Practice implicit differentiation. If you are curious, try to plot these curves on a computer (or google it).

1. [10] (cardioid) Let \((x^2 + y^2)^2 + 4x(x^2 + y^2) - 4y^2 = 0\) be a curve. Find \(y'\).

Implicit differentiate with respect to \(x\).

\[
2 \cdot (x^2 + y^2) \cdot (x^2 + y^2)' + 4 [(x^2 + y^2) + x \cdot (x^2 + y^2)'] - 8yy' = 0
\]

\[
(x^2 + y^2) \cdot (2x + 2yy') + 4(x^2 + y^2) + 4x(2x + 2yy') - 8yy' = 0
\]

\[
4(x^2 + y^2)x + 4(x^2 + y^2)yy' + 4(x^2 + y^2) + 8x^2 + 8xyy' - 8yy' = 0
\]

\[
(x^2 + y^2)x + (x^2 + y^2)yy' + (x^2 + y^2) + 2x^2 + 2xyy' - 2yy' = 0
\]

Re-arranging:

\[
(x^2 + y^2)yy' + 2xyy' - 2yy' = -(x^2 + y^2)x - 3x^2 - y^2
\]

\[
y' = - \frac{(x^2 + y^2)x + 3x^2 + y^2}{(x^2 + y^2)y + 2xy - 2y}
\]

2. [10] (astroid) Let \(x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2\) be a curve. Find the points on the curve where the tangent line is parallel to \(y = -x\).

Two lines are parallel when their slopes are equal. We are looking for the points where \(y' = -1\).

Implicit differentiate:

\[
\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}y' = 0
\]

\[
y' = - \left( \frac{y}{x} \right)^{\frac{1}{3}}
\]

We want \(y' = -1\):

\[- \left( \frac{y}{x} \right)^{\frac{1}{3}} = -1\]

\[y = x\]

Finally we want to find the intersection:

\[
\begin{align*}
x^{\frac{2}{3}} + y^{\frac{2}{3}} &= 2 \\
y &= x
\end{align*}
\]
We get substituting the second equation on the first equation:

\[ 2 \cdot x^3 = 2 \]

So \( x = 1 \) or \( x = -1 \).

Using the second equation we find the solution to be \((1, 1)\) or \((-1, -1)\).

3. [20] (quadrifolium) Let \((x^2 + y^2)^3 = (x^2 - y^2)^2\) be a curve. Find the points on the curve where the normal line is parallel to \(y = 0\).

If the normal line is horizontal then the tangent line must be vertical. So we are looking for point where \( \frac{dx}{dy} = 0 \).

Implicit differentiate with respect to \( y \):

\[
3(x^2 + y^2)^2 \cdot (x^2 + y^2)' = 2(x^2 - y^2) \cdot (x^2 - y^2)' \\
3(x^2 + y^2)^2(2xx' + 2y) = 2(x^2 - y^2)(2xx' - 2y) \\
6(x^2 + y^2)^2 xx' + 6(x^2 + y^2) y = 4(x^2 - y^2) xx' - 4(x^2 - y^2) y \\
6(x^2 + y^2) xx' - 4(x^2 - y^2) xx' = -6(x^2 + y^2)^2 y - 4(x^2 - y^2) y \\
[6(x^2 + y^2)^2 x - 4(x^2 - y^2) x] x' = -6(x^2 + y^2)^2 y - 4(x^2 - y^2) y \\
[3(x^2 + y^2)^2 x - 2(x^2 - y^2) x] x' = -3(x^2 + y^2)^2 y - 2(x^2 - y^2) y
\]

We obtain:

\[ x' = -\frac{[3(x^2 + y^2)^2 + 2(x^2 - y^2)] y}{[3(x^2 + y^2)^2 - 2(x^2 - y^2)] x} \]

whenever the denominator is non-zero. Observe, however, that when the denominator is zero then either \( x = 0 \) or the numerator is not zero (this is not a trivial observation). To avoid a lengthy argument, I recommend looking at a graph of this curve and observe that when \( x = 0 \) the tangent line cannot be vertical.

From now on we proceed on the assumption that the denominator is not zero.

For the numerator to be zero, we have two possibilities.

Case 1: \( y = 0 \). In this case we solve:

\[
\begin{cases}
(x^2 + y^2)^3 = (x^2 - y^2)^2 \\
y = 0
\end{cases}
\]

Which gives us \( x = 1 \) or \( x = -1 \). That is, at \((-1, 0)\) and \((1, 0)\) the tangent line is vertical hence the normal line is horizontal.

The second case is substantially more difficult so I will grant full marks for everyone who found the solution above.

Case 2: \( 3(x^2 + y^2)^2 + 2(x^2 - y^2) = 0 \). So we want to solve

\[
\begin{cases}
(x^2 + y^2)^3 = (x^2 - y^2)^2 \\
3(x^2 + y^2)^2 + 2(x^2 - y^2) = 0
\end{cases}
\]
To simplify observe that the solutions of this system have a symmetry, in fact, if \((x, y)\) is a solution then so are \((-x, y)\), \((x, -y)\) and \((-x, -y)\). In other words, we can assume that both \(x\) and \(y\) are positive then find all other solutions using the symmetry.

Put \(A = (x^2 + y^2)\) and \(B = (x^2 - y^2)\). We can re-write the system as

\[
\begin{cases}
A^3 = B^2 \\
3A^2 = -2B
\end{cases}
\]

This system is easier to solve. Take the square of the second line and substitute on the first to find \(A = \frac{4}{9}\) and then \(B = -\frac{8}{27}\). (Observe that \(A = B = 0\) is also a solution but it would imply \(x = y = 0\) which we already ruled out)

Now we need to find \(x\) and \(y\) by solving

\[
\begin{cases}
x^2 + y^2 = \frac{4}{9} \\
x^2 - y^2 = -\frac{8}{27}
\end{cases}
\]

Adding the two equations we obtain \(2x^2 = \frac{4}{27}\), so \(x = \frac{\sqrt{6}}{3\sqrt{3}} = \frac{\sqrt{6}}{9}\).

Substituting on the second equation again we obtain \(y^2 = \frac{10}{27}\), so \(y = \frac{\sqrt{10}}{3\sqrt{3}} = \frac{\sqrt{30}}{9}\).

Finally, using the symmetry, the solutions are: \((\frac{\sqrt{6}}{9}, \frac{\sqrt{30}}{9})\), \((-\frac{\sqrt{6}}{9}, \frac{\sqrt{30}}{9})\), \((\frac{\sqrt{6}}{9}, -\frac{\sqrt{30}}{9})\) and \((-\frac{\sqrt{6}}{9}, -\frac{\sqrt{30}}{9})\).

4. [10] Let \(x^y = x e^{2xy}\), find the normal line at \((1, 0)\).

First we use logarithmic differentiation to find the slope of the tangent line at \((1, 0)\). Apply \(\ln\) on both sides and use properties of logarithm to obtain \(y \ln(x) = \ln(x) + 2xy\).

Now implicit differentiate with respect to \(x\):

\[
y' \ln(x) + y \cdot (\ln(x))' = \frac{1}{x} + 2(y + xy')
\]

\[
y' \ln(x) + \frac{y}{x} = \frac{1}{x} + 2(y + xy')
\]

When \(x = 1\) and \(y = 0\) we have:

\[
y' \ln(1) + \frac{0}{1} = \frac{1}{1} + 2(0 + 1y')
\]

So \(y' = -\frac{1}{2}\). The slope of the normal line is then \(m = 2\) and its equation is:

\[y = 2(x - 1)\]