^0,0)_{s_0} \cdot (0,\mathbf{c}^0,0)_{s_0} + \\
&\qquad (0,\mathbf{c}^0,0)_{s_0} \cdot (\mathbf{c}^0\iota^{-1},0,0)_1 \cdot e_{\mathrm{id}} + e_{\mathrm{id}^{-1}} \cdot (\mathbf{c}^0\iota^{-1},0,0)_1 \cdot (0,\mathbf{c}^0,0)_{s_0} \qquad \text{(using (70))} \\
&= (0,\mathbf{c}^0,0)_{s_0} \cdot (0,\mathbf{c}^0,0)_{s_0} - \sum_{u\in\mathbb{F}_p^\times} \big(u^{-1}[(0,\mathbf{c}^0,0)_{s_0} \cdot \tau_{\omega_u} \cup \tau_{s_0} \cdot (\mathbf{c}^0\iota^{-1},0,0)_{\omega_u}] \\
&\qquad\qquad - u^{-1}[(\mathbf{c}^0\iota^{-1},0,0)_{\omega_u} \cdot \tau_{s_0} \cup \tau_{\omega_u}(0,\mathbf{c}^0,0)_{s_0}]\big) \\
&= (0,\mathbf{c}^0,0)_{s_0} \cdot (0,\mathbf{c}^0,0)_{s_0} + \sum_{u\in\mathbb{F}_p^\times} u^{-1}[(0,\mathbf{c}^0,0)_{s_0\omega_u} \cup (0,0,\mathbf{c}^0\iota^{-1})_{s_0\omega_u}] \\
&\qquad\qquad - \sum_{u\in\mathbb{F}_p^\times} u[(\mathbf{c}^0\iota^{-1},0,0)_{s_0\omega_u} \cup (0,\mathbf{c}^0,0)_{s_0\omega_u}] \ .
\end{aligned}$$

But by (11), there exists $\gamma_{s_0^2} \in H^2(I,\mathbf{X}(s_0^2))$ such that (see Lemma 3.12-ii)

$$\begin{aligned}
(0,\mathbf{c}^0,0)_{s_0} \cdot (0,\mathbf{c}^0,0)_{s_0} &= [(0,\mathbf{c}^0,0)_{s_0} \cdot \tau_{s_0} \cup \tau_{s_0} \cdot (0,\mathbf{c}^0,0)_{s_0}] + \gamma_{s_0^2} \\
&= [(-e_1 \cdot (0,\mathbf{c}^0,0)_{s_0} - e_{\mathrm{id}^{-1}} \cdot (\mathbf{c}^0\iota^{-1},0,0)_{s_0}) \\
&\qquad \cup ((0,-\mathbf{c}^0,0)_{s_0} \cdot e_1 + (0,0,\mathbf{c}^0\iota^{-1})_{s_0} \cdot e_{\mathrm{id}})] + \gamma_{s_0^2} \\
&= -[(e_1 \cdot (0,\mathbf{c}^0,0)_{s_0}) \cup ((0,0,\mathbf{c}^0\iota^{-1})_{s_0} \cdot e_{\mathrm{id}})] \\
&\qquad + [(e_{\mathrm{id}^{-1}} \cdot (\mathbf{c}^0\iota^{-1},0,0)_{s_0}) \cup ((0,\mathbf{c}^0,0)_{s_0} \cdot e_1)] \\
&\qquad - [(e_{\mathrm{id}^{-1}} \cdot (\mathbf{c}^0\iota^{-1},0,0)_{s_0}) \cup ((0,0,\mathbf{c}^0\iota^{-1})_{s_0} \cdot e_{\mathrm{id}})] + \gamma_{s_0^2} \\
&= -[(\sum_{u\in\mathbb{F}_p^\times} (0,\mathbf{c}^0,0)_{\omega_u s_0}) \cup (\sum_{v\in\mathbb{F}_p^\times} v^{-1}(0,0,\mathbf{c}^0\iota^{-1})_{s_0\omega_v})] \\
&\qquad + [(\sum_{u\in\mathbb{F}_p^\times} u^{-1}(\mathbf{c}^0\iota^{-1},0,0)_{\omega_u s_0}) \cup (\sum_{v\in\mathbb{F}_p^\times} (0,\mathbf{c}^0,0)_{s_0\omega_v})] \\
&\qquad - [(e_{\mathrm{id}^{-1}} \cdot (\mathbf{c}^0\iota^{-1},0,0)_{s_0}) \cup ((0,0,\mathbf{c}^0\iota^{-1})_{s_0} \cdot e_{\mathrm{id}})] + \gamma_{s_0^2} \\
&= -\sum_{u\in\mathbb{F}_p^\times} u^{-1}(0,\mathbf{c}^0,0)_{s_0\omega_u} \cup (0,0,\mathbf{c}^0\iota^{-1})_{s_0\omega_u} \\
&\qquad + \sum_{u\in\mathbb{F}_p^\times} u(\mathbf{c}^0\iota^{-1},0,0)_{s_0\omega_u} \cup (0,\mathbf{c}^0,0)_{s_0\omega_u} \\
&\qquad - [(e_{\mathrm{id}^{-1}} \cdot (\mathbf{c}^0\iota^{-1},0,0)_{s_0}) \cup ((0,0,\mathbf{c}^0\iota^{-1})_{s_0} \cdot e_{\mathrm{id}})] + \gamma_{s_0^2} \ .
\end{aligned}$$

So

$$\mathbf{x}_0 \cdot \mathbf{x}_0 = -[(e_{\mathrm{id}^{-1}} \cdot (\mathbf{c}^0\iota^{-1},0,0)_{s_0}) \cup ((0,0,\mathbf{c}^0\iota^{-1})_{s_0} \cdot e_{\mathrm{id}})] + \gamma_{s_0^2} \ .$$

Compute that

$$
\begin{aligned}
&(e_{\mathrm{id}^{-1}} \cdot (\mathbf{c}^0\iota^{-1}, 0, 0)_{s_0}) \cup ((0, 0, \mathbf{c}^0\iota^{-1})_{s_0} \cdot e_{\mathrm{id}}) \\
&\qquad = (\sum_{u\in\mathbb{F}_p^\times} u^{-1}(\mathbf{c}^0\iota^{-1}, 0, 0)_{\omega_u s_0}) \cup (\sum_{v\in\mathbb{F}_p^\times} v^{-1}(0, 0, \mathbf{c}^0\iota^{-1})_{s_0\omega_v}) \\
&\qquad = (\sum_{u\in\mathbb{F}_p^\times} u^{-1}(\mathbf{c}^0\iota^{-1}, 0, 0)_{\omega_u s_0}) \cup (\sum_{v\in\mathbb{F}_p^\times} v(0, 0, \mathbf{c}^0\iota^{-1})_{\omega_v s_0}) \\
&\qquad = \sum_{u\in\mathbb{F}_p^\times} (\mathbf{c}^0\iota^{-1}, 0, 0)_{\omega_u s_0} \cup (0, 0, \mathbf{c}^0\iota^{-1})_{\omega_u s_0} = -\sum_{u\in\mathbb{F}_p^\times} (0, \boldsymbol{\alpha}^0, 0)_{\omega_u s_0} \qquad \text{by (90)} \\
&\qquad = e_1 \cdot (0, \boldsymbol{\alpha}^0, 0)_{s_0} = -e_1 \cdot \boldsymbol{\alpha}^\star_{s_0} \qquad \text{by (122)}\ .
\end{aligned}
$$

Since $\mathbf{x}_0$ and $\boldsymbol{\alpha}^\star_{s_0}$ both lie in the kernel of the left action of $(\tau_{s_0}+e_1)$ (Remark 8.2) we obtain directly, using the formulas of Prop. 4.5, that $\gamma_{s_0^2} = 0$. So as expected $\mathbf{x}_0 \cdot \mathbf{x}_0 = e_1 \cdot \boldsymbol{\alpha}^\star_{s_0}$. The same result is valid with s_1 instead of s_0 by conjugation by ϖ (Remark 8.3 and proof of Proposition 3.18-v which says that $\Gamma_\omega(\mathbf{x}_0) = \mathbf{x}_1$).

□

8.2 The products $(\mathcal{C}_{E^i}(Z), \mathcal{C}_{E^{3-i}}(Z)) \to \mathcal{C}_{E^3}(Z)$ for $i = 1, 2$

For $\tau \in F^1H$, we have the homomorphisms of left, resp. right, H-modules

$$L_\tau : {}^{\mathscr{J}}H^{\mathscr{J}} \to {}^{\mathscr{J}}(F^1H)^{\mathscr{J}}, \quad h \mapsto h \cdot \tau = \mathscr{J}(\tau)h \quad \text{and} \quad R_\tau : {}^{\mathscr{J}}H^{\mathscr{J}} \to {}^{\mathscr{J}}(F^1H)^{\mathscr{J}}, \quad h \mapsto \tau \cdot h = h\mathscr{J}(\tau)$$

which by pullback give homomorphisms of right, resp. left, H-modules

$$L^*_\tau : {}^{\mathscr{J}}((F^1H)^\vee)^{\mathscr{J}} \to {}^{\mathscr{J}}(H^\vee)^{\mathscr{J}}, \quad \alpha \mapsto \alpha \circ L_\tau \quad \text{and} \quad R^*_\tau : {}^{\mathscr{J}}((F^1H)^\vee)^{\mathscr{J}} \to {}^{\mathscr{J}}(H^\vee)^{\mathscr{J}}, \quad \alpha \mapsto \alpha \circ R_\tau$$

such that $L^*_{x\tau y}(\alpha) = x \cdot (L^*_\tau(y \cdot \alpha))$ and $R^*_{x\tau y}(\alpha) = (R^*_\tau(\alpha \cdot x)) \cdot y$ for $x, y \in H$ and $\alpha \in {}^{\mathscr{J}}((F^1H)^\vee)^{\mathscr{J}}$. We therefore have natural homomorphisms of H-bimodules

$$
\begin{array}{ccc}
F^1H \otimes_H {}^{\mathscr{J}}((F^1H)^\vee)^{\mathscr{J}} & \longrightarrow & {}^{\mathscr{J}}(H^\vee)^{\mathscr{J}} \\
\tau \otimes \alpha & \longmapsto & -L^*_\tau(\alpha) = -\alpha(\mathscr{J}(\tau)_-)
\end{array}
$$

$$
\begin{array}{ccc}
{}^{\mathscr{J}}((F^1H)^\vee)^{\mathscr{J}} \otimes F^1H & \longrightarrow & {}^{\mathscr{J}}(H^\vee)^{\mathscr{J}} \\
\alpha \otimes \tau & \longmapsto & -R^*_\tau(\alpha) = -\alpha(_-\mathscr{J}(\tau))
\end{array}
$$

which respectively induce homomorphisms of H-bimodules

(126) $$F^1H \otimes_H {}^{\mathscr{J}}((F^1H)^{\vee,f})^{\mathscr{J}} \longrightarrow {}^{\mathscr{J}}(H^{\vee,f})^{\mathscr{J}}$$

(127) $${}^{\mathscr{J}}((F^1H)^{\vee,f})^{\mathscr{J}} \otimes_H F^1H \longrightarrow {}^{\mathscr{J}}(H^{\vee,f})^{\mathscr{J}}\ .$$

Proposition 8.6. *Assume that $G = \mathrm{SL}_2(\mathbb{Q}_p)$, $p \neq 2, 3$ and $\pi = p$. We have commutative diagrams of H-bimodules*

(128) $$
\begin{array}{ccc}
\mathcal{C}_{E^1}(Z) \otimes_H \mathcal{C}_{E^2}(Z) & \xrightarrow{\text{Yoneda product}} & \mathcal{C}_{E^3}(Z) = E^3 \\
{\scriptstyle (120)\otimes(121)}\downarrow\cong & & \cong\downarrow{\scriptstyle \Delta^3 \text{ (see (14))}} \\
F^1H \otimes_H {}^{\mathscr{J}}((F^1H)^{\vee,f})^{\mathscr{J}} & \xrightarrow{(126)} & {}^{\mathscr{J}}(H^{\vee,f})^{\mathscr{J}}
\end{array}
$$

$$\begin{array}{ccc} \mathcal{C}_{E^2}(Z)\otimes_H \mathcal{C}_{E^1}(Z) & \xrightarrow{\textit{Yoneda product}} & \mathcal{C}_{E^3}(Z)=E^3 \\ (121)\otimes(120)\Big\downarrow\cong & & \cong\Big\downarrow \Delta^3\,\textit{(see (14))} \\ {}^{\mathcal{J}}((F^1H)^{\vee,f})^{\mathcal{J}}\otimes_H F^1H & \xrightarrow{(127)} & {}^{\mathcal{J}}(H^{\vee,f})^{\mathcal{J}} \end{array} \tag{129}$$

Both these Yoneda product maps have image $\ker(\mathcal{S}^3)$*, namely the space of* ζ*-torsion in* E^3.

Proof. Preliminary observations:

A) For $s\in\{s_0,s_1\}$ and $w\in\widetilde{W}$, $\ell(w)\geq 1$, the map (126) sends $\tau_s\otimes\tau_w^\vee|_{F^1H}$ to $-\tau_s\cdot\tau_w^\vee\in{}^{\mathcal{J}}(H^{\vee,f})^{\mathcal{J}}$. and (127) sends $\tau_w^\vee|_{F^1H}\otimes\tau_s$ to $-\tau_w^\vee\cdot\tau_s\in{}^{\mathcal{J}}(H^{\vee,f})^{\mathcal{J}}$.

B) By Remark 6.1-iii, we have $\ker(g_1)\cdot\ker(f_2)\subseteq\ker(f_3)$ and likewise $\ker(f_2)\cdot\ker(g_1)\subseteq\ker(f_3)$. But $\ker(f_3)$ is a one dimensional vector space with basis $e_1\cdot\phi_1$ and supporting the character χ_{triv} of H (Lemma 6.2). Therefore, $e_\lambda\cdot\ker(g_1)\cdot\ker(f_2)=\{0\}$ and $e_\lambda\cdot\ker(f_2)\cdot\ker(g_1)=\{0\}$ for any $\lambda\neq 1$.

We now turn to the proof of the commutativity of the diagrams. The left H-module $\mathcal{C}_{E^1}(Z)$ is generated by $\mathbf{x_0}$ and $\mathbf{x}_1$. Hence, and observation A) and (46) above, it is enough to prove, for $\epsilon\in\{0,1\}$ and $w\in\widetilde{W}$, $\ell(w)\geq 1$:

$$\mathbf{x}_\epsilon\cdot\alpha_w^\star=-\tau_{s_\epsilon}\cdot\phi_w\ =\begin{cases}-\phi_{s_\epsilon w}+e_1\cdot\phi_w & \text{if } w\in\widetilde{W}^{1-\epsilon}\\ 0 & \text{if } w\in\widetilde{W}^{\epsilon}\end{cases}$$

$$\alpha_w^\star\cdot\mathbf{x}_\epsilon=-\phi_w\cdot\tau_{s_\epsilon}=\begin{cases}-\phi_{ws_\epsilon}+e_1\cdot\phi_w & \text{if } w^{-1}\in\widetilde{W}^{1-\epsilon}\\ 0 & \text{if } w^{-1}\in\widetilde{W}^{\epsilon}\end{cases}.$$

Using Remark 2.16, these identities show that the Yoneda product maps have image $\ker(\mathcal{S}^3)$.

By the proof of Proposition 3.18-iv, we know that $\mathcal{J}(\mathbf{x}_\epsilon)=-\tau_{s_\epsilon^2}\cdot\mathbf{x}_\epsilon$ and this is equal to $-\mathbf{x}_\epsilon\cdot\tau_{s_\epsilon^2}$ (since $f_{(\mathbf{x}_0,\mathbf{x}_1)}$) is a homomorphism of H-bimodules). By Remark 8.3-i we have, that $\mathcal{J}(\alpha_w^\star)=-\alpha_{w^{-1}}^\star$. Lastly, $\mathcal{J}(\phi_w)=\phi_{w^{-1}}$ by [OS3] (8.2). Since $\mathcal{J}$ is an anti-involution of the graded algebra E^*, it is therefore enough to prove the first identity above (namely we focus on the commutativity of (128)).

- Suppose $w\in\widetilde{W}^\epsilon$ with $\ell(w)\geq 1$. Then by Remark 8.2 we have $\alpha_w^\star=(\tau_{s_\epsilon}+e_1)\cdot\alpha^\star_{s_\epsilon^{-1}w}$. But $\mathbf{x}_\epsilon\cdot(\tau_{s_\epsilon}+e_1)=0$. Therefore $\mathbf{x}_\epsilon\cdot\alpha_w^\star=0$.
- Suppose $w\in\widetilde{W}^{1-\epsilon}$ with $\ell(w)\geq 1$. We know from (122) that

  $$\begin{cases}\alpha_w^\star\in-(0,\alpha^0,0)_w+e_{\gamma_0}\cdot\ker(f_2) & \text{if } \epsilon=0\\ \alpha_w^\star\in(0,\alpha^0,0)_w+e_{\gamma_0}\cdot\ker(f_2) & \text{if } \epsilon=1\end{cases}$$

  so by observation B) above, we have

  $$\mathbf{x}_\epsilon\cdot\alpha_w^\star=\begin{cases}-\mathbf{x}_\epsilon\cdot(0,\alpha^0,0)_w & \text{if } \epsilon=0\\ \mathbf{x}_\epsilon\cdot(0,\alpha^0,0)_w & \text{if } \epsilon=1.\end{cases}$$

  Therefore, when $\epsilon=0$ we compute, using Proposition 2.1 and Lemma 3.12-i,

  $$\begin{aligned}\mathbf{x}_0\cdot\alpha_w^\star=&((0,\mathbf{c}^0,0)_{s_0}+e_{\mathrm{id}^{-1}}\cdot(\mathbf{c}^0\iota^{-1},0,0)_1)\cdot(0,\alpha^0,0)_w\\ =&(0,\mathbf{c}^0,0)_{s_0}\cdot(0,\alpha^0,0)_w+e_{\mathrm{id}^{-1}}\cdot[(\mathbf{c}^0\iota^{-1},0,0)_1\cdot\tau_w\cup(0,\alpha^0,0)_w]\\ =&(0,\mathbf{c}^0,0)_{s_0}\cdot(0,\alpha^0,0)_w+e_{\mathrm{id}^{-1}}\cdot[(\mathbf{c}^0\iota^{-1},0,0)_w\cup(0,\alpha^0,0)_w]\\ =&(0,\mathbf{c}^0,0)_{s_0}\cdot(0,\alpha^0,0)_w\\ =&[(0,\mathbf{c}^0,0)_{s_0}\cdot\tau_w\cup\tau_{s_0}\cdot(0,\alpha^0,0)_w]+\mu_w\phi_{s_0w}\text{ where }\mu_w\in k.\end{aligned}$$

Now using Lemma 3.12-ii, Proposition 4.5, and (91), we compute

$$
\begin{aligned}
&(0,\mathbf{c}^0,0)_{s_0}\cdot\tau_w\cup\tau_{s_0}\cdot(0,\alpha^0,0)_w\\
&\quad=(e_1\cdot(0,\mathbf{c}^0,0)_w)\cup(e_1\cdot(0,\alpha^0,0)_w)-(e_{\mathrm{id}^{-1}}\cdot(\mathbf{c}^0\iota^{-1},0,0)_w)\cup(e_{\mathrm{id}}\cdot(2\iota(\alpha^0),0,0)_w)\\
&\quad=[\sum_{u,v\in\mathbb{F}_p^\times}(0,\mathbf{c}^0,0)_{\omega_u w}\cup(0,\alpha^0,0)_{\omega_v w}]-[\sum_{u,v\in\mathbb{F}_p^\times}u^{-1}(\mathbf{c}^0\iota^{-1},0,0)_{\omega_u w}\cup v(2\iota(\alpha^0),0,0)_{\omega_v w}]\\
&\quad=[\sum_{u\in\mathbb{F}_p^\times}(0,\mathbf{c}^0,0)_{\omega_u w}\cup(0,\alpha^0,0)_{\omega_u w}]-[\sum_{u\in\mathbb{F}_p^\times}(\mathbf{c}^0\iota^{-1},0,0)_{\omega_u w}\cup(2\iota(\alpha^0),0,0)_{\omega_u w}]\\
&\quad=[\sum_{u\in\mathbb{F}_p^\times}\phi_{\omega_u w}]-2[\sum_{u\in\mathbb{F}_p^\times}\phi_{\omega_u w}]=e_1\cdot\phi_w \text{ by (87)}.
\end{aligned}
$$

So $\mathbf{x}_0\cdot\alpha_w^\star=e_1\cdot\phi_w+\mu_w\phi_{s_0w}$. But $(\tau_{s_0}+e_1)\cdot(e_1\cdot\phi_w+\mu_w\phi_{s_0w})=e_1\cdot\phi_{s_0w}+\mu_we_1\cdot\phi_{s_0w}$ (see (46)) and $\mathbf{x}_0$ being in the kernel of $\tau_{s_0}+e_1$, we obtain $\mu_w=-1$. Therefore, as expected, $\mathbf{x}_0\cdot\alpha_w^\star=e_1\cdot\phi_w-\phi_{s_0w}=-\tau_{s_0}\cdot\phi_w$. The case when $\epsilon=1$ may then be obtained by conjugation by ϖ ((50), the proof of Proposition 3.18-v which says that $\Gamma_\omega(\mathbf{x}_0)=\mathbf{x}_1$), and 8.3-i).

□

Remark 8.7. For $w\in\widetilde{W}$ with length 1 and $\epsilon\in\{0,1\}$ the map (126) satisfies:

$$\tau_{s_\epsilon}\otimes\tau_w^\vee|_{F^1H}\longmapsto\begin{cases}0 & \text{if } w\in\widetilde{W}^\epsilon\\ -\psi_w & \text{if } w\in\widetilde{W}^{1-\epsilon}\end{cases}$$

where ψ_w was defined in Remark 2.16.

Together with Remark 8.4 and using Propositions 8.5 and 8.6, this completely describes the triple Yoneda product $\mathcal{C}_{E^1}(Z)\otimes_H\mathcal{C}_{E^1}(Z)\otimes_H\mathcal{C}_{E^1}(Z)\to\mathcal{C}_{E^3}(Z)=E^3$ with image the subspace $ke_1\cdot\psi_{s_0}\oplus ke_1\cdot\psi_{s_1}\subseteq\ker(\mathcal{S}^3)$.

9 Appendix

9.1 Proof of Proposition 2.1

This proposition is written in the general context of $G:=\mathbf{G}(\mathfrak{F})$ being the group of $\mathfrak{F}$-rational points of a connected reductive group $\mathbf{G}$ over $\mathfrak{F}$ which we assume to be $\mathfrak{F}$-split. The first point was proved in [OS3] Cor. 5.5. To prove the second point, we recall some notations of [OS3]. The affine Coxeter system (W_{aff},S_{aff}) attached to G was introduced in §2.1.3 *loc. cit.* Recall that W_{aff} is a subgroup of $W=N_G(T)/T^0$ and that $\widetilde{W}=N_G(T)/T^0$ (see §2).

The action of τ_ω where $\omega\in\widetilde{W}$ has length zero is given in [OS3] Prop. 5.6 (it is the same formula as (64)). Using this formula together with (9), we see that it is enough to prove the second point of Proposition 2.1 in the case when v is a lift in $N(T)/T^1$ of $s\in S_{aff}$. For $s\in S_{aff}$, we pick the element $n_s\in N(T)$ as defined in §2.1.6 *loc. cit.* and let $v:=n_sT^1$. Recall that each $s\in S_{aff}$ corresponds to an affine simple root of the form $(\alpha,\mathfrak{h})$. As in (2.13) *loc. cit.*, the corresponding cocharacter $\check{\alpha}$ carves out the finite subgroup $\check{\alpha}([\mathbb{F}_q^\times])=\{\check{\alpha}([z]),z\in\mathbb{F}_q^\times\}$ of T^0, where $[\_]:\mathbb{F}_q^\times\to\mathfrak{O}^\times$ denotes the multiplicative Teichmüller lift. By (2.18) *loc. cit.*, we have

$$n_sIn_s^{-1}I=I\,\dot\cup\,\dot{\bigcup}_{z\in\mathbb{F}_q^\times}x_\alpha(\pi^{\mathfrak{h}}[z])\check{\alpha}([z])n_s^{-1}I\subset I\,\dot\cup\bigcup_{z\in\mathbb{F}_q^\times}I\check{\alpha}([z])n_s^{-1}I=I\,\dot\cup\,\dot{\bigcup}_{\omega\in\check{\alpha}([\mathbb{F}_q^\times])}I\omega n_s^{-1}I$$

where $x_\alpha(\pi^\natural[z]) \in I$ is defined in *loc. cit.* (2.14). We choose a lift $\dot{w} \in N(T)$ of $w \in \widetilde{W}$. Because of the condition on the length (namely $\ell(vw) = \ell(w) - 1$), we know that $I\dot{w}I = In_s^{-1}In_s\dot{w}I$ and therefore

$$n_sI\dot{w}I = In_s\dot{w}I \,\dot\cup \bigcup_{z\in\mathbb{F}_q^\times} x_\alpha(\pi^\natural[z])\check\alpha([z])n_s^{-1}In_s\dot{w}I \tag{130}$$
$$\subseteq In_s\dot{w}I \,\dot\cup \bigcup_{z\in\mathbb{F}_q^\times} I\check\alpha([z])\dot{w}I = In_s\dot{w}I \,\dot\cup \dot\bigcup_{\omega\in\check\alpha([\mathbb{F}_q^\times])} I\omega\dot{w}I \ .$$

This shows a result which is more precise than the one announced in Proposition 2.1. Namely, when $v = n_sT^1$, we have

$$a\cdot b \in H^{i+j}(I, \mathbf{X}(vw)) \oplus \bigoplus_{\omega\in\check\alpha([\mathbb{F}_q^\times])} H^{i+j}(I, \mathbf{X}(\omega\dot{w})) \ .$$

Let $\omega \in \check\alpha([\mathbb{F}_q^\times])$ and $u_\omega := \omega\dot{w}$. We study the component c_{u_ω} of $a\cdot b$ in $H^{i+j}(I, \mathbf{X}(u_\omega))$. We have $n_s^{-1}Iu_\omega I \cap I\dot{w}I = n_s^{-1}(I\omega\dot{w}I \cap n_sI\dot{w}I)$. From (130) we obtain that

$$n_s^{-1}Iu_\omega I \cap I\dot{w}I = \bigcup_{z\in\mathbb{F}_q^\times, \check\alpha([z])=\omega} n_s^{-1}x_\alpha(\pi^\natural[z])\check\alpha([z])n_s^{-1}In_s\dot{w}I = \bigcup_{z\in\mathbb{F}_q^\times, \check\alpha([z])=\omega} I_{n_s}n_s^{-1}x_\alpha(\pi^\natural[z])u_\omega I$$

The second identity comes from the fact that $I_{n_s} = I_{n_s^{-1}}$ is normalized by J by Cor. 2.5-iii. and from (2.7) in Lemma 2.2 (still in [OS3]). Now suppose that $\mathbf{G}$ is semisimple and simply connected, then by the proof of Lemma 2.8 *loc. cit.*, the map $\check\alpha$ is injective. Therefore there is a unique $z \in \mathbb{F}_q^\times$ such that $\check\alpha([z]) = \omega$ and

$$n_s^{-1}Iu_\omega I \cap I\dot{w}I = I_{n_s}n_s^{-1}x_\alpha(\pi^\natural[z])u_\omega I \ .$$

To apply the formula of Prop. 5.3 of [OS3], we need to study the double cosets $I_{n_s}\backslash(n_s^{-1}Iu_\omega \cap I\dot{w}I)/I_{u_\omega^{-1}}$. But from Lemma 5.2 *loc. cit.* and the above identity, we obtain immediately:

$$n_s^{-1}Iu_\omega \cap I\dot{w}I = I_{n_s}n_s^{-1}x_\alpha(\pi^\natural[z])u_\omega I_{u_\omega^{-1}}$$

Let $h := n_s^{-1}x_\alpha(\pi^\natural[z])u_\omega$. We have $u_\omega h^{-1}Ihu_\omega^{-1} = x_\alpha(\pi^\natural[z])^{-1}n_sIn_s^{-1}x_\alpha(\pi^\natural[z])$. Since $x_\alpha(\pi^\natural[z]) \in I$ normalizes I_{n_s} and since $I_w \subset I_s$ (Lemma 2.2 *loc. cit.*), we obtain:

$$\begin{aligned} I_{u_\omega} \cap u_\omega h^{-1}Ihu_\omega^{-1} &= I \cap wIw^{-1} \cap \big(x_\alpha(\pi^\natural[z])^{-1}n_sIn_s^{-1}x_\alpha(\pi^\natural[z])\big) \\ &= x_\alpha(\pi^\natural[z])^{-1}I_{n_s}x_\alpha(\pi^\natural[z]) \cap wIw^{-1} = I_{n_s} \cap wIw^{-1} = I_s \cap I_w = I_w = I_{u_\omega} \end{aligned}$$

By Remark 5.4 *loc. cit.*, it implies that the component of $a\cdot b - a\cdot\tau_w \cup \tau_{n_s}\cdot b$ in $H^{i+j}(I, \mathbf{X}(u_\omega))$ is zero. So

$$a\cdot b - a\cdot\tau_w \cup \tau_{n_s}\cdot b \in H^{i+j}(I, \mathbf{X}(n_sw)).$$

This concludes the proof. We add the computation of this element. Using Lemma 2.2 and Lemma 5.2-i *loc. cit.*, we obtain the following. Let $u := n_s\dot{w}$. We have $n_s^{-1}In_s\dot{w} \subset I_{n_s}\dot{w}I$ therefore $n_s^{-1}In_s\dot{w}\, I \cap I\dot{w}I = I_{n_s}\dot{w}I$ and $I_{n_s}\backslash(n_s^{-1}Iu \cap I\dot{w}I)/I_{u^{-1}}$ is made of only one double coset $I_{n_s}\dot{w}I_{u^{-1}}$. We have $I_u = I_{n_s\dot{w}}$ and $I_u \cap u\dot{w}^{-1}I\dot{w}u = n_sI_wn_s^{-1}$ while $I \cap u\dot{w}^{-1}I\dot{w}u^{-1} = I_s$ and $uIu^{-1} \cap u\dot{w}^{-1}I\dot{w}u^{-1} = n_sI_wn_s^{-1}$. So, by Prop. 5.3 *loc. cit.*, the component $c_{n_s\dot{w}}$ in $H^{i+j}(I, \mathbf{X}(n_s\dot{w}))$ of $a\cdot b$ is given by

$$\mathrm{Sh}_{n_s\dot{w}}(c_{n_s\dot{w}}) = \mathrm{cores}_{I_{n_sw}}^{n_sI_wn_s^{-1}}\big(\mathrm{res}^{I_{n_s}}_{n_sI_wn_s^{-1}}\big(\mathrm{Sh}_{n_s}(a)\big) \cup \big(n_{s*}\mathrm{Sh}_w(b)\big)\big).$$

In particular if $\mathbf{G}$ is semisimple and simply connected, then the image by $\mathrm{Sh}_{n_s\dot{w}}$ of the element

$$a\cdot b - a\cdot\tau_w \cup \tau_{n_s}\cdot b \ ,$$

which lies in $H^{i+j}(I, \mathbf{X}(n_s\dot{w}))$, is

$$\operatorname{cores}_{I_{n_s w}}^{n_s I_w n_s^{-1}} \left(\operatorname{res}_{n_s I_w n_s^{-1}}^{I_{n_s}} \left(\operatorname{Sh}_{n_s}(a) \right) \cup \left(n_{s*} \operatorname{Sh}_w(b) \right) \right) \\ - \operatorname{cores}_{I_{n_s w}}^{n_s I_w n_s^{-1}} \left(\operatorname{res}_{n_s I_w n_s^{-1}}^{I_{n_s}} \left(\operatorname{Sh}_{n_s}(a) \right) \right) \cup \operatorname{cores}_{I_{n_s w}}^{n_s I_w n_s^{-1}} \left(n_{s*} \operatorname{Sh}_w(b) \right).$$

9.2 Computation of some transfer maps

We use notations introduced in §2.4.1 and §3.2, see in particular Remark 3.2.

Lemma 9.1. *Suppose $p \neq 2$ and $G = \mathrm{SL}_2(\mathfrak{F})$. Let $w \in \widetilde{W}$ with length $m := \ell(w)$ which we suppose ≥ 1. Let $s \in \{s_0, s_1\}$ be the unique element such that $\ell(sw) = \ell(w) - 1$.*

i. Suppose $\mathfrak{F} \neq \mathbb{Q}_p$. If $m \geq 2$ or $m = 1$ and $q \neq 3$, then the transfer map $(I_{sw})_\Phi \to (sI_w s^{-1})_\Phi$ is the zero map.

ii. Suppose that $\mathfrak{F} = \mathbb{Q}_p$. If $m \geq 2$ or $m = 1$ and $p \neq 3$ then the transfer map $(I_{sw})_\Phi \to (sI_w s^{-1})_\Phi$ is

$$\begin{pmatrix} 1+\pi x & y \\ \pi^m z & 1+\pi t \end{pmatrix} \mapsto \begin{pmatrix} 1 & py \\ 0 & 1 \end{pmatrix} \bmod \begin{pmatrix} 1+\pi^2\mathbb{Z}_p & \pi^2\mathbb{Z}_p \\ \pi^{m+1}\mathbb{Z}_p & 1+\pi^2\mathbb{Z}_p \end{pmatrix} \qquad \textit{if } s = s_0$$
$$\begin{pmatrix} 1+\pi x & \pi^{m-1} y \\ \pi z & 1+\pi t \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ p\pi z & 1 \end{pmatrix} \bmod \begin{pmatrix} 1+\pi^2\mathbb{Z}_p & \pi^m\mathbb{Z}_p \\ \pi^3\mathbb{Z}_p & 1+\pi^2\mathbb{Z}_p \end{pmatrix} \qquad \textit{if } s = s_1.$$

Proof. Compare with [OS2, Prop. 3.65]. We let $m := \ell(w)$. By conjugation by ϖ, it is enough to treat the case of the transfer map $(I_{m-1}^+)_\Phi \to (s_0 I_m^- s_0^{-1})_\Phi$ in both the proofs of i. and ii. We denote this map by tr. Recall that when $s = s_0$, then $I_w = I_m^-$ and $I_{sw} = I_{m-1}^+$ where

$$I_{m-1}^+ := \begin{pmatrix} 1+\mathfrak{M} & \mathfrak{O} \\ \mathfrak{M}^m & 1+\mathfrak{M} \end{pmatrix}, \quad s_0 I_m^- s_0^{-1} = \begin{pmatrix} 1+\mathfrak{M} & \mathfrak{M} \\ \mathfrak{M}^m & 1+\mathfrak{M} \end{pmatrix}.$$

By the Iwahori factorization of I_{m-1}^+, it suffices to compute the transfer of elements of the form $\left(\begin{smallmatrix} 1 & 0 \\ \pi^m v & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$ of I_{m-1}^+. Let $S \subseteq \mathfrak{O}$ be a set of representatives for the cosets in $\mathfrak{O}/\mathfrak{M}$. Then the matrices $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$, for $b \in S$, form a set of representatives in the right cosets $s_0 I_m^- s_0^{-1} \backslash I_{m-1}^+$.

- Since $\left(\begin{smallmatrix} 1 & 0 \\ \pi^m v & 1 \end{smallmatrix}\right) \in s_0 I_m^- s_0^{-1}$, which is normal in I_{m-1}^+, we have

$$\operatorname{tr}(\left(\begin{smallmatrix} 1 & 0 \\ \pi^m v & 1 \end{smallmatrix}\right)) \equiv \prod_{b \in S} \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ \pi^m v & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & -b \\ 0 & 1 \end{smallmatrix}\right) \equiv \prod_{b \in S} \begin{pmatrix} 1+b\pi^m v & -b^2\pi^m v \\ \pi^m v & 1-b\pi^m v \end{pmatrix} \mod \Phi(s_0 I_m^- s_0^{-1})$$

  where $\Phi(s_0 I_m^- s_0^{-1})$ denotes the Frattini subgroup of $s_0 I_m^- s_0^{-1}$. From [OS2, Prop. 3.62] we get $[s_0 I_m^- s_0^{-1}, s_0 I_m^- s_0^{-1}] = s_0 [I_m^-, I_m^-] s_0^{-1} = \begin{pmatrix} 1+\mathfrak{M}^{m+1} & \mathfrak{M}^2 \\ \mathfrak{M}^{m+1} & 1+\mathfrak{M}^{m+1} \end{pmatrix}$ so

$$(s_0 I_m^- s_0^{-1})_\Phi \cong \mathfrak{M}^m/\mathfrak{M}^{m+1} \times (1+\mathfrak{M}/((1+\mathfrak{M}^{m+1})(1+\mathfrak{M})^p) \times \mathfrak{M}/\mathfrak{M}^2.$$

  In this isomorphism the above element corresponds to

$$(q\pi^m v \bmod \mathfrak{M}^{m+1}, \prod_b (1+b\pi^m v) \bmod (1+\mathfrak{M}^{m+1})(1+\mathfrak{M})^p, -\pi^m v \sum_b b^2 \bmod \mathfrak{M}^2)$$
$$= (0,\ 1+\pi^m v \sum_b b \bmod (1+\mathfrak{M}^{m+1})(1+\mathfrak{M})^p,\ -\pi^m v \sum_b b^2 \bmod \mathfrak{M}^2)\ .$$

The zero coordinate comes from the fact that for any choice of $\mathfrak{F}$ we have $q\mathfrak{M}^m \subseteq \mathfrak{M}^{m+1}$.

View $b \longmapsto b$ and $b \longmapsto b^2$ as $\mathbb{F}_q$-valued characters of the group $\mathbb{F}_q^\times$ of order prime to p. By the orthogonality relation for characters the sum $\sum_{b\in\mathbb{F}_q^\times} b$, resp. $\sum_{b\in\mathbb{F}_q^\times} b^2$, vanishes if and only if the respective character is nontrivial if and only if $q \neq 2$, resp. $q \neq 2, 3$. Since we assume $p \neq 2$ the second component is zero whereas the last component is zero if either $m \geq 2$, or $m = 1$ and $q \neq 3$.

- For $t \in 1 + \mathfrak{M}$, the element $\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$ again lies in $s_0 I_m^- s_0^{-1}$ so we have

$$\mathrm{tr}(\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)) \equiv \prod_{b\in S} \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & -b \\ 0 & 1 \end{smallmatrix}\right) \equiv \prod_{b\in S} \begin{pmatrix} t & b(t^{-1}-t) \\ 0 & t^{-1} \end{pmatrix} \mod \Phi(s_0 I_m^- s_0^{-1}).$$

The above element seen in $(s_0 I_m^- s_0^{-1})_\Phi$ corresponds to

$$(0,\ t^q \bmod (1+\mathfrak{M}^{m+1})(1+\mathfrak{M})^p,\ (t^{-1} - t)\sum_b b \bmod \mathfrak{M}^2)\ .$$

Since t^q is a pth power, the second component is zero. The last component is zero since $q \neq 2$.

- To compute $\mathrm{tr}(\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right))$, where $u \in \mathfrak{O}$, we follow the argument of the proof of [OS2, Lemma 3.40.i.a)]. Let $U(\mathfrak{M}) := \left(\begin{smallmatrix} 1 & \mathfrak{M} \\ 0 & 1 \end{smallmatrix}\right)$ and $U(\mathfrak{O}) := \left(\begin{smallmatrix} 1 & \mathfrak{O} \\ 0 & 1 \end{smallmatrix}\right)$. Since $I_{m-1}^+ = U(\mathfrak{O}) s_0 I_m^- s_0^{-1}$ we obtain the commutative diagram ([NSW] Cor. 1.5.8)

$$\begin{array}{ccc} H^1(s_0 I_m^- s_0^{-1}, k) & \xrightarrow{\text{cores}} & H^1(I_{m-1}^+, k) \\ \downarrow{\scriptstyle \text{res}} & & \downarrow{\scriptstyle \text{res}} \\ H^1(U(\mathfrak{M}), k) & \xrightarrow{\text{cores}} & H^1(U(\mathfrak{O}), k). \end{array} \quad \text{or dually} \quad \begin{array}{ccc} U(\mathfrak{O})_\Phi & \longrightarrow & U(\mathfrak{M})_\Phi \\ \downarrow & & \downarrow \\ (I_{m-1}^+)_\Phi & \xrightarrow{\text{tr}} & (s_0 I_m^- s_0^{-1})_\Phi. \end{array}$$

The upper right horizontal arrow is the transfer map $U(\mathfrak{O})_\Phi \to U(\mathfrak{M})_\Phi$ and it coincides with the qth power map $g \longmapsto g^q$ ([Hup] Lemma IV.2.1). So we study the image of $u \in \mathfrak{O}$ under the map $\mathfrak{O} \longrightarrow \mathfrak{M}, x \mapsto qx$. If $\mathfrak{F} \neq \mathbb{Q}_p$, then we have $q\mathfrak{O} \subseteq \mathfrak{M}^2$. Therefore $\mathrm{tr}(\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)) \equiv 0 \bmod \Phi(s_0 I_m^- s_0^{-1})$. If $\mathfrak{F} = \mathbb{Q}_p$, then $\mathrm{tr}(\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)) \equiv \left(\begin{smallmatrix} 1 & pu \\ 0 & 1 \end{smallmatrix}\right) \bmod \Phi(s_0 I_m^- s_0^{-1})$.

Under the hypotheses $p \neq 2$, and $m \geq 2$ or $m = 1$ and $q \neq 3$ we have proved: if $\mathfrak{F} \neq \mathbb{Q}_p$ then the transfer map $(I_{m-1}^+)_\Phi \to (s_0 I_m^- s_0^{-1})_\Phi$ is trivial; if $\mathfrak{F} = \mathbb{Q}_p$, then the image of

$$\left(\begin{smallmatrix} 1+\pi x & y \\ \pi^m z & 1+\pi t \end{smallmatrix}\right) = \begin{pmatrix} 1 & 0 \\ \frac{\pi^m z}{1+\pi x} & 1 \end{pmatrix} \begin{pmatrix} 1+\pi x & 0 \\ 0 & (1+\pi x)^{-1} \end{pmatrix} \begin{pmatrix} 1 & \frac{y}{1+\pi x} \\ 0 & 1 \end{pmatrix} \in I_{m-1}^+$$

by the transfer map $(I_{m-1}^+)_\Phi \to (s_0 I_m^- s_0^{-1})_\Phi$ is $\left(\begin{smallmatrix} 1 & py \\ 0 & 1 \end{smallmatrix}\right) \bmod \Phi(s_0 I_m^- s_0^{-1})$. □

9.3 Proof of Proposition 3.9

Here $G = \mathrm{SL}_2(\mathbb{Q}_p)$ with $p \neq 2, 3$ and $\pi = p$. Let $w \in \widetilde{W}$ with length $m := \ell(w)$. For $s \in \{s_0, s_1\}$ we compute the action of τ_s on an element $c \in H^1(I, \mathbf{X}(w))$ seen as a triple $(c^-, c^0, c^+)_w$. Using Lemma 3.4 and knowing that the map (49) of conjugation by ϖ is compatible with the Yoneda product hence the action of H, it is enough to prove the formulas for $s = s_0$. We recall the following result from [OS3] Prop. 5.6. There we worked with n_{s_i} (instead of the matrices s_i of the current article) where $n_{s_0} = s_0$ (but $n_{s_1} = s_1^{-1}$). Recall $s_0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. We have either $\ell(s_0 w) = \ell(w) + 1$ and $\tau_{s_0} \cdot c \in h^1(s_0 w)$ with

$$\mathrm{Sh}_{s_0 w}(\tau_{s_0} \cdot c) = \mathrm{res}^{s_0 I_w s_0^{-1}}_{I_{s_0 w}} \left(s_{0*}\, \mathrm{Sh}_w(c)\right)\ , \tag{131}$$

or $\ell(s_0w) = \ell(w) - 1$ and

$$\tau_{s_0} \cdot c = \gamma_{s_0w} + \sum_{\omega\in\Omega} \gamma_{\omega w} \in h^1(s_0w) \oplus \bigoplus_{\omega\in\Omega} h^1(\omega w) \tag{132}$$

with

$$\mathrm{Sh}_{s_0w}(\gamma_{s_0w}) = \mathrm{cores}_{I_{s_0w}}^{s_0 I_w s_0^{-1}} \left(s_{0*}\, \mathrm{Sh}_w(c)\right) \text{ and} \tag{133}$$

$$\mathrm{Sh}_{\omega_u w}(\gamma_{\omega_u w}) = \left(s_0 \omega_u^{-1} \left(\begin{smallmatrix} 1 & [u]^{-1} \\ 0 & 1 \end{smallmatrix}\right) s_0{}^{-1}\right)_* \mathrm{Sh}_w(c) \ . \tag{134}$$

A) Case when $\ell(s_0w) = \ell(w) + 1$. It means that $w \in \widetilde{W}^0$, $I_w = I_m^+$ and $I_{s_0w} = I_{m+1}^-$. We compute the composite map $H^1(I_m^+, k) \xrightarrow{s_{0*}} H^1(s_0 I_m^+ s_0^{-1}, k) \xrightarrow{\mathrm{res}} H^1(I_{m+1}^-, k)$. Let $X = \left(\begin{smallmatrix} 1+px & p^{m+1}y \\ pz & 1+pt \end{smallmatrix}\right) \in I_{m+1}^-$. Then $s_0^{-1} X s_0 = \left(\begin{smallmatrix} 1+pt & -pz \\ -p^{m+1}y & 1+px \end{smallmatrix}\right)$. Its image in $(I_m^+)_\Phi$ (see (53)) corresponds to

$$(-y, 1-px, 0) = (-y, 1+pt, 0) \in \mathbb{Z}_p/p\mathbb{Z}_p \times (1+p\mathbb{Z}_p)/(1+p^2\mathbb{Z}_p) \times \mathbb{Z}_p/p\mathbb{Z}_p \ .$$

This proves that $\mathrm{Sh}_{s_0w}(\tau_{s_0} \cdot c)$ is given by $(y, 1+px, z) \mapsto -c^-(y) - c^0(1+px)$, namely

$$\tau_{s_0} \cdot c = (0, -c^0, -c^-)_{s_0w} \text{ if } m \geq 1$$

and if $m = 0$ then $\tau_{s_0} \cdot c = (0, 0, -c^-)_{s_0w}$.

B) Now suppose $\ell(s_0w) = \ell(w) - 1$. Then $\tau_{s_0} \cdot c$ has a component $\gamma_{s_0w} \in h^1(s_0w)$ and a component $\sum_{u\in\mathbb{F}_p^\times} \gamma_{\omega_u w} \in \bigoplus_{\omega\in\Omega} h^1(\omega w)$. Recall that ω_u was defined in (37).

1) We compute $\sum_{u\in\mathbb{F}_p^\times} \gamma_{\omega_u w} \in \bigoplus_{\omega\in\Omega} h^1(\omega w)$.

In fact, for all $u \in \mathbb{F}_p^\times$, we compute the elements $\varepsilon_u \in H^1(I_w, k)$ defined by

$$e_1 \cdot c + \sum_{u\in\mathbb{F}_p^\times} \gamma_{\omega_u w} = \sum_{u\in\mathbb{F}_p^\times} \mathrm{Sh}_{\omega_u w}^{-1}(\varepsilon_u) \in \bigoplus_{u\in\mathbb{F}_p^\times} h^1(\omega_u w) \ ,$$

namely $\varepsilon_u = \mathrm{Sh}_{\omega_u w}(\gamma_{\omega_u w}) - (\omega_u)_* \mathrm{Sh}_w(c)$.

Recall $I_{\omega w} = I_m^- = \left(\begin{smallmatrix} 1+p\mathbb{Z}_p & p^m\mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{smallmatrix}\right)$ for any $\omega \in \Omega$. Compute $s_0 \omega_u^{-1} \left(\begin{smallmatrix} 1 & [u]^{-1} \\ 0 & 1 \end{smallmatrix}\right) s_0^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ -[u] & 1 \end{smallmatrix}\right) \omega_u$. Therefore, by (134)

$$\mathrm{Sh}_{\omega_u w}(\gamma_{\omega_u w}) - (\omega_u)_* \mathrm{Sh}_w(c) : X \mapsto (\omega_u)_* \mathrm{Sh}_w(c)(\left(\begin{smallmatrix} 1 & 0 \\ -[u] & 1 \end{smallmatrix}\right)^{-1} X \left(\begin{smallmatrix} 1 & 0 \\ -[u] & 1 \end{smallmatrix}\right) X^{-1})$$

for any $X := \left(\begin{smallmatrix} 1+px & p^m y \\ pz & 1+pt \end{smallmatrix}\right) \in I_w$. We have

$$\left(\begin{smallmatrix} 1 & 0 \\ -[u] & 1 \end{smallmatrix}\right)^{-1} X \left(\begin{smallmatrix} 1 & 0 \\ -[u] & 1 \end{smallmatrix}\right) X^{-1} = \left(\begin{smallmatrix} 1+px-p^m y[u] & p^m y \\ pz+p(x-t)[u]-p^m y[u^2] & 1+pt+p^m y[u] \end{smallmatrix}\right) X^{-1}.$$

Via (58) the image of this element in $(I_m^-)_\Phi$ corresponds to

$$(2x[u] - p^{m-1}y[u]^2, 1 - p^m y[u], 0) \in \mathbb{Z}_p/p\mathbb{Z}_p \times (1+p\mathbb{Z}_p)/(1+p^2\mathbb{Z}_p) \times \mathbb{Z}_p/p\mathbb{Z}_p.$$

So for $u \in \mathbb{F}_p^\times$, we just computed that $\mathrm{Sh}_{\omega_u w}(\gamma_{\omega_u w}) - (\omega_u)_* \mathrm{Sh}_w(c)$ is the element ε_u in $\mathrm{Hom}(I_w, k)$ sending $X \in I_w$ to

$$\begin{aligned} (\omega_u)_* \mathrm{Sh}_w(c)((2x[u] - p^{m-1}y[u]^2, 1 - p^m y[u], 0)) &= \mathrm{Sh}_w(c)((2x[u]^{-1} - p^{m-1}y, 1 - p^m y[u], 0)) \\ &= c^-(2x[u]^{-1} - p^{m-1}y) + c^0(1 - p^m y[u]) \end{aligned}$$

If $m=1$, then ε_u sends X onto (see notation (59)):

$$[u]^{-1}2c^-\iota(1+px)\ -[u]^{-2}c^-([u]^2y)\ -[u]^{-1}c^0\iota^{-1}(y[u]^2)).$$

Using (73) we see that its preimage by $\mathrm{Sh}_{\omega_u w}$ is the component in $h^1(\omega_u w)$ of $e_{\mathrm{id}}\cdot(0,-2c^-\iota,0)_w+e_{\mathrm{id}^2}\cdot(0,0,c^-)_w+e_{\mathrm{id}}\cdot(0,0,c^0\iota^{-1})_w$ so when $m=1$, we have

$$\sum_{u\in\mathbb{F}_p^\times}\gamma_{\omega_u w}=-e_1\cdot(c^-,c^0,c^+)_w+e_{\mathrm{id}}\cdot(0,-2c^-\iota,c^0\iota^{-1})_w+e_{\mathrm{id}^2}(0,0,c^-)_w.$$

If $m\geq 2$, then the only remaining component of ε_u is $X\mapsto[u]^{-1}2c^-\iota(1+px)$ so we obtain

$$\sum_{u\in\mathbb{F}_p^\times}\gamma_{\omega_u w}=-e_1\cdot(c^-,c^0,c^+)_w+e_{\mathrm{id}}\cdot(0,-2c^-\iota,0)_w.$$

2) We compute $\gamma_{s_0w}\in h^1(s_0w)$. By (133) we have $\mathrm{Sh}_{s_0w}(\gamma_{s_0w})=\mathrm{cores}_{I_{s_0w}}^{s_0I_ws_0^{-1}}\big(s_{0*}\,\mathrm{Sh}_w(c)\big)$. By Lemma 9.1, the composite map $(I_{s_0w})_\Phi\xrightarrow{\mathrm{tr}}(s_0I_ws_0^{-1})_\Phi\xrightarrow{s_0^{-1}\_s_0}(I_w)_\Phi$ is

$$(z,1+px,y)\mapsto(-y,0,0)\in\mathbb{Z}_p/p\mathbb{Z}_p\times(1+p\mathbb{Z}_p)/(1+p^2\mathbb{Z}_p)\times\mathbb{Z}_p/p\mathbb{Z}_p$$

This shows that $\gamma_{s_0w}=(0,0,-c^-)_{s_0w}$.

9.4 Proof of Proposition 4.5

Let $w\in\widetilde{W}$ and $\alpha=(\alpha^-,\alpha^0,\alpha^+)_w\in h^1(w)^\vee\subset{}^{\mathfrak{J}}((E^1)^{\vee,f})^{\mathfrak{J}}$. We suppose that $s=s_0$, the case $s=s_1$ following by conjugation by ϖ (by the map (49) which is compatible with the Yoneda product).

- Suppose that $\ell(s_0w)=\ell(w)+1$. By (9) we know that $\tau_{s_0^{-1}}\cdot\alpha=\alpha(\tau_{s_0}\cdot\_)$ has support in $h^1(s_0^{-1}w)$. Let $c=(c^-,c^0,c^+)_{s_0^{-1}w}\in h^1(s_0^{-1}w)$. We compute $(\tau_{s_0^{-1}}\cdot\alpha)(c)=\alpha(\tau_{s_0}\cdot c)$. By Proposition 3.9, the component in $h^1(w)$ of $\tau_{s_0}\cdot c$ is $(0,0,-c^-)_w$. Therefore $(\tau_{s_0^{-1}}\cdot\alpha)(c)=\alpha((0,0,-c^-)_w)=-c^-(\alpha^+)$ and $\tau_{s_0^{-1}}\cdot\alpha=(-\alpha^+,0,0)_{s_0^{-1}w}$. Using (92), it gives $\tau_{s_0}\cdot\alpha=(-\alpha^+,0,0)_{s_0w}$.
- Suppose that $\ell(s_0w)=\ell(w)-1$. By Proposition 2.1 (or (9)) we know that $\tau_{s_0^{-1}}\cdot\alpha=\alpha(\tau_{s_0}\cdot\_)$ has support in $h^1(s_0^{-1}w)\oplus\bigoplus_{\omega\in\Omega}h^1(\omega w)$.
  - Compute its component in $(h^1(s_0^{-1}w))^\vee$:
    We compute $(\tau_{s_0^{-1}}\cdot\alpha)(c)=\alpha(\tau_{s_0}\cdot c)$ for $c=(c^-,c^0,c^+)_{s_0^{-1}w}\in h^1(s_0^{-1}w)$ with $c^0=0$ if $\ell(w)=1$. By Proposition 3.9, the element $\tau_{s_0}\cdot c$ lies in $h^1(w)$ and is equal to $(0,-c^0,-c^-)_w$. Therefore $(\tau_{s_0^{-1}}\cdot\alpha)(c)=-c^0(\alpha_0)-c^-(\alpha^+)$, and the component in $(h^1(s_0^{-1}w))^\vee$ of $\tau_{s_0^{-1}}\cdot\alpha$ is $(-\alpha^+,-\alpha_0,0)_{s_0^{-1}w}$ if $\ell(w)\geq 2$ and $(-\alpha^+,0,0)_{s_0^{-1}w}$ if $\ell(w)=1$. Using (92), the component in $(h^1(s_0w))^\vee$ of $\tau_{s_0}\cdot\alpha$ is $(-\alpha^+,-\alpha_0,0)_{s_0w}$ if $\ell(w)\geq 2$ and $(-\alpha^+,0,0)_{s_0w}$ if $\ell(w)=1$.
  - Compute the component $\sum_{u\in\mathbb{F}_p^\times}\beta_{\omega_u w}$ in $\oplus_{u\in\mathbb{F}_p^\times}(h^1(\omega_u w))^\vee$ of $\tau_{s_0^{-1}}\cdot\alpha$:
    The component in $(h^1(w))^\vee$ of $(\tau_{\omega_u{}^{-1}}\tau_{s_0^{-1}}\cdot\alpha)$ is $\tau_{\omega_u{}^{-1}}\cdot\beta_{\omega_u w}$. We therefore compute $(\tau_{\omega_u}^{-1}\cdot\beta_{\omega_u w})(c)=\alpha(\tau_{s_0}\tau_{\omega_u}\cdot c)$ for $c=(c^-,c^0,c^+)_w\in h^1(w)$. By Proposition 3.9 and the definition of the idempotents (36) (see also (2.12)), the component in $h^1(w)$ of $\tau_{s_0}\tau_{\omega_u}\cdot c=\tau_{\omega_u^{-1}}\tau_{s_0}\cdot c$ is

$$\begin{cases}(c^-,c^0,c^+)_w+u^{-1}(0,2c^-\iota,0)_w & \text{if } \ell(w)\geq 2,\\ (c^-,c^0,c^+)_w+u^{-1}(0,2c^-\iota,-c^0\iota^{-1})_w+u^{-2}(0,0,-c^-)_w & \text{if } \ell(w)=1\ .\end{cases}$$

Therefore

$$\alpha(\tau_{s_0}\tau_{\omega_u}\cdot c) = \begin{cases} c^-(\alpha^-) + c^0(\alpha^0) + c^+(\alpha^+) + u^{-1}2c^-\iota(\alpha^0) & \text{if } \ell(w)\geq 2,\\ c^-(\alpha^-) + c^0(\alpha^0) + c^+(\alpha^+) + u^{-1}2c^-\iota(\alpha^0) - u^{-1}c^0\iota^{-1}(\alpha^+) - u^{-2}c^-(\alpha^+) & \text{if } \ell(w) = 1\ . \end{cases}$$

So

$$\beta_{\omega_u w} = \begin{cases} \tau_{\omega_u}\cdot(\alpha^-,\alpha^0,\alpha^+)_w + u^{-1}\tau_{\omega_u}\cdot(2\iota(\alpha^0),0,0)_w & \text{if } \ell(w)\geq 2,\\ \tau_{\omega_u}\cdot(\alpha^-,\alpha^0,\alpha^+)_w + u^{-1}\tau_{\omega_u}\cdot(2\iota(\alpha^0),-\iota^{-1}(\alpha^+),0)_w - u^{-2}\tau_{\omega_u}\cdot(\alpha^+,0,0)_w & \text{if } \ell(w) = 1 \end{cases}$$

and

$$\sum_{u\in\mathbb{F}_p^\times}\beta_{\omega_u w} = \begin{cases} -e_1\cdot(\alpha^-,\alpha^0,\alpha^+)_w - e_{\mathrm{id}}\cdot(2\iota(\alpha^0),0,0)_w & \text{if } \ell(w)\geq 2,\\ -e_1\cdot(\alpha^-,\alpha^0,\alpha^+)_w - e_{\mathrm{id}}\cdot(2\iota(\alpha^0),-\iota^{-1}(\alpha^+),0)_w + e_{\mathrm{id}^2}\cdot(\alpha^+,0,0)_w & \text{if } \ell(w) = 1\ . \end{cases}$$

The component in $\oplus_{u\in\mathbb{F}_p^\times}(h^1(\omega_u w))^\vee$ of $\tau_{s_0}\cdot\alpha$ is

$$\tau_{s_0^2}\cdot\sum_{u\in\mathbb{F}_p^\times}\beta_{\omega_u w} = \begin{cases} -e_1\cdot(\alpha^-,\alpha^0,\alpha^+)_w + e_{\mathrm{id}}\cdot(2\iota(\alpha^0),0,0)_w & \text{if } \ell(w)\geq 2,\\ -e_1\cdot(\alpha^-,\alpha^0,\alpha^+)_w + e_{\mathrm{id}}\cdot(2\iota(\alpha^0),-\iota^{-1}(\alpha^+),0)_w + e_{\mathrm{id}^2}\cdot(\alpha^+,0,0)_w & \text{if } \ell(w) = 1\ . \end{cases}$$

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