ON THE ZERO-FREE REGION OF CERTAIN FAMILIES OF DEDEKIND ZETA-FUNCTIONS

by

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Abstract

In this thesis, we will study the analytical properties of subfamilies of Dedekind zeta-functions satisfying certain counting conditions. This was motivated by a result of Pierce, Turnage-Butterbaugh and Wood, giving a Chebotarev density result for subfamilies of Dedekind zeta-functions satisfying a zero-free region assumption. Their result is then applied to remove the requirement to assume the Generalised Riemann Hypothesis in a result of Ellenberg and Venkatesh towards the $\ell$-Torsion Conjecture. The Chebotarev density result of Pierce–Turnage-Butterbaugh–Wood is still conditional upon the validity of Strong Artin Conjecture. In particular, the reliance to the Strong Artin Conjecture is due to the application of an analytical method of Kowalski and Michel in their argument, which requires that the irreducible factors of the Dedekind zeta-functions be automorphic cuspidal. The goal of this thesis is to work towards the removal of the reliance to the Strong Artin Conjecture, by condensing down the conditional requirement to a requirement that poles (of pieces) of Dedekind zeta-functions satisfy some mild analytical condition in a family. This will pave a path towards an unconditional effective version of the result of Pierce–Turnage-Butterbaugh–Wood.

We will establish directly that certain families of Artin $L$-functions are amenable to the analytical method of Kowalski–Michel under the aforementioned mild analytical condition. Applying Kowalski–Michel, we will prove the analog of an effective Chebotarev density result for certain subfamilies of Dedekind zeta-functions. With regards to the result of Pierce–Turnage-Butterbaugh–Wood, this will show that the Strong Artin Conjecture can be replaced with a mild analytical condition in the assumption of Pierce–Turnage-Butterbaugh–Wood.
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Chapter 1

INTRODUCTION

1.1 HISTORICAL BACKGROUND

To understand where the branch of mathematics that is analytic number theory comes from, it is perhaps most informative to start from the beginning of the subject.

Let us consider the positive integers \( \mathbb{N} \) and its multiplicative structure. One of the first things one learns is the existence of prime numbers, and how they are the building block for \( \mathbb{N} \) via the unique prime factorisation of integers. As such, to study \( \mathbb{N} \), the most basic and visible of mathematical objects, a study of prime numbers would yield many properties of \( \mathbb{N} \).

Given their importance, the natural question one might ask is the following.

Question. How many prime numbers are there in \( \mathbb{N} \)?

Of course, with a basic course in number theory, one would be able to reproduce the arguments originally utilised by Euclid to show that there are infinitely many prime numbers. Instead, we will like to direct the readers’ attention to the proof of Euler regarding the infinitude of prime numbers. Central in Euler’s argument is the divergence of the harmonic series

\[
f(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}
\]

at \( k = 1 \). In particular, Euler’s proof exploits the analytical properties of the function \( f \) to deduce an arithmetic result. This is the essence of analytical number theory: using analytical properties to deduce arithmetic results.

This line of investigation bore a number of arithmetical results under the study of Dirichlet, who was interested in primes in arithmetic progressions. Utilising character theory, Dirichlet rephrase the question of infinitude of primes in arithmetic progression to the question of divergence of Dirichlet series associated to Dirichlet characters, which will then yield results of infinitude of primes in arithmetic progression. This study of Dirichlet would then firmly establish analytical methods in multiplicative number theory.

With the infinitude of prime numbers, one might then ask the following question.

Question. What is the distribution of prime numbers in \( \mathbb{N} \)?

Let us rephrase the question: Let \( \pi(x) \) be the prime-counting function, i.e.

\[
\pi(x) = |\{ \text{primes } p \leq x \}|
\]

How does \( \pi(x) \) grows as \( x \) grows?

In a landmark paper [Rie59], motivated by his study for formulas for the prime-counting function, Riemann did groundbreaking work on the study of \( \zeta(s) \), the Riemann zeta-functions. In his paper, Riemann expanded \( f(k) \) as a function on the complex plane \( \mathbb{C} \), which he denotes as \( \zeta(s) \), and proceed
to deduce many analytical properties of $\zeta(s)$. Of particular note is Riemann’s sketch, later proved by von Mangoldt, of his explicit formula for the (normalised) prime-counting function involving non-trivial zeros of $\zeta(s)$. Riemann thus divised a general program for studying $\pi(x)$, by studying the non-trivial zeros of $\zeta(s)$. This is greatly expanded upon by Hadamard and de la Valée Poussin, culminating in the celebrated Prime Number Theorem.

**Prime Number Theorem.** Let $\pi(x)$ be the number of primes less than or equal to $x$. Then

$$\pi(x) \sim \frac{x}{\log x}$$

Returning to Riemann’s paper [Rie59], Riemann proceeded to study the non-trivial zeros of $\zeta(s)$. Riemann calculated the first few zeroes of $Z(s)$, and discovered that they all lie on the critical line $Re(s) = \frac{1}{2}$. From this, Riemann conjectured the most famous open conjecture in mathematics.

**Riemann Hypothesis.** All the non-trivial zeroes of $\zeta(s)$ has real part $\frac{1}{2}$.

The fame of Riemann Hypothesis is not unwarranted. Besides its notoriety as an extremely inadmissible open problem, its veracity also greatly improve the Prime Number Theorem, by explicitly quantifying the error term in the Prime Number Theorem to be of the magnitude of $\sqrt{x}\log x$. This cements the role of the study of non-trivial zeroes of $\zeta(s)$ in analytical number theory as the premier direction towards the study of the distribution of prime numbers.

Let us shift our focus. In light of the development of algebraic number theory, we can do an analogous study that we have been doing for $\mathbb{Z}$ to a more general setting. Let $K$ be a number field, and let $O_K$ be the integral closure of $\mathbb{Z}$ in $K$. Even though there is no notion of prime element in $O_K$, we still have the notion of prime ideal in $O_K$ and that the unique prime factorisation of ideals holds in $O_K$.

In this general case, analogous to the $\mathbb{Q}$ case, we have now Dedekind zeta-functions $\zeta_K(s)$ playing the role of $\zeta(s)$. Dedekind zeta-functions are very natural generalisation of $\zeta(s)$ to general number fields, and they share similar properties. In fact, a version of Riemann Hypothesis for Dedekind zeta-functions is conjectured to hold.

**Generalised Riemann Hypothesis.** All the non-trivial zeroes of $\zeta_K(s)$ has real part $\frac{1}{2}$.

The analogous question of prime distribution in $O_K$ will be the following.

**Question.** What is the distribution of prime ideals in $O_K$?

We will rephrase the question similar to the case we have done for $\mathbb{Z}$, but more generally, as follows: Let $F$ be a number field, and consider a normal extension $K/F$ with Galois group $Gal(K/F)$. Let $$\mathcal{P}(F) = \{ p \triangleleft O_F \}$$

For $C \subseteq Gal(K/F)$ a conjugacy class, consider the set $$\mathcal{P}(C, K/F) = \{ p \triangleleft O_F; \text{ p is unramified in } K, \sigma_p = C \}$$

We define the following prime counting function in $O_F$, given by

$$\pi(x, C, K/F) = |\{ p \in \mathcal{P}(C, K/F); N_{Q}^{F}(p) \leq x \}|$$

How does the density of $\mathcal{P}(C, K/F)$ in $\mathcal{P}(F)$ varies as $C$ varies? How does the growth of $\pi(x, C, K/F)$ changes as $C$ varies?

The solution to the questions above is first given by Chebotarev, in [Tsc26]. Chebotarev proved the following celebrated result which describes the density of the set $\mathcal{P}(C, K/F)$.
**Chebotarev Density Theorem.** Let $F$ be a number field, and consider a normal extension $K/F$ with Galois group $\text{Gal}(K/F)$. For $\mathcal{C} \subseteq \text{Gal}(K/F)$ a conjugacy class, the set $\mathcal{P}(\mathcal{C}, K/F)$ has density
\[
\lim_{x \to \infty} \frac{\left| \{ \mathfrak{p} \in \mathcal{P}(\mathcal{C}, K/F) ; \ N_{Q}^{F}(\mathfrak{p}) \leq x \} \right|}{\left| \{ \mathfrak{p} \in \mathcal{P}(F) ; \ N_{Q}^{F}(\mathfrak{p}) \leq x \} \right|} = \frac{|\mathcal{C}|}{|\text{Gal}(K/F)|}
\]

Chebotarev Density Theorem is a celebrated result in analytic number theory, being the principal tool in deducing many analytic property of subset of primes. It is also a generalisation of Dirichlet’s theorem on primes in arithmetic progressions.

One can rephrase the Chebotarev Density Theorem as
\[
\pi(x, \mathcal{C}, K/F) \sim \frac{|\mathcal{C}|}{|\text{Gal}(K/F)|} \text{Li}(x) \sim \frac{|\mathcal{C}|}{|\text{Gal}(K/F)|} \frac{x}{\log x}
\]
where we denote
\[
\text{Li}(x) = \int_{2}^{x} \frac{1}{\log t} \, dt
\]
Observe that when $K = F = Q$, we get the Prime Number Theorem.

Here, in the general case, the Generalised Riemann Hypothesis plays the same role as Riemann Hypothesis in the Prime Number Theorem, providing a description of the error term in Chebotarev Density Theorem conditional on the veracity of the Generalised Riemann Hypothesis. This was proven by Lagarias and Odlyzko in [LO77]. We will quote here the refined version by Serre in [Ser81].

**Conditional Effective Chebotarev Density Theorem.** Let $F$ be a number field, and consider a normal extension $K/F$ with Galois group $\text{Gal}(K/F)$. Let the absolute discriminant of $K$ be $|\Delta_{K/Q}|$. If the Generalised Riemann Hypothesis holds for $\zeta_{K}(s)$, then there exists a constant $C$ such that for $\mathcal{C} \subseteq \text{Gal}(K/F)$ a conjugacy class, and for $x \geq 2$, we have that
\[
\left| \pi(x, \mathcal{C}, K/F) - \frac{|\mathcal{C}|}{|\text{Gal}(K/F)|} \text{Li}(x) \right| \leq C \frac{|\mathcal{C}|}{|\text{Gal}(K/F)|} x^{\frac{1}{2}} \log(|\Delta_{K/Q}| x^{[K:Q]})
\]

Let us focus on the reliance of Generalised Riemann Hypothesis of these result. Given that the Generalised Riemann Hypothesis can be thought of as a result about the zero-free region of Dedekind zeta-functions, we can formulate the following question.

**Question.** Is there an unconditional result zero-free region on Dedekind zeta-functions which can replace Generalised Riemann Hypothesis in the Effective Chebotarev Density Theorem?

Towards this question, Kowalski and Michel developed in [KM02] results about the zeroes of families of cuspidal automorphic $L$-functions. These results can be used in certain number-theoretic applications to replace the Generalised Riemann Hypothesis. Pierce, Turnage-Butterbaugh and Wood showed in [PTBW17] that by assuming the Strong Artin Conjecture, one can port the results of Kowalski and Michel to the case of Artin $L$-functions and replace the reliance of Generalised Riemann Hypothesis, giving an effective Chebotarev density result which does not rely on the veracity of Generalised Riemann Hypothesis. This comes at the cost of have the resulting effective Chebotarev density result to be true for all but a quantifiable number of exceptions.
1.2 Summary of Results

In this thesis, we will mainly be interested in working towards the removal of the reliance of Strong Artin Conjecture in the result of Pierce, Turnage-Butterbaugh and Wood. We will replace this reliance with an analytical condition involving the poles of Artin $L$-functions. To achieve that, we will need to reframe the result of Kowalski and Michel in a manner which is applicable to a suitable family of Artin $L$-function, along with some notion of a convolution of Galois representations which mirrors the Rankin-Selberg convolution for automorphic $L$-functions.

We will first define convolution of representations in §2.2.4 of Chapter 2, proving some auxiliary results in the process.

In Chapter 3, we will proceed to define a family of Artin $L$-functions which is susceptible to the methods of Kowalski–Michel. We will call such a family a Kowalski–Michel ameanable family, and it mimics all the required properties of a family of cuspidal automorphic $L$-functions. Crucially, to remove the reliance of Strong Artin Conjecture, we require Kowalski–Michel ameanable family to be well-behaved under the convolution we introduced. We will then proceed as in [KM02], obtaining the following proposition.

**Proposition 3.2.8** Fix $z \geq 1$ large enough and $0 < \delta < \frac{1}{2}$, and let $F$ be a Kowalski–Michel $\mathbb{Q}$-ameanable family. Let $N \geq R^2$. Then, there exists a function $G_F(X, N, R)$ such that for $\varepsilon > 0$ satisfying

$$N^{\frac{1}{2} - \varepsilon} > X^{d + \frac{m_a}{2}} R^{2 + 2\delta \log R}$$

there exists a constant $M(\varepsilon)$ such that for $a_n \in \mathbb{C},$

$$\sum_{L(s, \varphi, F/\mathbb{Q}) \in F(X)} \frac{1}{s \varphi(\rho, R)} \sum_{\rho \in \mathcal{R}_{\alpha, \varphi}(\rho)} \frac{1}{|\psi_{\rho, r}(n)|} \left| \sum_{n=N} \alpha_n \psi_{\rho, r}(n)([n] f(s, \rho)) \right|^2 \leq M(\varepsilon) N G_F(X, N, R) \sum_{n=N} |a_n|^2$$

We conjecture the following, which will remove the reliance to Strong Artin Conjecture if it is indeed proven to be true.

**Conjecture 3.2.9** Fix $z \geq 1$ large enough, and let $F$ be a Kowalski–Michel $\mathbb{Q}$-ameanable family. Let $N \geq R^2$. Then, the function $G_F(X, N, R)$ in Proposition 3.2.8 satisfy

$$G_F(X, N, R) \ll 1$$

Proceeding as in [KM02] under the assumption of the above conjecture, we deduce the theorem of Kowalski–Michel in our setting.

**Theorem 3.3.2** Let $F$ be Kowalski–Michel $\mathbb{Q}$-ameanable family of Artin $L$-functions, and assume Conjecture 3.2.9. Let $\alpha \geq \frac{3}{4}$ and $T \geq 2$. Then, there exists a constant $B > 0$, depending on the family parameters, such that for all $c > 10 m[F : \mathbb{Q}] a + 4d$, there exists a constant $M(c)$ such that

$$N_{F, \alpha, T}(X) \leq M(c) T^B X^{c \frac{1}{d - 1}}$$

for all $X \in \mathbb{N}$.

In Chapter 4, we will start by studying faithful representations and $\nu$-regular representations of $G$, defined as

$$\text{reg}_G^\nu = \bigoplus_{\rho \in \mathcal{G}_{\text{ker}(\rho) = \{e\}}} (\dim \rho) \rho$$

We will then prove a variant of Aramata–Brauer Theorem for Artin $L$-functions associated to $\nu$-regular representations.
**Theorem 4.1.25.** Let \( K/F \) be a Galois extension. The Artin \( L \)-functions associated to \( \nu \)-regular representations of \( \text{Gal}(K/F) \) is entire.

We will also prove a variant of Aramata–Brauer Theorem for Artin \( L \)-functions associated to convolution of \( \nu \)-regular representations.

**Theorem 4.1.27.** Let \( K, K'/F \) be Galois extensions such that \( \text{Gal}(K/F) \cong G \) and \( \text{Gal}(K'/F) \cong G \). Then, the Artin \( L \)-functions associated to convolution of \( \nu \)-regular representations of \( \text{Gal}(K/F) \) and \( \text{Gal}(K'/F) \) is entire if and only if \( K \neq K' \). If \( K = K' \), then it has a pole only at \( s = 1 \).

We then reframed the results of Pierce–Turnage-Butterbaugh–Wood in this language.

**Theorem 4.2.10.** Let \( F/Q \) be Galois, and assume Conjecture 3.2.9. Let \( H \) be a sterile subfamily of \( \mathcal{F}_{F,G} \) with contamination index \( \tau \). Then, for any \( 0 < \Delta < 1 - \frac{\tau}{d_{F,G}(H)} \) and \( \eta < \frac{1}{4} \), there exists \( B \) (depending on the Kowalski–Michel family parameter of \( \mathcal{F}_{F,G}^\nu (H) \) for each \( N < G \)), a constant \( M(G, a_{F,G}(H), d_{F,G}(H), \Delta, \tau) \) and \( 0 < \delta \leq \frac{1}{4} \), given by

\[
\delta = \frac{\varepsilon}{20m[F : Q]a_{Q,G}(H) + 8d_{Q,G}(H) + 4\varepsilon}
\]

with \( m = \max_{N < G} \dim \text{reg}_{G/N}^\nu \) and

\[
\Delta = 1 - \frac{\tau}{d_{F,G}(H)} - \frac{\varepsilon}{2d_{F,G}(H)}
\]

such that for all \( X \in \mathbb{N} \), at most \( M(G, a_{F,G}(H), d_{F,G}(H), \Delta, \tau) X^{1 - (1 - \eta)\Delta d_{F,G}(H)} \) members of \( \mathcal{H}(X) \) that can have a zero in the region

\[
[1 - \delta, 1] \times \left[ -X^{\frac{\eta \Delta d_{F,G}(H)}{\nu}}, X^{\frac{\eta \Delta d_{F,G}(H)}{\nu}} \right]
\]

We then reframed the results of Pierce–Turnage-Butterbaugh–Wood in this language.

Finally, in Chapter 5, after briefly sketching the effective Chebotarev density result of Pierce–Turnage-Butterbaugh–Wood conditional on a zero-free region, we prove the their results holds unconditionally for set of fields which describes a sterile subfamily.

**Theorem 5.3.1.** Let \( F/Q \) be Galois, and assume Conjecture 3.2.9. Let

\[
Z \subseteq \left\{ K/F; [K : F] = n, \text{Gal}(K^F/F) = G \right\}
\]

Suppose \( \mathcal{H}(Z) \) is a sterile subfamily of \( \mathcal{F}_{F,G} \) with contamination index \( \tau \). Let \( A \geq 2 \). Then, for every \( 0 < \varepsilon < \frac{1}{2} \) with

\[
\Delta = 1 - \frac{\tau}{d_{F,G}(\mathcal{H}(Z))} - \frac{\varepsilon}{2d_{F,G}(\mathcal{H}(Z))} > 0
\]
such that

\[
\delta = \frac{\varepsilon}{20m[F : \mathbb{Q}]a_{F,G}(\mathcal{H}) + 8d_{F,G}(\mathcal{H}) + 4\varepsilon} \leq \frac{1}{2A}
\]

there exists effectively computable absolute constants \(C_1, C_2, C_3, C_4\), a constant

\[
D = D(Z, \tau, \varepsilon, A, |G|, c_F, \beta_0^{(F)}, [F : \mathbb{Q}], C_1, C_2, C_3)
\]

such that for \(X \in \mathbb{N}\), all but \(DX^{\tau+\varepsilon}\) fields \(K/F\) in

\[
\left\{ K/F \in Z; |\Delta_{K/F}^F| < X \right\}
\]

has that there exists constants

\[
k_i = k_i(\delta, A, |G|, c_F, \beta_0^{(F)}, |\Delta_F/\mathbb{Q}|, [F : \mathbb{Q}], |K^F/\mathbb{Q}|, C_3, C_4)
\]

for \(i = 1, 2, 3\) with for any conjugacy class \(C \subseteq G\), we have for all \(x \geq k_1e^{k_2(k_1\log|\Delta_{K^F/\mathbb{Q}}^{k_3}|^2)}\),

\[
\left| \pi(x, C, K^F/F) - \frac{|C|}{|G|} \text{Li}(x) \right| \leq \frac{|C|}{|G|} \frac{x}{(\log x)^A}
\]

Specialising to \(F = \mathbb{Q}\) and utilising their work on field-counting, we thus remove the reliance of Strong Artin Conjecture in their main effective Chebotarev density result under the assumption of Conjecture \(3.2.9\).
Chapter 2

A BRIEF OVERVIEW OF ARTIN L-FUNCTIONS

In this chapter, we will provide a basic overview on the study of Artin L-functions. In particular, we will place emphasis on reviewing properties of Artin L-functions that we will need later on in this thesis. We will not be exhaustive in our discussions, nor will we provide a novel approach to the subject except in §2.2.4. Experts can safely skip this chapter except §2.2.4 where we introduce convolution of Galois representations, and only referring back to this chapter as needed.

Our exposition will mainly follow the exposition of Cogdell in [Cog07], with references to [IK04] and [MM12]. We also refer to other sources such as [CF67], [Neu86], [Neu13] and [Lan13a] for technical details and other number-theoretic background.

2.1 L-Functions Before Artin

Although we will not be thorough in listing the history behind the study of Artin L-functions, we will nevertheless sketch a brief timeline, to help motivate the definition that follows. Readers who are interested in the history of the development of L-functions should consult [Cog07] and its references.

2.1.1 Riemann Zeta-Function

Let us start from the beginning: Consider the Riemann zeta-function,

\[ \zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{n^s} \]

\( \zeta(s) \) is initially defined as a function with integer domain. It is known since Euler’s time that one can write the Riemann zeta-function as an Euler product

\[ \zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \]

for \( s > 1 \). Using the divergence of harmonic series, Euler then deduced the infinitude of rational primes.

In a landmark paper [Rie59], motivated by his study for formulas for the prime-counting function, Riemann did groundbreaking work on the study of \( \zeta(s) \), which now bore his name, and formulated the beginnings of analytic number theory. Riemann first expanded \( \zeta(s) \) as a function on the complex plane \( \mathbb{C} \), and proceeded showed the existence of a meromorphic continuation of \( \zeta(s) \) to the half-plane \( \text{Re}(s) > 0 \) with a simple pole at \( s = 1 \) of residue 1. By setting

\[ Z(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) \]

Riemann also showed that \( Z(s) \) has a meromorphic continuation to the entire complex plane with simple poles \( s = 0 \) and \( s = 1 \), and that it satisfies the functional equation

\[ Z(s) = Z(1 - s) \]
Riemann also proved that \( \zeta(s) \) has infinitely many zeroes, which all non-trivial ones lie in the critical strip \( 0 \leq \text{Re}(s) \leq 1 \). By calculating the first few zeroes of \( Z(s) \), and discovering that they all lie on the critical line \( \text{Re}(s) = \frac{1}{2} \), Riemann proceeded to make his most famous conjecture, after supposedly “vergeblichen Versuchen (futile attempts)”, which still remains very much open.

**Riemann Hypothesis.** *All the non-trivial zeroes of \( \zeta(s) \) has real part \( \frac{1}{2} \).*

Riemann also sketched a proof, later proved rigorously by von Mangoldt, for his explicit formula for the (normalised) prime-counting function, of which involves non-trivial zeroes of \( \zeta(s) \). This is greatly expanded upon by Hadamard and de la Valée Poussin, culminating in the celebrated Prime Number Theorem.

**Prime Number Theorem.** *Let \( \pi(x) \) be the number of primes less than or equal to \( x \). Then

\[
\pi(x) \sim \frac{x}{\log x}
\]

With the veracity of the Riemann Hypothesis, the error term is conjectured to be of the magnitude of \( \sqrt{x} \log x \).

### 2.1.2 Dedekind Zeta-Functions

Dedekind extended the notion of zeta-function to the context of an arbitrary number field.

**Definition 2.1.1.** *Let \( K \) be a number field. A Dedekind zeta-function of \( K \) is defined by

\[
\zeta_K(s) = \sum_{a \subseteq \mathcal{O}_K} \frac{1}{N_{K/Q}(a)^s}
\]

where \( a \subseteq \mathcal{O}_K \) means that \( a \) is an ideal of \( \mathcal{O}_K \).*

Hecke proved that Dedekind zeta-functions has properties very similar to \( \zeta(s) \), which we will henceforth rename as \( \zeta_Q(s) \). Specifically,

1. Dedekind zeta-functions \( \zeta_K(s) \) has an analytic continuation to \( \text{Re}(s) > 1 - \frac{1}{|K:Q|} \) except a simple pole at \( s = 1 \) with residue

\[
\frac{2^{r_1}(2\pi)^{r_2}R}{m\sqrt{|\Delta_{K/Q}|}} h
\]

where \( h \) is the class number of \( K \), \( r_1 \) is the number of real places of \( K \), \( r_2 \) is the number of complex places of \( K \), \( R \) is the regulator, \( m \) is the number of roots of unity in \( K \), and \( |\Delta_{K/Q}| \) is the absolute discriminant of \( K \).

2. Dedekind zeta-functions \( \zeta_K(s) \) has the Euler identity for \( \text{Re}(s) > 1 \) of

\[
\zeta_K(s) = \prod_{p \subseteq \mathcal{O}_K} (1 - N_{Q/K}^K(p)^{-s})^{-1}
\]

where \( p \subseteq \mathcal{O}_K \) means that \( p \) is a prime ideal of \( \mathcal{O}_K \).

3. Dedekind zeta-functions \( \zeta_K(s) \) has a meromorphic continuation to the entire \( \mathbb{C} \), and satisfy a functional equation.

Indeed, we even have a version of Riemann Hypothesis for Dedekind zeta-functions, which we will state here as Generalised Riemann Hypothesis.

**Generalised Riemann Hypothesis.** *All the non-trivial zeroes of \( \zeta_K(s) \) has real part \( \frac{1}{2} \).*
2.1.3 Dirichlet Series and Dirichlet \( L \)-Functions

Let us go back to the original definition of \( \zeta_{\mathbb{Q}}(s) \). Dirichlet, in his study of infinitude of primes in arithmetic progression, expanded on the series form of \( \zeta_{\mathbb{Q}}(s) \) and considered a general form of the series we define below.

**Definition 2.1.2.** A *Dirichlet series* is a series of the form

\[
L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}
\]

for \( s \in \mathbb{C} \).

To be precise, Dirichlet considered Dirichlet series with \( a_n = \chi(n) \), where \( \chi : (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^* \) is a multiplicative character, called Dirichlet characters. In modern terminology, these are Dirichlet \( L \)-functions associated to Dirichlet characters \( \chi : (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^* \), defined as

\[
L(s, \chi) = \sum_{n \in \mathbb{N}} \frac{\chi(n)}{n^s} = \prod_{p \text{ prime}} \left( 1 - \chi(p)p^{-s} \right)^{-1}
\]

Dirichlet then showed the convergence and non-vanishing of \( L(s, \chi) \) at \( \text{Re}(s) = 1 \) associated to non-principal Dirichlet characters, and deduced the infinitude of primes in arithmetic progression.

2.1.4 Hecke \( L \)-Functions

The work of Weber and Hecke pushed the definitions further, resulting in a definition of a \( L \)-function of a number field \( F/\mathbb{Q} \) associated to a ray-class character \( \chi \) of \( F \). These are hugely important in the early developments of abelian class field theory, and are a direct generalisation of Dirichlet \( L \)-functions.

Due to elements of \( \mathcal{O}_F \) not having unique prime factorisation, the naive generalisation of \( \mathbb{Z}/m\mathbb{Z} \) to \( \mathcal{O}_F/\mathfrak{a} \) breaks down quite significantly. However, basic algebraic number theory has provided us with another option, which is to instead consider characters of ideal class groups.

Let \( \mathfrak{m} \) be a modulus of \( F \). We let \( I_{\mathfrak{m}} \) be the group of nonzero fractional ideals of \( F \) which do not divide \( \mathfrak{m} \), and let \( P_{\mathfrak{m}} \) be the subgroup generated by principal fractional ideals such that it is generated by an element \( \alpha \) with for all \( \mathfrak{p} \triangleleft \mathcal{O}_F \),

- \( \text{ord}_\mathfrak{p}(\alpha - 1) \geq \text{ord}_\mathfrak{p} \mathfrak{m}_0 \)
- \( \tau(\alpha) > 0 \) for all \( \tau \in \mathfrak{m}_\infty \).

We call \( C_{\mathfrak{m}} = I_{\mathfrak{m}}/P_{\mathfrak{m}} \) to be the ray class group of modulus \( \mathfrak{m} \), and this will now take the role of \( \mathbb{Z}/m\mathbb{Z} \) in the case of \( F = \mathbb{Q} \). We call the characters of \( C_{\mathfrak{m}} \) a Hecke character, which will now take the role of Dirichlet characters in the case of \( F = \mathbb{Q} \).

There is then a finite abelian extension \( F(\mathfrak{m})/F \) for each modulus \( \mathfrak{m} \), which is called the ray class field of \( F \) of modulus \( \mathfrak{m} \). By Artin–Takagi, we get that

\[
C_{\mathfrak{m}} \cong \text{Gal}(F(\mathfrak{m})/F)
\]

and that every finite abelian extension of \( F \) is contained in some ray class field of \( F \). This means that we have successfully given a \( L \)-function of an abelian extension associated to its Galois representations.
Weber and Hecke then defined the following: For a modulus \(m\) of \(F\) and a Hecke character \(\chi\), the associated \(L\)-function is defined as

\[
L(s, \chi; F) = \sum_{a \in \mathcal{O}_F} \frac{\chi(a)}{N_{\mathbb{Q}^F}(a)^s} = \prod_{p \in \mathcal{O}_F} (1 - \chi(p)N_{\mathbb{Q}^F}(p)^{-s})^{-1}
\]

Hecke then proved that these \(L\)-functions has an analytic continuation to the entire \(\mathbb{C}\) for non-principal \(\chi\), and satisfy an analogous functional equation. With this, we can state the following theorem, which relates Dedekind zeta-functions to Hecke \(L\)-functions.

**Theorem 2.1.3.** Let \(K/F\) be an abelian extension, and let \(m\) be a modulus of \(F\) such that \(K \subseteq F(m)\). Let \(H\) be the corresponding normal subgroup of \(C_m\). Then,

\[
\zeta_K(s) = \prod_{\chi \in C_m/H} L(s, \chi; F)
\]

In particular, \(\frac{\zeta_K(s)}{\zeta_F(s)}\) is entire.

## 2.2 Artin \(L\)-Functions

Class field theory gives a way of defining \(L\)-functions of an abelian extension \(K/F\) of number fields associated to (characters of) its Galois representations, by relating it to the Hecke \(L\)-functions associated to a ray class character of \(F\). We will now extend this, and define \(L\)-functions associated to Galois representations of Galois extensions, first introduced by Artin in a series of papers written in 1923 and in the 1930s.

For this section, let us set the following: Let \(K/F\) be a Galois extension of number fields, and let \(\mathcal{O}_K\) and \(\mathcal{O}_F\) be ring of integers of \(K\) and \(F\) respectively. Let \(G := \text{Gal}(K/F)\), and denote by its set of irreducible representations by \(\hat{G}\). Let \(\rho : G \rightarrow \text{GL}(V)\) be a representation of \(G\), with character

\[
\chi_\rho(g) := \text{Tr}(\rho(g))
\]

In this thesis, we will only consider complex representations.

### 2.2.1 Decomposition Groups, Inertia Groups and Ramification Groups

We begin by giving a brief exposition on decomposition groups, inertia groups and ramification groups, which are ingredients central in the definition of Artin \(L\)-functions. Readers should consult, for example, [CF67] or [Lan13a] for more details and the number-theoretical backgrounds.

Let \(p \in \mathcal{O}_F\) be a prime ideal. Recall that the Galois group \(G\) acts transitively on the set of primes in \(\mathcal{O}_K\) lying over \(p\). We define the following:

**Definition 2.2.1.** Let \(\mathfrak{p} \triangleleft \mathcal{O}_K\) be a prime lying over \(p\). The **decomposition group** of \(\mathfrak{p}\) is the subgroup

\[
D_\mathfrak{p} = \{ \sigma \in G; \sigma \mathfrak{p} = \mathfrak{p} \} = \text{Stab}_G(\mathfrak{p})
\]

The following lemma shows how the decomposition groups of all primes lying over \(p\) are related.

**Lemma 2.2.2.** The decomposition group of primes in \(\mathcal{O}_K\) lying over \(p\) are conjugates.
Proof. Let $\mathfrak{p} \triangleleft \mathcal{O}_K$ be a prime lying over $p$. Since $G$ acts transitively on primes lying over $p$, we can write any other prime $\mathfrak{p}' \triangleleft \mathcal{O}_K$ lying over $p$ as $\mathfrak{p}' = \tau \mathfrak{p}$, where $\tau \in G$. Now,

$$\sigma \in D_{\tau \mathfrak{p}} \iff \sigma \tau \mathfrak{p} = \tau \mathfrak{p} \iff \tau^{-1} \sigma \tau \mathfrak{p} = \mathfrak{p} \iff \tau^{-1} \sigma \in D_{\mathfrak{p}}$$

and hence we get that

$$D_{\tau \mathfrak{p}} = \tau D_{\mathfrak{p}} \tau^{-1}$$

and thus we get that the decomposition group of primes in $\mathcal{O}_K$ lying over $p$ are all conjugates. \qed

In the following, let $\mathfrak{p} \triangleleft \mathcal{O}_K$ be a prime lying over $p$. Consider the subfield $F \subseteq K^{D_{\mathfrak{p}}} \subseteq K$ fixed by $D_{\mathfrak{p}}$. We show that

Lemma 2.2.3. $\mathfrak{p}$ is the unique prime lying over $\mathfrak{p} \cap \mathcal{O}_{K^{D_{\mathfrak{p}}}}$. Furthermore, the injection of residue fields

$$\mathcal{O}_F / p \hookrightarrow \mathcal{O}_{K^{D_{\mathfrak{p}}}} / \mathfrak{p} \cap \mathcal{O}_{K^{D_{\mathfrak{p}}}}$$

is an isomorphism.

Proof. First, note that $D_{\mathfrak{p}} = \text{Gal}(K/K^{D_{\mathfrak{p}}})$ acts transitively on primes in $\mathcal{O}_L$ lying over $\mathfrak{p} \cap \mathcal{O}_{K^{D_{\mathfrak{p}}}}$, which includes $\mathfrak{p}$. However, since $D_{\mathfrak{p}}$ fixes $\mathfrak{p}$, we get that therefore $\mathfrak{p}$ is the only prime lying over $\mathfrak{p} \cap \mathcal{O}_{K^{D_{\mathfrak{p}}}}$.

Further, first note that for any $\sigma \in G \setminus D_{\mathfrak{p}}$, we get that $\sigma \mathfrak{p} \neq \mathfrak{p}$, which implies that

$$\sigma \mathfrak{p} \cap \mathcal{O}_{K^{D_{\mathfrak{p}}}} \neq \mathfrak{p} \cap \mathcal{O}_{K^{D_{\mathfrak{p}}}}$$

Hence, for any $x \in \mathcal{O}_{K^{D_{\mathfrak{p}}}}$, there exists a $y \in \mathcal{O}_{K^{D_{\mathfrak{p}}}}$ such that

$$y \equiv x \pmod{\mathfrak{p} \cap \mathcal{O}_{K^{D_{\mathfrak{p}}}}} \quad \text{and} \quad y \equiv 1 \pmod{\mathfrak{p} \cap \mathcal{O}_{K^{D_{\mathfrak{p}}}}}$$

$$\implies y \equiv x \pmod{\mathfrak{p}} \quad \text{and} \quad \sigma y \equiv 1 \pmod{\mathfrak{p}}$$

for all $\sigma \in G \setminus D_{\mathfrak{p}}$. In particular, for $\bar{\sigma} \in G / D_{\mathfrak{p}}$ with $\bar{\sigma} \neq 1$, we get that

$$\bar{\sigma}(y) \equiv 1 \pmod{\mathfrak{p}}$$

Thus, $N_{F/K^{D_{\mathfrak{p}}}}(y) \in \mathcal{O}_F$ such that

$$N_{F/K^{D_{\mathfrak{p}}}}(y) = \prod_{\bar{\sigma} \in G / D_{\mathfrak{p}}} \bar{\sigma}(y) \equiv x \pmod{\mathfrak{p}}$$

and this shows the surjectivity of $\mathcal{O}_F / p \hookrightarrow \mathcal{O}_{K^{D_{\mathfrak{p}}}} / \mathfrak{p} \cap \mathcal{O}_{K^{D_{\mathfrak{p}}}}$. \qed

Note that by Orbit-Stabiliser Theorem and since $G$ acts transitively on primes in $\mathcal{O}_L$ lying over $p$, we get that the number of primes in $\mathcal{O}_L$ lying over $p$ is $[G : D_{\mathfrak{p}}] = [K : K^{D_{\mathfrak{p}}}]$. We thus get the following proposition.

Proposition 2.2.4. $p$ splits completely and does not ramify in $K^{D_{\mathfrak{p}}}$. 

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CHAPTER 2. A BRIEF OVERVIEW OF ARTIN L-FUNCTIONS

Here, we see the usefulness of the decomposition group, by allowing us to study the extension $K/F$ as a tower of extensions where the splitting behaviour of $p$ in each extension is well-understood.

Consider now the residue fields $k_{\mathfrak{p}} = \mathcal{O}_K/\mathfrak{p}$ and $F_p = \mathcal{O}_F/p$. For any $\sigma \in D_{\mathfrak{p}}$, $\sigma$ descends to a well-defined automorphism of $k_{\mathfrak{p}}$ which fixes $F_p$, and hence there exists a natural homomorphism

$$D_{\mathfrak{p}} \to \text{Gal}(k_{\mathfrak{p}}/F_p)$$

In fact, more is true, as demonstrated by the following proposition.

**Proposition 2.2.5.** The natural homomorphism

$$D_{\mathfrak{p}} \to \text{Gal}(k_{\mathfrak{p}}/F_p)$$

is surjective.

Let us define the following.

**Definition 2.2.6.** Let $\mathfrak{p} \triangleleft \mathcal{O}_K$ a prime lying over $p$. The **inertia group** of $\mathfrak{p}$ is the group

$$I_{\mathfrak{p}} = \ker(D_{\mathfrak{p}} \to \text{Gal}(k_{\mathfrak{p}}/F_p))$$

Along with Proposition 2.2.5, we get the following exact sequence

$$1 \longrightarrow I_{\mathfrak{p}} \longrightarrow D_{\mathfrak{p}} \longrightarrow \text{Gal}(k_{\mathfrak{p}}/F_p) \longrightarrow 1$$

It is also therefore clear that $|I_{\mathfrak{p}}| = e(\mathfrak{p}/p)$, the ramification index of $\mathfrak{p}$ over $p$. We also make the following observation.

**Proposition 2.2.7.** The inertia group of $\mathfrak{p}$ is the subgroup of $G$ defined by

$$I_{\mathfrak{p}} = \{ \sigma \in G; \sigma(a) \equiv a \pmod{\mathfrak{p}}, \forall a \in \mathcal{O}_K \}$$

**Proof.** For $\sigma \notin D_{\mathfrak{p}}$, we get that $\sigma^{-1} \mathfrak{p} \neq \mathfrak{p}$. Since $\mathfrak{p}$ and $\sigma^{-1} \mathfrak{p}$ are maximal in $\mathcal{O}_K$, we get an element $a \in \mathfrak{p}$ such that $a \notin \sigma^{-1} \mathfrak{p}$. Thus, $\sigma(a) \neq a \pmod{\mathfrak{p}}$, and the result follows. $\square$

Indeed, the inertia groups of primes in $\mathcal{O}_K$ lying over $p$ are intimately related such as the case of its decomposition groups.

**Lemma 2.2.8.** The inertia group of primes in $\mathcal{O}_K$ lying over $p$ are conjugates.

**Proof.** Let $\mathfrak{p}, \mathfrak{p}' \triangleleft \mathcal{O}_K$ be primes lying over $p$. By Lemma 2.2.2, let $\sigma \in G$ be such that $\sigma \mathfrak{p} = \mathfrak{p}'$ and $D_{\mathfrak{p}'} = \sigma D_{\mathfrak{p}} \sigma^{-1}$. Then, by Proposition 2.2.7, we get that

$$\tau \in I_{\mathfrak{p}'} \iff \tau \sigma(a) \equiv \sigma(a) \pmod{\mathfrak{p}'}, \forall a \in \mathcal{O}_K$$

$$\iff \sigma^{-1} \tau \sigma(a) \equiv a \pmod{\mathfrak{p}}, \forall a \in \mathcal{O}_K \iff \sigma^{-1} \tau \sigma \in I_{\mathfrak{p}}$$

and hence, we get that $I_{\mathfrak{p}'} = \sigma I_{\mathfrak{p}} \sigma^{-1}$. $\square$

Now, recall that $\text{Gal}(k_{\mathfrak{p}}/F_p)$ is cyclic with the generator $x \mapsto x^{N\mathfrak{p}/p}$. We define the following.

**Definition 2.2.9.** The **Frobenius element** at $\mathfrak{p}$ is an element $\sigma_{\mathfrak{p}} \in D_{\mathfrak{p}}$ that is mapped to $x \mapsto x^{N\mathfrak{p}/p}$ under the map $D_{\mathfrak{p}} \to \text{Gal}(k_{\mathfrak{p}}/F_p)$. 

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Note that the Frobenius element is only defined mod $I_{\mathfrak{p}}$. If however $p$ is unramified in $K$, then $I_{\mathfrak{p}} = \{ e \}$, and in that case the Frobenius elements are defined, and further they are in the same conjugacy class by Lemma 2.2.2. In particular, this is true for all but finitely many primes of $K$. In this case, we can make the following well-defined definition.

**Definition 2.2.10.** Suppose $p \nmid O_F$ is unramified in $K$. The **Frobenius element** $\sigma_p$ of $p$ is the conjugacy class of $\sigma_{\mathfrak{p}}$, where $\mathfrak{p} \nmid O_K$ lies over $p$.

For our purpose in Section 2.2.4, we shall also define the following.

**Definition 2.2.11.** Let $\mathfrak{p} \nmid O_K$ be a prime lying over $p$. Let $i \geq -1$ be an integer. The $i$-th **ramification group** of $\mathfrak{p}$ is the group

$$G_i(\mathfrak{p}/p) = \{ \sigma \in G; \sigma(a) \equiv a \pmod{\mathfrak{p}^{i+1}}, \forall a \in O_K \}$$

We note that $G_{-1}(\mathfrak{p}/p) = G$ and $G_0(\mathfrak{p}/p) = I_{\mathfrak{p}}$, and that each $G_i(\mathfrak{p}/p)$ are normal in $G$. We also note that ramification groups are generalisations of the inertia group via Proposition 2.2.7. In later sections, we will consider the descending series of ramification groups

$$G = G_{-1}(\mathfrak{p}/p) \supseteq G_0(\mathfrak{p}/p) \supseteq G_1(\mathfrak{p}/p) \supseteq \cdots$$

For some background in ramification groups, c.f. [CF67] and [Lan13a] for an exposition. For our purpose, we wish to highlight the following result, regarding ramification groups of composite fields.

**Proposition 2.2.12.** Let $K/F$ and $K'/F$ be Galois extensions of number fields, with $K \subseteq K'$. Let $p$ be a prime in $F$ and let $\mathfrak{p}$ be a prime in $K'$ lying over $p$. Then, for $i \geq 0$,

$$\frac{|G_0(\mathfrak{p}/p)|}{|G_i(\mathfrak{p}/p)|} \geq \frac{|G_0((\mathfrak{p} \cap O_K)/p)|}{|G_i((\mathfrak{p} \cap O_K)/p)|}$$

**Proof.** The quotient map

$$q : \text{Gal}(K'/F) \to \text{Gal}(K/F)$$

induces a surjection

$$q : G_0(\mathfrak{p}/p) \to G_0((\mathfrak{p} \cap O_K)/p)$$

Composing with the quotient map

$$\pi_{G_i((\mathfrak{p} \cap O_K)/p)} : G_0((\mathfrak{p} \cap O_K)/p) \to G_0((\mathfrak{p} \cap O_K)/p)/G_i((\mathfrak{p} \cap O_K)/p)$$

we get an isomorphism

$$G_0(\mathfrak{p}/p)/\ker(\pi_{G_i((\mathfrak{p} \cap O_K)/p)} \circ q) \cong G_0((\mathfrak{p} \cap O_K)/p)/G_i((\mathfrak{p} \cap O_K)/p)$$

Finally, note that since $q(G_i(\mathfrak{p}/p)) \subseteq G_i((\mathfrak{p} \cap O_K)/p)$, we get that

$$G_i(\mathfrak{p}/p) \subseteq \ker(\pi_{G_i((\mathfrak{p} \cap O_K)/p)} \circ q)$$

and thus the result follows.

We conclude this discussion with an example, produced using [LMF13].
Example 2.2.1. Consider the extension $K/\mathbb{Q}$, where $K$ is the splitting field of the polynomial
\[ f(x) = x^4 + x + 1 \]
Then, $|\Delta_K| = 229$, and so by basic number theory, the only prime of $\mathbb{Q}$ that ramifies in $K$ is the prime 229.

Let $r_1, r_2, r_3, r_4$ be the four distinct roots of $f(x)$. Now, we compute the Galois group $\text{Gal}(K/\mathbb{Q}) \leq S_4$ of $K/\mathbb{Q}$. Clearly, the order of the Galois group is divisible by 4. Furthermore, the cubic resolvent of $f(x)$ is $x^3 - 4x + 1$, which is irreducible mod 3, hence it is irreducible. Thus, $K$ contains the cubic field $\mathbb{Q}(r_1r_2 + r_3r_4)$, and therefore the order of the $\text{Gal}(K/\mathbb{Q})$ is also divisible by 3. This means that $\text{Gal}(K/\mathbb{Q})$ is either $A_4$ or $S_4$. Finally, since the discriminant of $f(x)$ is 229, which is crucially not a square, we therefore must have that $\text{Gal}(K/\mathbb{Q}) = S_4$.

By brute force computation, we find that $K = \mathbb{Q}(\alpha)$, where $\alpha$ is the root of
\[ f(x) = 266962921 + 2114616240x^2 + 67773664x^4 + 74190340x^6 + 50899280x^8 - 462400x^{10} \\
- 973378x^{12} + 23120x^{14} + 7520x^{16} + 340x^{18} - 80x^{20} + x^{24} \]
Reducing $f(x)$ modulo 229 gives us that
\[ f(x) = (x^2 + 29)^2(x^2 + 32)(x^2 + 71x + 174)^2(x^2 + 87x + 32)^2(x^2 + 142x + 32)^2(x^2 + 158x + 174)^2 \pmod{229} \]
Hence, (229) in $\mathcal{O}_K$ factorises as
\[ (229) = \prod_{i=1}^{6} \mathfrak{p}_i^2 \]
where
\begin{align*}
\mathfrak{p}_1 &= (229) + (\alpha^2 + 29) \\
\mathfrak{p}_2 &= (229) + (\alpha^2 + 32) \\
\mathfrak{p}_3 &= (229) + (\alpha^2 + 71\alpha + 174) \\
\mathfrak{p}_4 &= (229) + (\alpha^2 + 87\alpha + 32) \\
\mathfrak{p}_5 &= (229) + (\alpha^2 + 142\alpha + 32) \\
\mathfrak{p}_6 &= (229) + (\alpha^2 + 158\alpha + 174) 
\end{align*}
Thus, we can now conclude that the ramification and residual index of (229) are both 2. Therefore, choosing $\mathfrak{p}$ to be any of the six primes lying over (229), we get that $I_\mathfrak{p} = \mathbb{Z}/2\mathbb{Z}$ and $D_\mathfrak{p}$ is one of $(\mathbb{Z}/2\mathbb{Z})^2$ or $\mathbb{Z}/4\mathbb{Z}$ (it turns out to be the former).

### 2.2.2 Unramified Factors of Artin $L$-Functions

In this section, we suppose the prime $\mathfrak{p}$ is unramified in $K$. Then, we get a well-defined Frobenius element $\sigma_\mathfrak{p}$. As $\sigma_\mathfrak{p}$ is a conjugacy class, we get that any class function $\chi$ of $G$ is constant on $\sigma_\mathfrak{p}$, and hence we denote this value by $\chi(\sigma_\mathfrak{p})$.

We can now define the unramified factor of an Artin $L$-function.

**Definition 2.2.13.** Let $\rho : \text{Gal}(K/F) \rightarrow \text{GL}(V)$ be a representation of $\text{Gal}(K/F)$, and let $\chi_\rho$ be its associated character. We define
\[ L_\mathfrak{p}(s, \rho; K/F) = \det (I - \rho(\sigma_\mathfrak{p}) N^F_{\mathbb{Q}}(\mathfrak{p})^{-s})^{-1} \]
and
\[
L_{\text{unram}}(s, \rho; K/F) = L_{\text{unram}}(s, \chi_{\rho}; K/F) = \prod_{p \in \mathcal{O}_F \text{ unramified in } K} L_p(s, \rho; K/F) = \prod_{p \in \mathcal{O}_K \text{ unramified in } L} \det(I - \rho(\sigma_p)N_{\mathbb{Q}}^F(p)^{-s})^{-1}
\]

where \(\rho(\sigma_p) = \rho(\mathfrak{p})\) for any prime \(\mathfrak{p}\) lying over \(p\).

We note that there is no ambiguity in the definition: for any \(\tau \in G\), we get that
\[
\det(I - \rho(\tau \sigma_{\mathfrak{p}}^{-1})N_{\mathbb{Q}}^F(p)^{-s}) = \det(\rho(\tau)) \det(I - \rho(\mathfrak{p})N_{\mathbb{Q}}^F(p)^{-s}) \det(\rho(\tau))^{-1} = \det(I - \rho(\mathfrak{p})N_{\mathbb{Q}}^F(p)^{-s})
\]

Remark 2.2.14: Note that, since \(\rho(\sigma_p)\) is a linear transformation of finite order on \(V\), the eigenvalues \(\lambda_i(p)\) of \(\rho(\sigma_p)\) are thus roots of unity. Hence, we get that
\[
L_p(s, \rho; K/F) = \det(I - \rho(\sigma_p)N_{\mathbb{Q}}^F(p)^{-s})^{-1} = \prod_{i=1}^{\dim \rho} (1 - \lambda_i(p)N_{\mathbb{Q}}^F(p)^{-s})^{-1}
\]

where \(|\lambda_i(p)| = 1\).

For each unramified prime \(p\), note that
\[
\det(I - \rho(\sigma_p)N_{\mathbb{Q}}^F(p)^{-s})^{-1} = \det \left( e^{-\log(I - \rho(\sigma_p)N_{\mathbb{Q}}^F(p)^{-s})} \right) = \det \left( e^{-\sum_{l \geq 1} \frac{\rho(\sigma_p^l)}{N_{\mathbb{Q}}^F(p^l)^s}} \right) = e^{-\sum_{l \geq 1} \frac{\rho(\sigma_p^l)}{N_{\mathbb{Q}}^F(p^l)^s}} = e^{-\sum_{l \geq 1} \chi_{\rho}(\sigma_p^l)}
\]

\[\implies \log \det(I - \rho(\sigma_p)N_{\mathbb{Q}}^F(p)^{-s})^{-1} = -\sum_{l \geq 1} \frac{\chi_{\rho}(\sigma_p^l)}{N_{\mathbb{Q}}^F(p^l)^s}\]

and here we see the similarities of the above with the Hecke \(L\)-functions. In fact, this is probably where Artin started, leading to the definition we quoted above in his 1924 paper [Art24].

Example 2.2.2. Consider the trivial representation \(1\) of \(G\). Then, we get that
\[
L_{\text{unram}}(s, 1, K/F) = \prod_{p \in \mathcal{O}_K \text{ unramified in } L} (1 - N_{\mathbb{Q}}^F(p)^{-s})^{-1}
\]

which, up to finite number of Euler products, is \(\zeta_K(s)\). Indeed, once we define the Euler products at the ramified primes, we will see that we do get \(\zeta_K(s)\).

The unramified factors of Artin \(L\)-functions satisfy desired properties, as demonstrated by the next few statements.

Proposition 2.2.15. \(L_{\text{unram}}(s, \rho; K/F)\) is regular for \(\Re(s) > 1 + \delta\), for all \(\delta > 0\).
Proposition 2.2.16. $L_{\text{unram}}$ are additive: for representations $\rho_1, \rho_2$, we get that

$$L_{\text{unram}}(s, \rho_1 \oplus \rho_2; K/F) = L_{\text{unram}}(s, \rho_1; K/F)L_{\text{unram}}(s, \rho_2; K/F)$$

Proof. We recall that $\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$. The result follows by considering $\log L_{\text{unram}}(s, \rho_1 \oplus \rho_2; K/F)$. □

Proposition 2.2.17. $L_{\text{unram}}$ satisfies inflation: Let $E/K/F$ be extensions such that $E/K$ is also Galois. Let

$$\pi_{\text{Gal}(E/K)} : \text{Gal}(E/F) \twoheadrightarrow G$$

be the natural surjection. For $\rho : G \rightarrow \text{GL}(V)$ a representation of $G$, let $\tilde{\rho}$ be the representation of $\text{Gal}(E/K)$ given by

$$1 \xrightarrow{} \text{Gal}(E/K) \xrightarrow{} \text{Gal}(E/F) \xrightarrow{\pi_{\text{Gal}(E/K)}} G \xrightarrow{} 1$$

That is, $\tilde{\rho} = \rho \circ \pi_{\text{Gal}(E/K)}$. Then, we get that

$$L_{\text{unram}}(s, \rho; K/F) = L_{\text{unram}}(s, \tilde{\rho}, E/F) = L(s, \rho \circ \pi_{\text{Gal}(E/L)}; E/F)$$

Proposition 2.2.18. $L_{\text{unram}}$ is inductive: Let $E$ be an intermediate field, i.e. $F \subseteq E \subseteq K$. For $\rho : \text{Gal}(K/E) \rightarrow \text{GL}(V)$ a representation of $\text{Gal}(K/E) \leq G$, and consider the induced representation $\text{Ind}^G_{\text{Gal}(K/E)} \rho$. Then, we get that

$$L_{\text{unram}}(s, \rho; K/E) = L_{\text{unram}}(s, \text{Ind}^G_{\text{Gal}(K/E)} \rho, K/F)$$

We will not give a proof of Proposition 2.2.18 here, refering the interested reader to, for example, Lang’s [Lan13a] or Heilbronn’s essay in [CF67].

We conclude this section by remarking the propositions above are really the guiding principles behind Artin’s definition of his $L$-functions, along with the demand that it agrees with the Hecke $L$-functions in the abelian case.

2.2.3 **Ramified and Archimedean Factors of Artin $L$-Functions**

In the previous section, we defined Artin $L$-functions at unramified primes of $K$. This accounts for all but finitely many Euler factors, but we still need to define the factors at the finitely many primes of $F$ which ramifies in $K$ to complete the definition of Artin $L$-functions.

However, at a ramified prime $p$ with $\mathfrak{p}$ a prime lying over $p$, we note that the inertia group $I_{\mathfrak{p}}$ of $\mathfrak{p}$ is non-trivial, as the size of the inertia group corresponds to the ramification index $e(\mathfrak{p}/p)$. Thus, the Frobenius element $\sigma_{\mathfrak{p}}$ is a coset, rather than an element of the decomposition group $D_{\mathfrak{p}}$.

Thus, in order to have a good definition of the Euler product at $p$, we need to instead consider a subspace of $V$ such that $I_{\mathfrak{p}}$ acts trivially on this subspace, and that it is compatible with our definition of Euler products at unramified primes. For this, we simply consider the subspace

$$V^{I_{\mathfrak{p}}} := \{ v \in V; \rho(g)v = v, \forall g \in I_{\mathfrak{p}} \}$$
and we note that this is indeed compatible with the definition of Euler product at unramified primes, for in that case
\[ I_\mathfrak{p} = \{ e \} \implies V^I_\mathfrak{p} = V \]
Thus, for any \( g_1, g_2 \in \sigma_\mathfrak{p} \), we get that
\[ \rho(g_1)|_{V^I_\mathfrak{p}} = \rho(g_2)|_{V^I_\mathfrak{p}} \]
so we define \( \rho(\sigma_\mathfrak{p})|_{V^I_\mathfrak{p}} \) to be this value. Finally, for another \( \mathfrak{p}' \) lying over \( \mathfrak{p} \), since \( \sigma_\mathfrak{p} \) and \( \sigma_{\mathfrak{p}'} \) are conjugates (mod \( I_\mathfrak{p} \)), we get that equality of their characteristic polynomial
\[
\det \left( xI - \rho(\sigma_\mathfrak{p})|_{V^I_\mathfrak{p}} \right) = \det \left( xI - \rho(\sigma_{\mathfrak{p}'})|_{V^I_{\mathfrak{p}'}} \right)
\]
Hence, we arrive at the definition given by Artin in [Art31b].

**Definition 2.2.19.** Let \( \rho : \text{Gal}(K/F) \to \text{GL}(V) \) be a representation of \( \text{Gal}(K/F) \), and let \( \chi_\rho \) be its associated character. The **Artin L-function** associated to \( \rho \) is
\[
L(s, \rho; K/F) = L(s, \chi_\rho; K/F) = \prod_{\mathfrak{p} \subset \mathcal{O}_K} \det \left( I - \rho(\sigma_\mathfrak{p})|_{V^I_\mathfrak{p}} \right) N^F_\mathfrak{p} (p)^{-s})^{-1}
\]
where \( \mathfrak{p} \) is any prime lying over \( \mathfrak{p} \). We denote
\[
L_\mathfrak{p}(s, \rho; K/F) = \det \left( I - \rho(\sigma_\mathfrak{p})|_{V^I_\mathfrak{p}} \right) N^F_\mathfrak{p} (p)^{-s})^{-1}
\]

**Remark 2.2.20:** As in Remark 2.2.14 note that, since \( \rho(\sigma_\mathfrak{p})|_{V^I_\mathfrak{p}} \) is a linear transformation of finite order on \( V^I_\mathfrak{p} \), the eigenvalues \( \lambda_i(\mathfrak{p}) \) of \( \rho(\sigma_\mathfrak{p})|_{V^I_\mathfrak{p}} \) are thus roots of unity. Hence, we get that
\[
L_\mathfrak{p}(s, \rho; K/F) = \det \left( I - \rho(\sigma_\mathfrak{p})|_{V^I_\mathfrak{p}} \right) N^F_\mathfrak{p} (p)^{-s})^{-1} = \prod_{i=1}^{\dim \rho} (1 - \lambda_i(\mathfrak{p}) N^F_\mathfrak{p} (p)^{-s})^{-1}
\]
where \( |\lambda_i(\mathfrak{p})| = 1 \).

Artin then proceeded to prove the following proposition, mirroring those of \( L_{\text{unram}} \).

**Proposition 2.2.21.** The Artin L-functions are additive, satisfy inflation and are inductive.

This result can be deduced similarly to the case of \( L_{\text{unram}} \).

As a consequence of additivity and inductivity, we get the following pleasing result.

**Corollary 2.2.22.** Let \( \text{reg}_G \) be the regular representation of \( G \). Then
\[
\zeta_K(s) = L(s, 1_G; K/K) = L(s, \text{reg}_G; K/F) = \prod_{\rho \in G} L(s, \rho; K/F)^{\chi_\rho(e)} = \zeta_F(s) \prod_{\rho \in G \setminus \{ 1_G \}} L(s, \rho; K/F)^{\chi_\rho(e)}
\]

For completion, we will define the completed Artin L-functions, where Artin defined by trying to generalise the L-function for abelian extension. First, we define the Archimedean factors: Let \( v \) be an Archimedean prime of \( F \), with \( w \) a prime lying over \( v \) in \( K \). We set
\[
\gamma_v(s, \rho; K/F) = \begin{cases} (\pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) )^{n+} \left( \pi^{-\frac{s+1}{2}} \Gamma \left( \frac{s+1}{2} \right) \right)^{n-}, & \text{if } v \text{ is real} \\ ((2\pi)^{-s} \Gamma(s))^{\dim(\rho)}, & \text{if } v \text{ is complex} \end{cases}
\]
where \( n_+ \) and \( n_- \) is the dimension of the eigenspace of \( V \) associated the the eigenvalues 1 and -1 of \( \rho(g) \) respectively, \( g \) the generator of \( D_w \) (of order 1 or 2). We denote

\[
\gamma(s, \rho; K/F) = \prod_{v \mid \infty} \gamma_v(s, \rho; K/F)
\]

Next, let \( p \) be a prime of \( F \), with \( \mathfrak{p} \) a prime lying over \( p \) in \( K \). Let

\[
G_0(\mathfrak{p}/p) \supseteq G_1(\mathfrak{p}/p) \supseteq \cdots
\]

be the decreasing series of ramification groups of \( \mathfrak{p} \). We define

\[
n(\rho, p; K/F) = \sum_{i=0}^{\infty} \frac{|G_i(\mathfrak{p}/p)|}{|G_0(\mathfrak{p}/p)|} \operatorname{codim} V^{G_i(\mathfrak{p}/p)}
\]

where \( V^{G_i(\mathfrak{p}/p)} \) is the subspace of \( V \) fixed by \( \rho(g) \), for all \( g \in G_i(\mathfrak{p}/p) \). Artin showed in [Art31a] that \( n(\rho, p; K/F) \) is a well-defined integer, 0 for all but finitely many of them.

We can now define the following.

**Definition 2.2.23.** The **Artin conductor** of \( \rho \) is the ideal

\[
f(\rho; K/F) = \prod_{p} p^{n(\rho, p; K/F)}
\]

The **conductor** of \( L(s, \rho; K/F) \) is defined as

\[
A(\rho; K/F) = |\Delta_{F/Q}|^{\dim_{F} \rho} N_{\mathbb{Q}}(f(\rho; K/F))
\]

where \( \Delta_{F/Q} \) is the absolute discriminant of \( F \).

One can show that the Artin conductor satisfy the same relations as its associated Artin \( L \)-functions.

**Proposition 2.2.24.** The Artin conductor is additive and satisfy inflation. It is also inductive, in the following sense: Let \( E \) be an intermediate field, i.e. \( F \subseteq E \subseteq K \). For \( \rho : \text{Gal}(K/E) \to \text{GL}(V) \) a representation of \( \text{Gal}(K/E) \leq G \), and consider the induced representation \( \text{Ind}_{\text{Gal}(K/E)}^{G}(\rho) \). Then, we get that

\[
f(\text{Ind}_{\text{Gal}(K/E)}^{G}(\rho); K/F) = \Delta_{E/F}^{\dim_{E} \rho} N_{F}^{E}(f(\rho, K/E))
\]

where \( \Delta_{E/F} \) is the relative discriminant of \( E/F \).

A proof of this can be found in [Neu13]. An important corollary, as a consequence of inducitivity applied to \( 1_{\text{Gal}(K/K)} \) and additivity, is the following famous formula of Artin [Art31a].

**Conductor-Discriminant Formula.** For Galois extension \( K/F \) with Galois group \( G \), we get that

\[
\Delta_{K/F} = f(\text{reg}_{G}; K/F) = \prod_{\rho \in G} f(\rho; K/F)^{\dim \rho}
\]

We also get the following corollary regarding the conductor of Artin \( L \)-functions.

**Corollary 2.2.25.** The conductor of Artin \( L \)-function is additive, satisfies inflation and is inductive.
Definition 2.2.26. The completed Artin L-function associated to $\rho$ is

$$\Lambda(s, \rho; K/F) = A(\rho; K/F)^{\frac{1}{2}}\gamma(s, \rho; K/F)L(s, \rho; K/F)$$

Here, we summarise the result of Artin in [Art31a].

Theorem 2.2.27. The completed Artin L-function $\Lambda(s, \rho; K/F)$ is additive, satisfies inflation, inductive, and satisfies the functional equation

$$\Lambda(s, \rho; K/F) = W(\rho)\Lambda(1-s, \bar{\rho}; K/F)$$

where $W(\rho)$ is a constant of absolute value 1. Furthermore, $\Lambda(s, \rho; K/F)$ has a meromorphic continuation to the whole complex plane.

A proof of this fact can be found in [Mar77].

We conclude this section by remarking about the number $W(\rho)$.

Definition 2.2.28. The Artin root number of $\rho$ is $W(\rho)$.

The Artin root number is a mysterious quantity. $W(\rho)$ carries deep arithmetic information, such as information regarding its Galois module structure. Its factorisation was found by Deligne in 1974 and Langlands, much after they were described by Artin. We refer the interested reader to [Frö12] and [Mar77].

2.2.4 Convolution of Artin L-Functions

We wish to now define a convolution on Galois representations, such that the associated Artin L-function "mirrors" the Rankin-Selberg convolution under the assumption of Strong Artin Conjecture. We thus define the following.

Definition 2.2.29. Let $K/F, K'/F'$ be Galois extensions of number fields, and let $\rho : \text{Gal}(K/F) \to \text{GL}(V)$ and $\tau : \text{Gal}(K'/F') \to \text{GL}(W)$ be representations. For number fields $\mathcal{R} \subseteq F \cap F'$, the convolution over $\mathcal{R}$ of $\rho$ and $\tau$ is defined as

$$\text{Conv}_{\mathcal{R}}(\rho, \tau) = \text{Res}_{\text{Gal}(\overline{K}/\mathcal{R}) \times \text{Gal}(\overline{K'}/\mathcal{R})} \left( \text{Ind}_{\text{Gal}(\overline{K}/\mathcal{R})}^{\text{Gal}(\overline{K}/F)} \rho \circ \pi_{\text{Gal}(\overline{K}/K)} \right) \otimes \left( \text{Ind}_{\text{Gal}(\overline{K'}/\mathcal{R})}^{\text{Gal}(\overline{K'}/F')} \tau \circ \pi_{\text{Gal}(\overline{K'}/K')} \right)$$

where $\overline{K}$, $\overline{K'}$ are the Galois closure of $K$ and $K'$ respectively over $\mathcal{R}$.

The definition is a little tedious, but it comes from the following (hopefully) more intuitive picture.

![](image.png)

Figure 2.2.1: Hasse Diagram For Definition of Conv$_{\mathcal{R}}$
Here and henceforth, we denoted
\[ \overline{\rho^\tau} = \Ind_{\Gal(K'/\mathcal{R})}^{\Gal(K'/F')} \rho \circ \Ind_{\Gal(K'/K)}^{\Gal(K'/\mathcal{R})} \] and
\[ \overline{\tau^\rho} = \Ind_{\Gal(K'/\mathcal{R})}^{\Gal(K'/F')} \tau^\rho \circ \Ind_{\Gal(K'/K)}^{\Gal(K'/\mathcal{R})} \]

We will say more about this convolution in the later chapters. For now, we wish to prove an upper bound for the conductor of convolutions. The heart of the matter is the following proposition.

**Proposition 2.2.30.** Let $K/F, K'/F'$ be Galois extensions of number fields, and let $\rho : \Gal(K/F) \to \GL(V)$ and $\tau : \Gal(K'/F') \to \GL(W)$ be representations. Let $p$ be a prime in $\mathcal{O}_\mathcal{R}$, and let $\mathfrak{P}$ be a prime in $\mathcal{O}_{K'/K}$ lying over $p$. Then, we have that
\[ n(\text{Conv}_\mathcal{R}(\rho, \tau), p; \overline{K^\tau/K^\rho} / \mathcal{R}) \leq (\dim \tau)[F' : \mathfrak{R}] n(\overline{\rho^\tau}, p; \overline{K^\tau/\mathcal{R}}) + (\dim \rho)[F : \mathfrak{R}] n(\overline{\tau^\rho}, p; \overline{K^\rho/\mathcal{R}}) \]

**Proof.** Without loss of generality, we assume that $\mathfrak{R} = F$, so that $\overline{K^\tau/K^\rho} = K$, $\overline{K^\rho/K} = K'$. First, we observe that for $i \geq 0$, the natural quotient map
\[ q_K : \Gal(KK'/F) \to \Gal(K/F) \]
gives a corresponding surjection
\[ G_i(\mathfrak{P}/p) \to G_i((\mathfrak{P} \cap \mathcal{O}_K)/p) \]
As such, by considering $\Gal(KK'/F) \leq \Gal(K/F) \times \Gal(K'/F)$, we get that
\[ G_i(\mathfrak{P}/p) \leq G_i((\mathfrak{P} \cap \mathcal{O}_K)/p) \times G_i((\mathfrak{P} \cap \mathcal{O}_{K'})/p) \]
\[ \implies (V \otimes W^*)^{G_i(\mathfrak{P}/p)} \subseteq (V \otimes W^*)^{G_i((\mathfrak{P} \cap \mathcal{O}_K)/p) \times G_i((\mathfrak{P} \cap \mathcal{O}_{K'})/p)} \subseteq V^{G_i((\mathfrak{P} \cap \mathcal{O}_K)/p)} \times (W^*)^{G_i((\mathfrak{P} \cap \mathcal{O}_{K'})/p)} \]
\[ \implies \text{codim}((V \otimes W^*)^{G_i(\mathfrak{P}/p)}) \leq (\dim V)(\dim W^*) - \dim(V^{G_i((\mathfrak{P} \cap \mathcal{O}_K)/p)}) \dim((W^*)^{G_i((\mathfrak{P} \cap \mathcal{O}_{K'})/p)}) \]
\[ \leq \text{codim}(V^{G_i((\mathfrak{P} \cap \mathcal{O}_K)/p)}) \dim W + \text{codim}((W^*)^{G_i((\mathfrak{P} \cap \mathcal{O}_{K'})/p)}) \dim V \]

Thus, combining with Proposition 2.2.12, we get that
\[ n(\rho \otimes \tau^*, p; KK'/F) = \sum_{i=0}^{\infty} \frac{|G_i((\mathfrak{P} \cap \mathcal{O}_K)/p)|}{|G_0((\mathfrak{P} \cap \mathcal{O}_K)/p)|} \text{codim}((V \otimes W^*)^{G_i(\mathfrak{P}/p)}) \]
\[ \leq \dim W \sum_{i=0}^{\infty} \frac{|G_i((\mathfrak{P} \cap \mathcal{O}_K)/p)|}{|G_0((\mathfrak{P} \cap \mathcal{O}_K)/p)|} \text{codim}(V^{G_i((\mathfrak{P} \cap \mathcal{O}_K)/p)}) \]
\[ + \dim V \sum_{i=0}^{\infty} \frac{|G_i((\mathfrak{P} \cap \mathcal{O}_{K'})/p)|}{|G_0((\mathfrak{P} \cap \mathcal{O}_{K'})/p)|} \text{codim}((W^*)^{G_i((\mathfrak{P} \cap \mathcal{O}_{K'})/p)}) \]
\[ = (\dim \tau)n(\rho, p; K/F) + (\dim \rho)n(\tau^*, p; K'/F) \]

which is what we wished to prove. \( \square \)

We will point out that the above inequality is probably not sharp. See \[BH97\] for a possible candidate for a sharp upper bound. It will however be sufficient for our purposes.

A direct consequence of the preceding proposition is the following result regarding upper bounds on the conductor of convolutions.
Corollary 2.2.31. Let $K/F, K'/F'$ be Galois extensions of number fields, and let $\rho : \text{Gal}(K/F) \to \text{GL}(V)$ and $\tau : \text{Gal}(K'/F') \to \text{GL}(W)$ be representations. For number fields $\mathfrak{R} \subseteq F \cap F'$, we have that

$$A(\text{Conv}_\mathfrak{R}(\rho, \tau); K^{\mathfrak{R} \cap \mathfrak{R}} / \mathfrak{R}) \leq A(\rho; K/F)^{(\dim \tau)[F': \mathfrak{R}]} A(\tau^*; K'/F')^{(\dim \rho)[F: \mathfrak{R}]}$$

Proof. The preceding proposition immediately gives

$$A(\text{Conv}_\mathfrak{R}(\rho, \tau); K^{\mathfrak{R} \cap \mathfrak{R}} / \mathfrak{R}) \leq A(\rho; K/F)^{(\dim \tau)[F': \mathfrak{R}]} A(\tau^*; K'/F')^{(\dim \rho)[F: \mathfrak{R}]}$$

The result then follows via inductivity and inflation property of conductors.

Lastly, given $K \subseteq \mathfrak{R} \subseteq F \cap F'$, we wish to prove a relation between $\text{Conv}_K$ and $\text{Conv}_\mathfrak{R}$. Before proceeding, we will briefly recall Mackey Decomposition Theorem, an important result in the representation theory of finite groups. For a proof of this, see for example [Ser12].

Mackey Decomposition Theorem. Let $S, R \subseteq G$, and let $(V, \rho)$ be a representation of $R$. For $g \in G$, define a representation $(V, \rho^g)$ of $R^g$ by $\rho^g(r) = \rho(g^{-1}rg)$ for all $r \in R^g$. Then

$$\text{Res}_S^G \text{Ind}_R^G \rho = \bigoplus_{SgR} \text{Ind}_S^R \text{Res}_{S \cap R^g} \rho^g$$

where the sum is taken over all double cosets $SgR$.

In general, it is currently unknown if there is a closed-form relation between $\text{Conv}_K$ and $\text{Conv}_\mathfrak{R}$. The following diagram gives the picture of the Hasse diagram involved in general.

![Hasse Diagram](image)

We will, however, prove a relation between $\text{Conv}_K(\rho, \tau)$ and $\text{Conv}_\mathfrak{R}(\rho, \tau)$ when $\mathfrak{R}/K$ is Galois, in which case $K^{\mathfrak{R}} = K^\mathfrak{R}$ and $K^{\mathfrak{R} \cap \mathfrak{R}} = K^{\mathfrak{R}}$. In this case, the Hasse diagram involved is much simpler, as shown below.
A Galois extension, we have that

\[ K^K K^\rho = K^\eta K^{\rho}\eta \]

Observe that since \( \text{Gal} \) and \( \text{Conv} \) are representations. For number fields \( K \subseteq \mathfrak{R} \subseteq F \cap F' \) with \( \mathfrak{R}/K \) a Galois extension, we have that

\[ \text{Conv}_K(\rho, \tau) = [\mathfrak{R} : K]\text{Ind}_{\text{Gal}(\mathfrak{R}/K)}^{\text{Gal}(\mathfrak{R}/\mathfrak{R}/K)} \text{Conv}_\mathfrak{R}(\rho, \tau) \]

**Proof.** We note that since \( K^K = K^\eta \), we get that

\[ \bar{\rho}^K = \text{Ind}_{\text{Gal}(\mathfrak{R}/F)}^{\text{Gal}(\mathfrak{R}/\mathfrak{R}/F)} \rho \circ \pi_{\text{Gal}(\mathfrak{R}/K)} = \text{Ind}_{\text{Gal}(\mathfrak{R}/\mathfrak{R}/F)}^{\text{Gal}(\mathfrak{R}/\mathfrak{R}/\mathfrak{R}/F)} \bar{\rho}_\mathfrak{R} \]

We thus have that

\[ \bar{\rho}^K \otimes \bar{\tau}^\mathfrak{R}^K = \left( \text{Ind}_{\text{Gal}(\mathfrak{R}/\mathfrak{R}/F)}^{\text{Gal}(\mathfrak{R}/\mathfrak{R}/F)} \rho \right) \otimes \left( \text{Ind}_{\text{Gal}(\mathfrak{R}/\mathfrak{R}/F)}^{\text{Gal}(\mathfrak{R}/\mathfrak{R}/F)} \pi_{\text{Gal}(\mathfrak{R}/K)} \right) = \text{Ind}_{\text{Gal}(\mathfrak{R}/\mathfrak{R}/F) \times \text{Gal}(\mathfrak{R}/\mathfrak{R}/F)}^{\text{Gal}(\mathfrak{R}/\mathfrak{R}/F) \times \text{Gal}(\mathfrak{R}/\mathfrak{R}/F)} \bar{\rho}^\mathfrak{R} \otimes \bar{\tau}^\mathfrak{R} \]

Observe that since \( \text{Gal}(\mathfrak{R}/\mathfrak{R}) \times \text{Gal}(\mathfrak{R}/\mathfrak{R}) \leq \text{Gal}(\mathfrak{R}/K) \times \text{Gal}(\mathfrak{R}/K) \), we have

\[ \left( \text{Gal}(\mathfrak{R}/\mathfrak{R}) \times \text{Gal}(\mathfrak{R}/\mathfrak{R}) \right)^{(\sigma_1, \sigma_2)} = \text{Gal}(\mathfrak{R}/\mathfrak{R}) \times \text{Gal}(\mathfrak{R}/\mathfrak{R}), \forall (\sigma_1, \sigma_2) \in \text{Gal}(\mathfrak{R}/K) \times \text{Gal}(\mathfrak{R}/K) \]

\[ \Rightarrow \text{Gal}(\mathfrak{R}/\mathfrak{R}) \cap \left( \text{Gal}(\mathfrak{R}/\mathfrak{R}) \times \text{Gal}(\mathfrak{R}/\mathfrak{R}) \right)^{(\sigma_1, \sigma_2)} = \text{Gal}(\mathfrak{R}/\mathfrak{R}) \]

and the order of the double coset, for each \( (\sigma_1, \sigma_2) \in \text{Gal}(\mathfrak{R}/K) \times \text{Gal}(\mathfrak{R}/K) \), is

\[ \left| \text{Gal}(\mathfrak{R}/\mathfrak{R} \times \mathfrak{R}/\mathfrak{R})(\sigma_1, \sigma_2) \left( \text{Gal}(\mathfrak{R}/\mathfrak{R}) \times \text{Gal}(\mathfrak{R}/\mathfrak{R}) \right) \right| = \frac{\left| \text{Gal}(\mathfrak{R}/\mathfrak{R}) \times \mathfrak{R}/\mathfrak{R} \right|}{\left| \text{Gal}(\mathfrak{R}/\mathfrak{R}) \right|} \cdot \frac{\left| \text{Gal}(\mathfrak{R}/\mathfrak{R}) \times \text{Gal}(\mathfrak{R}/\mathfrak{R}) \right|}{\left| \text{Gal}(\mathfrak{R}/\mathfrak{R}) \right|} = [\mathfrak{R} : K] \left| \text{Gal}(\mathfrak{R}/\mathfrak{R}) \times \text{Gal}(\mathfrak{R}/\mathfrak{R}) \right| \]
Thus, by Mackey Decomposition Theorem, we get that
\[
\text{Conv}_K(\rho, \tau) = \text{Res}^{\Gal(K^s/K)}_{\Gal(K^s/\mathcal{K})} \text{Ind}^{\Gal(K^s/\mathcal{K})}_{\Gal(K^s/\mathcal{R}^s/\mathcal{K})} \tilde{\rho}^\mathcal{R} \otimes \tau^\mathcal{R}
\]
\[
= \bigoplus_{S(\sigma_1, \sigma_2) \mathcal{R}} \text{Ind}^{\Gal(K^s/\mathcal{R}^s/\mathcal{K})}_{\Gal(K^s/\mathcal{R}^s/\mathcal{R})} \text{Res}^{\Gal(K^s/\mathcal{R}^s/\mathcal{R})}_{\Gal(K^s/\mathcal{R}^s/\mathcal{K})} \tilde{\rho}^\mathcal{R} \otimes \tau^\mathcal{R}
\]
\[
= \frac{|\Gal(K^s/\mathcal{K}) \times \Gal(K^s/\mathcal{R}^s/\mathcal{K})|}{|\mathcal{R} : \mathcal{K}| \Gal(K^s/\mathcal{R}^s/\mathcal{R}) \times \Gal(K^s/\mathcal{R}^s/\mathcal{K})|} \text{Ind}^{\Gal(K^s/\mathcal{R}^s/\mathcal{K})}_{\Gal(K^s/\mathcal{R}^s/\mathcal{R})} \text{Conv}_\mathcal{R}(\rho, \tau)
\]
\[
= [\mathcal{R} : \mathcal{K}] \text{Ind}^{\Gal(K^s/\mathcal{R}^s/\mathcal{K})}_{\Gal(K^s/\mathcal{R}^s/\mathcal{R})} \text{Conv}_\mathcal{R}(\rho, \tau) \quad \square
\]

2.3 Analyticity of Artin L-Functions

In this section, we will give a brief exposition on the analyticity of Artin L-functions. We will primarily be interested in the holomorphicity of Artin L-functions. We refer interested readers to [MM12] for explicit discussions about zeroes and poles of Artin L-functions.

2.3.1 Induction Theorems of Artin and Brauer

After defining the (unramified) Artin L-function in [Art24], Artin proceeded to establish the continuation and functional equation of his L-functions. Even though Artin will not have his Reciprocity Law to rely upon, Artin nevertheless realised that he needed to relate his L-functions to the abelian case, where he will have results of Hecke to draw upon, after he proved his Reciprocity Law later.

This meant that Artin was looking for a way to push results from the abelian case to the nonabelian case. The way to make sense of this is via the inductivity of Artin L-functions. Specifically, with this in mind and assuming Artin Reciprocity Law, the question becomes

**Question.** Can the representations of a finite group be expressed as a linear combination of representation induced from an abelian subgroup of the group?

Since, over characteristic zero, representations are uniquely determined by its character, we can instead consider the question in terms of characters of a group and thus expand the scope of the question.

Artin then proceeded to prove in [Art24] a beautiful representation-theoretic result, from a paper in number theory.

**Artin Induction Theorem.** Let $G$ be a finite group. Then, every character of $G$ is a rational linear combinations of characters induced from characters of cyclic subgroups of $G$.

Even more is true: for every character $\chi$ of $G$, $|G|\chi$ is a integer linear combinations of characters induced from characters of cyclic subgroups of $G$.

Along with Artin Reciprocity Law and inductivity, we get the following corollary.

**Corollary 2.3.1.** Let $K/F$ be a Galois extension, and let $\rho : \Gal(K/F) \to V$ be a representation. Then, there exists representation $\rho_i$ of cyclic subgroup $C_i$ of $\Gal(K/F)$ such that

$$L(s, \rho; K/F)|^{\Gal(K/F)} = \prod_{i=1}^{k} L(s, \rho_i; K/K^{C_i})^{n_i}$$

where $n_i \in \mathbb{Z}$.
In 1947, Brauer improved on Artin’s result in [Bra47a]. For this, we need the following definition from group theory.

**Definition 2.3.2.** Let \( p \) be a (rational) prime. A group is a **\( p \)-elementary group** is a direct product of a cyclic group of order prime to \( p \) and a \( p \)-group. A group is an **elementary group** if it is a \( q \)-elementary group for some (rational) prime \( q \).

We refer the reader to [Ser12] for properties of elementary groups especially with regards to Brauer Induction Theorem. We highlight the following property of elementary groups.

**Proposition 2.3.3.** Every character of an elementary group is monomial, i.e. it is induced from a 1-dimensional character of a subgroup.

Brauer is then able to prove the following in [Bra47a].

**Brauer Induction Theorem.** Let \( G \) be a finite group. Then, every character of \( G \) is an integer linear combinations of characters induced from one-dimensional characters of elementary subgroups of \( G \).

With this, we get the more pleasing corollary.

**Corollary 2.3.4.** Let \( K/F \) be a Galois extension, and let \( \rho: \text{Gal}(K/F) \rightarrow V \) be a representation. Then, there exists one-dimensional representation \( \rho_i \) of elementary subgroup \( H_i \) of \( \text{Gal}(K/F) \) such that

\[
L(s, \rho; K/F) = \prod_{i=1}^{k} L\left(s, \rho_i; K/K^{H_i}\right)^{n_i}
\]

where \( n_i \in \mathbb{Z} \). Thus, \( L(s, \rho; K/F) \) admits a meromorphic continuation to the complex plane.

We conclude this section with an explicit example of Brauer Induction Theorem at work.

**Example 2.3.1.** The following example is quoted from [CF67]. Consider the group \( S_3 \), with its three irreducible representations: the trivial representation \( 1_{S_3} \), the sign representation \( \text{sgn} \), and the standard representation \( \rho \) of \( S_3 \) which is 2-dimensional. The following is the table of its characters.

<table>
<thead>
<tr>
<th>( \chi ) ( S_3 )</th>
<th>( \chi_{\text{sgn}} )</th>
<th>( \chi_{\rho} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>((123), (132))</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>((12), (13), (23))</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>

Table 2.3.1: Table of Characters of \( S_3 \)

We now wish to express \( \text{sgn} \) and \( \rho \) as direct sum of induced representations of an elementary subgroups of \( S_3 \).

1. For \( \rho \), consider the non-trivial 1-dimensional representation \( \tau \) of \( A_3 \cong \mathbb{Z}/3\mathbb{Z} \), which is elementary.
   One can check that the character of \( \text{Ind}^{S_3}_{A_3}\tau \) matches \( \chi_{\rho} \), and thus we have that
   \[
   \chi_{\rho} = \text{Ind}^{S_3}_{A_3}\chi_{\tau} \implies \rho = \text{Ind}^{S_3}_{A_3}\tau
   \]

2. For \( \text{sgn} \), consider the non-trivial 1-dimensional representation \( \omega \) of \( H = \{1, (12)\} \), which is elementary. One can check that the character of \( \text{Ind}^{S_3}_{H}\omega \) matches \( \chi_{\rho} + \chi_{\text{sgn}} \), and thus we have that
   \[
   \chi_{\text{sgn}} = \text{Ind}^{S_3}_{H}\chi_{\omega} - \text{Ind}^{S_3}_{A_3}\chi_{\tau}
   \]
2.3.2 Artin and Dedekind Conjecture

Via Artin and Brauer Induction Theorems, we get a stronger relation between Artin $L$-functions and Hecke $L$-functions. By observing that Hecke $L$-functions of non-principal characters are holomorphic, one expects the corresponding statement to be true of Artin $L$-function in general. This is, as of yet, an open conjecture, made by Artin.

**Artin Holomorphy Conjecture.** Let $K/F$ be a Galois extension, and let $\rho$ be an irreducible representation of $\text{Gal}(K/F)$. Then, $L(s, \rho; K/F)$ is entire.

Artin Holomorphy Conjecture has only been verified when the Galois group of the extension is one of the following [CF67]:

- The symmetric groups $S_3$ and $S_4$.
- Groups of square free order.
- Groups of prime power order.
- Groups with abelian commutator subgroup.

In particular, Artin Holomorphy Conjecture has not been verified for the simple group $A_5$.

We show an example where Artin Holomorphy Conjecture is known to hold.

**Example 2.3.2.** Consider $S_3$, with its three irreducible representations: the trivial representation $1_{S_3}$, the sign representation $\text{sgn}$, and the standard representation $\rho$ of $S_3$ which is 2-dimensional. Let $K/\mathbb{Q}$ be a Galois extension with $\text{Gal}(K/\mathbb{Q}) \cong S_3$. We have already worked out in Example 2.3.1 that

$$\rho = \text{Ind}_{A_3}^{S_3} \tau$$

so by inductivity,

$$L(s, \rho; K/\mathbb{Q}) = L(s, \tau; K/K^{A_3})$$

which is therefore entire. Further, we have that

$$\text{sgn} = \omega \circ \pi_H$$

and by inflation,

$$L(s, \text{sgn}; K/\mathbb{Q}) = L(s, \omega; K^H/\mathbb{Q})$$

which is therefore entire. We thus have the veracity of Artin Holomorphy Conjecture in this case.

Now, with Galois extension $K/F$ with Galois group $G$, by Corollary 2.2.22 consider the factorisation

$$\zeta_K(s) = \zeta_F(s) \prod_{\rho \in G \setminus \{1_G\}} L(s, \rho; K/F)^{\dim \rho}$$

Then, the Artin Holomorphy Conjecture would imply that $\frac{\zeta_K(s)}{\zeta_F(s)}$ is entire. We will see in the next section that in fact Artin Holomorphy Conjecture is not needed to arrive at this conclusion.

One can now ask if the above is still true for an arbitrary finite extension. Let $K/F$ be a finite extension, and let $E/F$ be its normal closure. Let $G = \text{Gal}(E/F)$ and $H = \text{Gal}(E/K)$. Then, we have by inductivity that

$$\zeta_K(s) = L(s, 1_H; E/K) = L(s, \text{Ind}_H^G 1_H; E/F)$$
CHAPTER 2. A BRIEF OVERVIEW OF ARTIN L-FUNCTIONS

Since \( \text{Ind}_H^G 1_H \) is certainly a representation of \( G \), and more precisely the permutation representation for the action of coset space \( G/H \), we write

\[
\text{Ind}_H^G 1_H = 1_G \oplus \bigoplus_{\rho \in \hat{G} \setminus \{1_G\}} n_\rho \rho
\]

where \( n_\rho \in \mathbb{N} \). Thus, we get that

\[
\zeta_K(s) = L(s, \text{Ind}_H^G 1_H; E/F) = \zeta_F(s) \prod_{\rho \in \hat{G} \setminus \{1_G\}} L(s, \rho; E/F)^{n_\rho}
\]

Assuming the veracity of Artin Holomorphicity Conjecture, we get again that \( \frac{\zeta_K(s)}{\zeta_F(s)} \) is entire. This special case of Artin Holomorphicity Conjecture has actually been noted by Dedekind prior, where he formed his conjecture.

**Dedekind Conjecture.** For any finite extension \( K/F \) of number fields, \( \frac{\zeta_K(s)}{\zeta_F(s)} \) is entire.

Dedekind Conjecture is known to hold when \( K/F \) is Galois, which we will explore in the next section, and when the normal closure \( E/F \) is solvable, by Uchida [Uch75] and van der Waall [vdW75]. For the latter, we refer the reader to [MM12].

### 2.3.3 Aramata–Brauer Theorem

Even though we do not yet know the Artin Holomorphicity Conjecture in general, we can still deduce some analytical results of certain Artin \( L \)-functions. In this section, we will provide a proof of Aramata–Brauer Theorem, a central result that start off the major result of this thesis.

In this section, let \( G \) be a finite group. Let \( \rho, \omega, \lambda \) be representations of \( G \) such that \( \rho = \omega \oplus \lambda \). We denote \( \rho \ominus \omega := \lambda \). We also denote the inner product on characters of \( G \) by \( \langle \cdot, \cdot \rangle_G \).

We begin by proving the following lemma regarding the representations of \( G \), following [MM12].

**Lemma 2.3.5.** Let \( C \) be the set of all cyclic subgroups of \( G \). Then,

\[
|G| (\text{reg}_G \ominus 1_G) = \bigoplus_{C \in C} \bigoplus_{\psi \in \hat{C} \setminus \{1_C\}} n_{C,\psi} \text{Ind}_C^G \psi
\]

where \( n_{C,\psi} \in \mathbb{N} \).

**Proof.** We will prove this on the level of characters. For a cyclic group \( C \), we define a character \( \theta_C : C \to \mathbb{C} \) by

\[
\theta_C(g) = \begin{cases} |C|, & \text{if } C = \langle g \rangle \\ 0, & \text{otherwise} \end{cases}
\]

and let

\[
\chi_C = \phi(|C|) \chi_{\text{reg}_C} - \theta_C
\]

where \( \phi \) is the Euler totient function. In particular,

\[
\phi(|C|) = \sum_{\substack{g \in C \, \langle g \rangle = C}} 1
\]
Now, for any \( g \in C \),
\[
\lambda_C(g) = \begin{cases} 
\phi(|C|)|C|, & \text{if } g = e \\
-\theta_C(g), & \text{otherwise}
\end{cases}
\]

For any \( \psi \in \hat{C} \), we get that
\[
\langle \lambda_C, \chi_\psi \rangle_C = \langle \phi(|C|) \chi_{\text{reg}_C}, \chi_\psi \rangle_C - \langle \theta_C, \chi_\psi \rangle_C
\]
\[
= \phi(|C|) - \frac{1}{|C|} \sum_{g \in C} \overline{\theta_C(g)} \overline{\chi_\psi(g)} = \phi(|C|) - \sum_{\langle g \rangle = C, \psi} \overline{\chi_\psi(g)}
\]

If \( \psi = 1_C \), then the above is 0. Also, note that if \( \psi \) is non-trivial (so in particular \( |C| > 1 \)), then
\[
\{ \chi_\psi(g); \ g \in C, \langle g \rangle = C \}
\]
is the complete set of \( |C| \)-th root of unity, and hence
\[
\langle \lambda_C, \chi_\psi \rangle_C = \phi(|C|) - \sum_{\langle g \rangle = C} \overline{\chi_\psi(g)} = \phi(|C|) - \mu(|C|) \in \mathbb{N}
\]

where \( \mu \) is the M"obius function. Therefore, we get that
\[
\lambda_C = \sum_{\psi \in \hat{C} \setminus \{1_C\}} n_\psi \chi_\psi
\]

where \( n_\psi \in \mathbb{N} \). Finally, we note that for any irreducible representation \( \rho \) of \( G \), by Frobenius Reciprocity we get that
\[
\left\langle \sum_{C \in \mathcal{C}} \text{Ind}^G_C \lambda_C, \chi_\rho \right\rangle_G = \sum_{C \in \mathcal{C}} \left\langle \text{Ind}^G_C \lambda_C, \chi_\rho \right\rangle_G
\]
\[
= \sum_{C \in \mathcal{C}} \langle \lambda_C, \chi_\rho \rangle_C
\]
\[
= \sum_{C \in \mathcal{C}} \left( \phi(|C|) \chi_\rho(e) - \sum_{\langle g \rangle = C} \overline{\chi_\rho(g)} \right) = \chi_\rho(e) \sum_{C \in \mathcal{C}} \phi(|C|) - \sum_{\langle g \rangle = G} \overline{\chi_\rho(g)}
\]

and by noting that
\[
\sum_{C \in \mathcal{C}} \phi(|C|) = \sum_{C \in \mathcal{C}} \sum_{\langle g \rangle = C} 1 = \sum_{g \in G} 1 = |G|
\]

and thus
\[
\left\langle \sum_{C \in \mathcal{C}} \text{Ind}^G_C \lambda_C, \chi_\rho \right\rangle_G = \chi_\rho(e) \sum_{C \in \mathcal{C}} \phi(|C|) - \sum_{\langle g \rangle = G} \overline{\chi_\rho(g)} = \chi_\rho(e) |G| - \sum_{\langle g \rangle = G} \overline{\chi_\rho(g)} = |G| \langle \chi_{\text{reg}_G}, \chi_\rho \rangle_G
\]
\[
\implies |G| \langle \chi_{\text{reg}_G} - \chi_{1_G}, \chi_\rho \rangle_G = \sum_{C \in \mathcal{C}} \text{Ind}^G_C \lambda_C = \sum_{C \in \mathcal{C}} \sum_{\psi \in \hat{C} \setminus \{1_C\}} n_C \psi \chi_\psi
\]
With this lemma, we can prove Arama–Brauer Theorem, which was first proven by Arama [Ara33] and later by Brauer [Bra47b] independently.

**Aramata–Brauer Theorem.** Let $K/F$ be a Galois extension. Then, $\frac{\zeta_K(s)}{\zeta_F(s)}$ is entire.

**Proof.** By Lemma 2.3.5 and by inductivity, we get that

$$\left(\frac{\zeta_K(s)}{\zeta_F(s)}\right)^{|G|} = \prod_{C\in C} \prod_{\psi\in \hat{C}\setminus \{1\}} L(s, \psi; K/K^C)^{n_{C,\psi}}$$

where each $n_{C,\psi} \in \mathbb{N}$. We already know that each $L(s, \psi; K/K^C)$ is entire, and thus $\left(\frac{\zeta_K(s)}{\zeta_F(s)}\right)^{|G|}$ is as well. Since $\frac{\zeta_K(s)}{\zeta_F(s)}$ is meromorphic, we get that it must be entire.

As a closing remark to this section, we will like to point out the usage of the theory of Heilbronn characters, first introduced by Heilbronn in [Hei73], in the study of Artin Holomorphicity Conjecture. We define the following.

**Definition 2.3.6.** Let $K/F$ be a Galois extension with Galois group $G$. Let $s_0 \in \mathbb{C}\setminus \{1\}$. The Heilbronn character of $G$ at $s_0$ is the virtual character

$$\theta_{G,s_0} = \sum_{\rho \in \hat{G}} \left(\text{ord}_{s=s_0} L(s, \rho; K/F)\right) \chi_{\rho}$$

An immediate consequence is the following.

**Proposition 2.3.7.** Let $K/F$ be a Galois extension with Galois group $G$. Every Artin $L$-function associated to representations of $G$ is analytic at $s_0$ if and only if $\theta_{G,s_0}$ is either identically zero or is a (proper) character of $G$.

One can also use the theory of Heilbronn characters to prove a generalisation of Arama–Brauer Theorem. Specifically, one can prove the following:

**Theorem 2.3.8.** Let $K/F$ be a Galois extension with Galois group $G$. For every irreducible representation $\rho \in \hat{G}$ of $G$, we get that $\frac{\zeta_K(s)}{L(s, \rho; K/F)}$ are entire.

For more details, we suggest [FGM15] as a good starting point.

### 2.4 Artin $L$-Functions and Automorphic $L$-Functions

In the work of Pierce, Turnage-Butterbaugh and Wood in [PTBW17], they assumed the veracity of the Strong Artin Conjecture to port the results of Kowalski and Michel in [KM02] for the automorphic cuspidal $L$-functions to Artin $L$-functions. Since the main result of this thesis is the relaxation of that assumption, we will spend this section to give a very barebones introduction to automorphic $L$-functions and Strong Artin Conjecture.
2.4.1 Automorphic L-Functions

In order to define automorphic L-functions, one must have a good understanding of automorphic representations. The subject of automorphic representation is a vast one, one that witness a flurry of research work being done today. We will however not have the capacity to give a proper treatment of automorphic representation, lest we lengthen this chapter to an unreasonable length.

We proceed by assuming some knowledge of automorphic representations. We will only need the theory of automorphic L-functions to state the Strong Artin Conjecture and to remark on the result of Kowalski and Michel in [KM02] in the next chapter. Thus, one can also treat these automorphic representation as a “black box”, as we will not use any properties of automorphic representations in this thesis beyond what is needed for the aforementioned purposes. We will mainly follow the exposition of [Mic07].

Let $\pi = \otimes_p \pi_p$ be a unitary irreducible cuspidal automorphic representation of $GL_n(A_F)$. Godement and Jacquet in [GJ06], and later Jacquet and Shalika in [JS81], attached a $L$-function to $\pi$. (See also [Lan71].)

**Definition 2.4.1.** The automorphic L-function associated to $\pi$ is

$$L(s, \pi) = \prod_p L_p(s, \pi_p) = \prod_p \prod_{i=1}^n \left( 1 - \alpha_\pi(p, i) N_F(p)^{-s} \right)^{-1}$$

where $\alpha_\pi(p, i)$ is the local parameter of $\pi$ at $p$.

For us, we will note that $\alpha_\pi(p, i)$ for unramified $p$ are eigenvalues which parametrise a semisimple conjugacy class associated to $\pi_p$ [PTBW17].

An important example is the following.

**Example 2.4.1.** When $n = 1$, then we get that $L(s, \pi)$ is a Dirichlet $L$-function, where now $\pi$ is a Dirichlet character of conductor the conductor of $L(s, \pi)$. This shows that automorphic $L$-functions are, in this sense, a non-abelian generalisation of Dirichlet $L$-functions.

We can, as in the case of Artin L-functions, define the Euler factors at the infinite prime, to get an extended or completed automorphic $L$-function which satisfy a functional equation similar to those satisfied by the completed Artin $L$-functions.

We will highlight one more property of automorphic L-functions, before concluding our very barebones overview of the subject. It is known that for $1 \leq i \leq n$, by writing $N_F(p) = p^k$ for some $k$,

$$\log_p |\alpha_\pi(p, i)| \leq \frac{1}{2}$$

for all finite primes, and further for unramified primes

$$\log_p |\alpha_\pi(p, i)| = \frac{1}{2}$$

However, even more is conjectured to be true.

**Ramanujan-Petersson Conjecture.** At all finite primes,

$$|\alpha_\pi(p, i)| \leq 1$$

with equality when $p$ is unramified.

Here, we only quoted the conjecture for finite primes, as those are the ones we are mainly interested in in this thesis.
2.4.2 Strong Artin Conjecture

We noted previously in Example 2.4.1 that automorphic $L$-functions are non-abelian generalisations of Dirichlet $L$-functions. However, recall also that abelian Artin $L$-functions are also Dirichlet $L$-functions. If automorphic $L$-functions are to truly be a generalisation of Dirichlet $L$-function in a sensible way, we expect some relation between Artin $L$-functions and automorphic $L$-functions. A more precise statement is the following.

**Strong Artin Conjecture.** Let $K/F$ be a finite Galois extension, with $\text{Gal}(K/F) = G$. Let $\rho$ be an $n$-dimensional representation of $G$. Then, there exists an automorphic representation $\pi(\rho)$ on $\text{GL}_n(\mathbb{A}_F)$ such that the $L$-functions

$$L(s, \rho; K/F) = L(s, \pi(\rho))$$

except at a finite number of primes. Moreover, if $\rho$ is irreducible, then $\pi$ is cuspidal.

In fact, we have a stronger conjecture, by combining with the following theorem.

**Theorem 2.4.2.** Let $\pi$ be cuspidal automorphic on $\text{GL}_n(\mathbb{A}_F)$, and $\rho$ a $n$-dimensional representation of $G$. If for all but finitely many primes, we have that

$$L_p(s, \rho; K/F) = L(s, \pi)$$

then, we have that $L(s, \rho; K/F) = L(s, \pi)$.

The assumption of Strong Artin Conjecture is what allowed Pierce et al. to justify the usage of the results of Kowalski and Michel in [KM02], originally for the setting of automorphic $L$-functions, in the setting of Artin $L$-functions in [PTBW17]. The Strong Artin Conjecture was also used to get a convexity bound on Artin $L$-functions, by transferring known convexity bounds for automorphic $L$-functions via this conjecture.

We conclude this section by quoting [PTBW17], and note that the Strong Artin Conjecture has been verified for the following cases.

- One-dimensional representations, by Artin [Art31b].
- Nilpotent Galois extensions, by Arthur and Clozel [AC16].
- $A_4$, by Langlands [Lan80].
- $S_4$, by Tunnell [Tun81].
- Dihedral groups, by Langlands [Lan80].

2.5 Convexity Bound on Artin $L$-Functions

We conclude this brief primer on Artin $L$-functions with a discussion regarding its convexity bound. We first need a standard fact in complex analysis.

In short, we need a version of the maximus modulus principle for bounding holomorphic functions on vertical strips. This generalisation we needed is the various versions of the Phragmén–Lindelöf principle, which has been covered extensively in many texts on complex analysis such as [Lan13b]. We will quote the following version, regarding the bound of a holomorphic function on vertical strips.
**Phragmén–Lindelöf Principle.** Let \( f \) be continuous on the strip
\[
S = \{ s \in \mathbb{C}; \, \sigma_1 \leq \text{Re}(s) \leq \sigma_2 \}
\]
with \( f \) being holomorphic on the interior and satisfying the finite order inequality on \( S \), i.e. there exists some \( \lambda > 0 \) such that
\[
f(s) = O(e^{\lambda |s|})
\]
If there exists an \( M > 0 \) such that
\[
f(\sigma_i + it) = O(|t|^M)
\]
then we have that for all \( s = \sigma + it \in S, \)
\[
f(s) = O(|t|^M)
\]
We wish to point out that the implied constant depends only on \( M \).

A consequence of Phragmén–Lindelöf principle is the following convexity result.

**Theorem 2.5.1.** Let \( f \) be holomorphic on the strip
\[
S = \{ s \in \mathbb{C}; \, \sigma_1 \leq \text{Re}(s) \leq \sigma_2 \}
\]
and satisfying the finite order inequality on \( S \), i.e. there exists some \( \lambda > 0 \) such that
\[
f(s) = O(e^{\lambda |s|})
\]
Suppose \( f \) grows at most like a power of \( |t| \) on \( \text{Re}(s) = \sigma_i \), for both \( i = 1, 2 \), and let \( \psi(\sigma) \) be the least number such that for \( s = \sigma + it, \)
\[
f(s) = O_{\varepsilon}(|t|^{|\psi(\sigma)|+\varepsilon})
\]
Then, \( \psi \) is convex in \([\sigma_1, \sigma_2]\).

Note that through the proof of the theorem, the implied constant is the maximum of the constant implied on the side of the strip. Further, we can find a constant which allows one to bound the function along the vertical strip \( \text{Re}(s) = \sigma \).

We can now describe a convexity bound on Artin \( L \)-function.

**Lemma 2.5.2.** For any Artin \( L \)-function \( L(s, \rho; K/F) \) which is holomorphic, we get that for any \( \varepsilon > 0 \), there exists a constant \( C(\dim \rho, \varepsilon) \) such that
\[
|L(s, \rho; K/F)| \leq C(\dim \rho, \varepsilon) \left( A(\rho; K/F)(|t| + 2)^{\dim \rho} \right)^{\frac{1 - \text{Re}(s)}{2} + \varepsilon}
\]

**Proof.** By considering the Dirichlet series of \( L(s, \rho; K/F) \) for \( \text{Re}(s) = 1 + 2\varepsilon \), we get that \( L(s, \rho; K/F) \) is bounded along the strip \( \text{Re}(s) = 1 + 2\varepsilon \). Recalling the Stirling formula (see for example [IK04]) and the functional equation, we get that there exists a constant \( C'(\dim \rho, \varepsilon) \) such that on the strip \( \text{Re}(s) = -2\varepsilon, \)
\[
|L(s, \rho; K/F)| \leq C'(\dim \rho, \varepsilon) \left( A(\rho; K/F)(|t| + 2)^{\dim \rho} \right)^{\frac{1}{2} + 2\varepsilon}
\]
Thus, by the convexity corollary of Phragmén–Lindelöf principle, we get the desired convexity result. \( \square \)
Chapter 3

Kowalski–Michel Ameanable Families

In this chapter, we will define and study families of Artin $L$-functions which satisfy certain technical hypothesis. We will then follow the spirit of Kowalski and Michel in [KM02] in showing that such families admit analytical results about their zeroes akin to those obtained by [KM02] for families of cuspidal automorphic $L$-functions. In particular, we do not assume the Strong Artin Conjecture in the following, choosing instead to (re)prove all the analytical results by hand.

3.1 Families of Artin $L$-Functions

In this section, we will describe a special class of family of Artin $L$-functions. This special class will be the families such that the analytical method of Kowalski–Michel regarding the zero-region of the family is applicable, in the case of Artin $L$-functions instead of the case of cuspidal automorphic $L$-functions as was considered in [KM02].

3.1.1 Definition of Families of Artin $L$-Functions

Following [KM02] and [PTBW17], we define the following.

Definition 3.1.1. A family of Artin $L$-functions of degree $m$ is a set

$$\mathcal{F} = \{ \mathcal{F}(X); \ X \in \mathbb{N} \}$$

where for each $X \in \mathbb{N}$, $\mathcal{F}(X)$ is a finite collection of Artin $L$-functions associated to $m$-dimensional representations satisfying:

- There exists constants $a > 0$ and $C_{\text{cond}}$ such that for every $X \in \mathbb{N}$,

$$A(\rho; K/F) \leq C_{\text{cond}} X^a$$

for all $L(s, \rho; K/F) \in \mathcal{F}(X)$.

- There exists constants $d > 0$ and $C_{\text{order}}$ such that for every $X \in \mathbb{N}$,

$$|\mathcal{F}(X)| \leq C_{\text{order}} X^d$$

We say an Artin $L$-function $L(s, \rho; K/F)$ belongs to $\mathcal{F}$, denoted $L(s, \rho; K/F) \in \mathcal{F}$, if $L(s, \rho; K/F) \in \mathcal{F}(X)$ for some $X \in \mathbb{N}$.

On first glance, the above definition seems to differ slightly from the definitions of [KM02] and Condition 6.1 of [PTBW17]. Specifically, comparing with Condition 6.1 of [PTBW17], the property that the family satisfies Ramanujan-Petersson Conjecture and a uniform convexity bound is missing from our
definition. However, Remark 2.2.14 and 2.2.20 shows that Artin $L$-functions satisfies Ramanujan-
Petersson Conjecture, and so that condition could be dropped from the definition of a family. The
uniform convexity bound, on the other hand, will be handled later.

We also note the usage of a finite collection instead of a finite set. We wish to point out that it could
be the case that for two fields $K/F$ and $K'/F$, not necessarily Galois but with the same Galois group,
their Artin $L$-functions associated to some representation $\rho$ might be the same. Here is an example,
noted by PTBW17.

**Example 3.1.1.** Recall that two fields $K, K'$ are arithmetically equivalent if their Dedekind zeta-
functions $\zeta_K = \zeta_{K'}$. Of course, this does not happen in the Galois setting, i.e. $K/Q$ and $K'/Q$ are Galois,
which is the setting we will later be in. We still maintain our definition of a family, to allow for such
generality.

As we are mainly interested in the zero-region of Artin $L$-functions, we will define a zero-counting
function, both on the level of individual Artin $L$-functions and on the level of families. For an Artin
$L$-function $L(s, \rho; K/F)$, we define

$$Z(\alpha, T, \rho; K/F) = \{ s \in \mathbb{C}; \Re(s) \geq \alpha, \ |\Im(s)| \leq T, L(s, \rho; K/F) = 0 \}$$

and

$$N(\alpha, T, \rho; K/F) = \sum_{s \in Z(\alpha, T, \rho; K/F)} \ord L(s, \rho; K/F)$$

Note in particular that $N(\alpha, T, \rho; K/F)$ counts the zeroes with multiplicities. On the level of families,
we define $N_{F, \alpha, T}$ to be the function on $\mathbb{N}$ defined by

$$N_{F, \alpha, T}(X) = \sum_{L(s, \rho; K/F) \in F(X)} N(\alpha, T, \rho; K/F)$$

The following is the major example of this thesis.

**Example 3.1.2.** Let $G$ be a finite group, and let

$$\mathcal{F}_{F, G} = \{ \mathcal{F}_{F, G}(X); X \in \mathbb{N} \}$$

where for each $X \in \mathbb{N}$,

$$\mathcal{F}_{F, G}(X) = \{ \zeta_K(s); K/F \text{ Galois}, \text{Gal}(K/F) = G, N^F_Q(\Delta_K/F) < X \}$$

- For any $\zeta_K(s) = L(s, \text{reg}_G; K/F) \in \mathcal{F}_{F, G}(X)$, we get by the Conductor-Discriminant Formula that
  the conductor of $\zeta_K(s)$ is
  $$A(\text{reg}_G; K/F) = |\Delta_{F/Q}|^{\text{dim reg}_G} N^F_Q(f(\text{reg}_G; K/F)) = |\Delta_{F/Q}|^{[K:F]} N^F_Q(\Delta_K/F) \leq |\Delta_{F/Q}|^{\lvert G \rvert} X$$

- For every $X \in \mathbb{N}$, we get that
  $$|\mathcal{F}_{F, G}(X)| \leq N_{F,G}^{\Gal}(X)$$
  where $N_{F,G}^{\Gal}(X)$ is the number of Galois extensions of $F$ of degree $n$ with absolute discriminant at
  most $X$, i.e.
  $$N_{F,n}^{\Gal}(X) := \{ K/F \text{ Galois}; \ [K:F] = n, N^F_Q(\Delta_K/F) < X \}$$

By a well-known result of Schmidt Sch95, we get that

$$|\mathcal{F}_{F, G}(X)| \leq N_{F,G}^{\Gal}(X) \leq N_{F,G}(X) := \{ K/F; [K:F] = n, N^F_Q(\Delta_K/F) < X \} \leq C(F, |G|) X^{\frac{|G| + 2}{4}}$$

where $C(F, n)$ is a constant depending on $F$ and $n$.  

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With the constants \( a = 1, \) \( C_{\text{cond}} = |\Delta_{F/Q}|^{|G|}, \) \( d = \frac{|G|+2}{4} \) and \( C_{\text{order}} = C(F, |G|), \) this makes \( \mathcal{F}_{F,G} \) a family of Artin \( L \)-functions of degree \(|G|\).

To conclude, we will define the notion of a subfamily.

**Definition 3.1.2.** Let \( \mathcal{F} \) be a family of Artin \( L \)-functions of degree \( m \). A set
\[
\mathcal{H} = \{ \mathcal{H}(X); \; X \in \mathbb{N} \}
\]
is a **subfamily** of \( \mathcal{F} \) if for all \( X \in \mathbb{N} \), we have that
\[
\mathcal{H}(X) \subseteq \mathcal{F}(X)
\]
Note that it is clear by definition that a subfamily is also a family of Artin \( L \)-functions of the same degree, with possibly tighter parameters.

### 3.1.2 Kowalski–Michel Ameanable Families

The definition of family of Artin \( L \)-functions is not strong enough to have the analytical properties needed for the application of Kowalski–Michel type estimate of the zeroes of the family. Here, we will define a special class of families which we will show have the suitable analytical properties to apply Kowalski–Michel type estimates.

**Definition 3.1.3.** Let \( \mathcal{F} = \{ \mathcal{F}(X); \; X \in \mathbb{N} \} \) be a family of Artin \( L \)-functions of degree \( m \), and let \( \mathcal{K} \) be a number field. \( \mathcal{F} \) is said to be **Kowalski–Michel \( \mathcal{K} \)-ameanable** if for each \( X \in \mathbb{N} \), \( \mathcal{F}(X) \) is a finite collection of holomorphic Artin \( L \)-functions associated to \( m \)-dimensional representations such that

- For every \( L(s, \rho; K/F) \in \mathcal{F} \), we have that \( \mathcal{K} \subseteq F \), with the same \([F : \mathcal{K}] \). We denote this by \([\mathcal{F} : \mathcal{K}]\).
- \( \mathcal{F} \) satisfy uniform convexity bound: For every \( \varepsilon > 0 \), there exists a constant \( C_{\text{convex}}(\varepsilon) \) such that for all \( L(s, \rho; K/F) \in \mathcal{F} \), \( L(s, \rho; K/F) \) satisfy the convexity bound
  \[
  |L(s, \rho; K/F)| \leq C_{\text{convex}}(\varepsilon) \left( A(\rho; K/F)(|t| + 2)^{\dim \rho} \right)^{\frac{1-\text{Re}(s)}{2} + \varepsilon}
  \]
  for \( 0 \leq \text{Re}(s) \leq 1 \).
- \( \mathcal{F} \) is well-behaved under convolution over \( \mathcal{K} \), that is
  - For \( L(s, \rho; K/F), L(s, \tau; K'/F') \in \mathcal{F} \), their convolution over \( \mathcal{K} \)
    \[
    L \left( s, \text{Conv}_{\mathcal{K}}(\rho, \tau); \frac{K\mathcal{K}}{K'} / k \right)
    \]
    where \( \text{Conv}_{\mathcal{K}}(\rho, \tau) \) is given by
    \[
    \text{Conv}_{\mathcal{K}}(\rho, \tau) = \text{Res}_{\mathcal{K}} \left( \frac{\text{Gal}(K/F) \times \text{Gal}(K'/F')}{\mathcal{K}} \mathcal{K} \otimes \tau^* \mathcal{K} \right)
    \]
    is entire if and only if \( K \neq K' \) and \( \rho \neq \tau \), with the only pole at \( s = 1 \) of the same order when \( K = K', F = F' \) and \( \rho \simeq \tau \).
for every $\varepsilon > 0$, there exists a constant $C'_{\text{convex}}(\varepsilon)$ such that for all $L(s, \rho; K/F), L(s, \tau; K'/F') \in \mathcal{F}$, their convolution over $\mathfrak{R}$ satisfy the convexity bound

$$
\left| L \left( s, \text{Conv}_\mathfrak{R}(\rho, \tau); \overline{K^\mathfrak{R}} K^{\mathfrak{R}'} / \mathfrak{R} \right) \right| \leq C'_{\text{convex}}(\varepsilon) \left( A \left( \text{Conv}_\mathfrak{R}(\rho, \tau); \overline{K^\mathfrak{R}} K^{\mathfrak{R}'} / \mathfrak{R} \right) \right)^{1 - \text{Re}(s) + \varepsilon}
$$

for $0 \leq \text{Re}(s) \leq 1$.

We also define the tuple of data attached to such families.

**Definition 3.1.4.** Let $\mathcal{F}$ be a Kowalski–Michel $\mathfrak{R}$-ameanable family of Artin $L$-function of degree $m$. Its **Kowalski–Michel $\mathfrak{R}$-family parameters** is the tuple

$$(m, \mathfrak{R}, a, C_{\text{cond}}, d, C_{\text{order}}, C_{\text{convex}}, C'_{\text{convex}})$$

where $C_{\text{convex}}, C'_{\text{convex}}$ are thought of as functions in $\varepsilon$. We also denote $\kappa(\mathcal{F})$ to be the order of the residue at $s = 1$ of any $L \left( s, \text{Conv}_\mathfrak{R}(\rho, \rho); \overline{K^\mathfrak{R}} K^{\mathfrak{R}'} / k \right) \in \mathcal{F}$.

Note that the property of well-behavedness under convolution corresponds to the behavior under (unramified) Rankin-Selberg of automorphic $L$-function used in [KM02]. Indeed, under the assumption of the Strong Artin Conjecture, the convolution we described is the corresponding Rankin-Selberg of the automorphic $L$-function on the level of Galois representation, via functoriality of the correspondence. Since [PTBW17] assumes Strong Artin Conjecture, this property of being well-behaved under convolution can be safely dropped. However, we do not assume Strong Artin Conjecture, so we can only produce Kowalski–Michel estimates for families that do have this property.

We also note that the uniform convexity bounds was used by [PTBW17] to replace the assumption of [KM02] that all members of the family having the same gamma factor at infinity. [PTBW17] notes that since the assumption of [KM02] is only used to establish a uniform convexity bound, there is no harm in assuming that directly instead. We will follow the cue of [PTBW17] in this regard.

Here, we will highlight a non-example of a Kowalski–Michel ameanable family that is closely related to a family to be introduced later that is Kowalski–Michel ameanable.

**Example 3.1.3.** Consider the family $\mathcal{F}_{FG}$ in Example 3.1.2. Now, since each $\zeta_L(s) \in \mathcal{F}_{FG}$ is not entire, $\mathcal{F}_{FG}$ can not be Kowalski–Michel ameanable. But this is really the only technical obstruction to applying Kowalski–Michel type estimate. We will introduce a family in Chapter 4 as a fix to this obstruction, and thus have an example of a Kowalski–Michel ameanable family.

We also highlight the following straightforward result.

**Proposition 3.1.5.** Let $\mathcal{F}$ be a Kowalski–Michel $\mathfrak{R}$-ameanable family. Then, any subfamily of $\mathcal{F}$ is also Kowalski–Michel $\mathfrak{R}$-ameanable with the same $\mathfrak{R}$-family parameters.

We conclude by proving the following lemma, which will allow us to reduce the results of special cases to the case of Kowalski–Michel $\mathbb{Q}$-ameanable families.

**Lemma 3.1.6.** Let $\mathfrak{R}$ be a number field, and let $\mathcal{F}$ be a Kowalski–Michel $\mathfrak{R}$-ameanable family of Artin $L$-function of degree $n$, with Kowalski–Michel $\mathfrak{R}$-family parameters $(m, \mathfrak{R}, a, C_{\text{cond}}, d, C_{\text{order}}, C_{\text{convex}}, C'_{\text{convex}})$. Then, for any number field $K \subseteq \mathfrak{R}$ such that $\mathfrak{R}/K$ is Galois, $\mathcal{F}$ is also Kowalski–Michel $K$-ameanable, with Kowalski–Michel $K$-family parameters $(m, K, a, C_{\text{cond}}, d, C_{\text{order}}, C_{\text{convex}}, \left(C'_{\text{convex}} \right)^{|K|/|\mathfrak{R}|})$. 

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We also get that for every member \( L(s, \rho; K/F), L(s, \tau; K'/F') \in \mathcal{F} \), by Proposition 2.2.32, we get that
\[
\text{Conv}_K(\rho, \tau) = [\mathcal{R} : \mathcal{K}] \text{Ind}^{\text{Gal}(\mathcal{R}/\mathcal{K})}_{\text{Gal}(\mathcal{R}/\mathcal{K})} \text{Conv}_\mathcal{R}(\rho, \tau)
\]
By noting that since \( \mathcal{R}/\mathcal{K} \) is Galois,
\[
\mathcal{K}^\mathcal{R} = \mathcal{K}^\mathcal{K}
\]
and by inductivity and additivity of Artin \( L \)-function, we get that
\[
L \left( s, \text{Conv}_K(\rho, \tau); \mathcal{K}^\mathcal{R} \mathcal{K}^\mathcal{R}/\mathcal{K} \right) = L \left( s, \text{Conv}_\mathcal{R}(\rho, \tau), \mathcal{K}^\mathcal{R} \mathcal{K}^\mathcal{R}/\mathcal{R} \right)^{[\mathcal{R} : \mathcal{K}]}
\]
is entire if and only if \( K \neq K' \) and \( \rho \neq \tau \), with the only pole at \( s = 1 \) when \( K = K', F = F' \) and \( \rho \equiv \tau \).

For every \( \varepsilon > 0 \), there exists a constant \( C'_\text{convex}(\varepsilon) \) such that for all \( L(s, \rho; K/F), L(s, \tau; K'/F') \in \mathcal{F} \), their convolution over \( \mathcal{K} \) satisfy the convexity bound
\[
\left| L \left( s, \text{Conv}_K(\rho, \tau); \mathcal{K}^\mathcal{R} \mathcal{K}^\mathcal{R}/\mathcal{K} \right) \right|^{[\mathcal{R} : \mathcal{K}]}
\]
\[
= \left| L \left( s, \text{Conv}_\mathcal{R}(\rho, \tau), \mathcal{K}^\mathcal{R} \mathcal{K}^\mathcal{R}/\mathcal{R} \right) \right|^{[\mathcal{R} : \mathcal{K}]}
\]
\[
\leq C'_\text{convex}(\varepsilon)^{[\mathcal{R} : \mathcal{K}]}
\left( A \left( \text{Conv}_\mathcal{R}(\rho, \tau), \mathcal{K}^\mathcal{R} \mathcal{K}^\mathcal{R}/\mathcal{R} \right)^{[\mathcal{R} : \mathcal{K}]}
\left( |t| + 2 \right)^{m_2[\mathcal{F}, \mathcal{R}]^2[\mathcal{R} : \mathcal{K}]}
\right)^{\frac{1 - \text{Re}(s)}{2} + \varepsilon}
\]
\[
\leq C'_\text{convex}(\varepsilon)^{[\mathcal{R} : \mathcal{K}]}
\left( A \left( \text{Conv}_K(\rho, \tau), \mathcal{K}^\mathcal{R} \mathcal{K}^\mathcal{R}/\mathcal{K} \right)^{[\mathcal{R} : \mathcal{K}]}
\left( |t| + 2 \right)^{m_2[\mathcal{F}, \mathcal{K}]^2}
\right)^{\frac{1 - \text{Re}(s)}{2} + \varepsilon}
\]
for \( 0 \leq \text{Re}(s) \leq 1 \).

We thus get that \( \mathcal{F} \) is also Kowalski–Michel \( \mathcal{K} \)-ameanable, with Kowalski–Michel \( \mathcal{K} \)-family parameters \((m, \mathcal{K}, a, \text{Cond}, d, \text{Order}, \text{Convex}, (C'_\text{convex})^{[\mathcal{R} : \mathcal{K}])}\).

\[
\mathcal{F} \]

\section{Components of Kowalski–Michel Estimate}

We now introduce pre-requisites needed to deduce the Kowalski–Michel estimate for the zero-region of Kowalski–Michel \( \mathbb{Q} \)-ameanable families. In this section, we set \( \mathcal{F} \) a Kowalski–Michel \( \mathbb{Q} \)-ameanable family. For every member \( L(s, \rho; K/F) \in \mathcal{F} \), by inflation and inductivity of Artin \( L \)-functions we get that
\[
L(s, \rho; K/F) = L(s, \rho \circ \pi_{\text{Gal}(\mathbb{Q}/K)}; \overline{K}/F) = L \left( s, \text{Ind}^{\text{Gal}(\mathbb{Q}/K)}_{\text{Gal}(\mathbb{Q}/K)} \rho \circ \pi_{\text{Gal}(\mathbb{Q}/K)}; \overline{K}/\mathbb{Q} \right)
\]
where \( \overline{K}^\mathbb{Q} \) is the Galois closure of \( K \) over \( \mathbb{Q} \). Thus, for simplicity, we think of \( \mathcal{F} \) as a degree \( m = \text{deg}(\mathcal{F})[\mathcal{F} : \mathbb{Q}] \) family, and every member of \( \mathcal{F} \) will be written as \( L(s, \rho; F/Q) \) with \( F/Q \) being Galois. We also get that
\[
\text{Conv}_\mathbb{Q}(\rho, \tau) = \text{Res}^{\text{Gal}(F/Q) \times \text{Gal}(F'/Q)}_{\text{Gal}(FF'/Q)} \rho \otimes \tau^*
\]
3.2.1 Dirichlet Series Expansion

Let $L(s, \rho; F/Q) \in \mathcal{F}$. Consider its Dirichlet series expansion

$$L(s, \rho; F/Q) = \sum_{n=1}^{\infty} \lambda_n(\rho)n^{-s}$$

For a general Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

we denote

$$[n]f(s) := a_n$$

which is a notation adapted from enumerative combinatorics. In particular,

$$[n]L(s, \rho; F/Q) := \lambda_n(\rho)$$

We note that since $L(s, \rho; F/Q)$ is defined via an Euler product, $[\cdot]L(s, \rho; F/Q) : \mathbb{N} \to \mathbb{C}$ is a multiplicative arithmetic function.

There are minor issues with $[p]L(s, \rho; F/Q)$ for $p \mid A(\rho; F/Q)$, that is the primes which ramifies in $F$. To overcome these, we define the following.

**Definition 3.2.1.** The **unramified Dirichlet series** associated to $L(s, \rho; F/Q)$ is the Dirichlet series $f(s, \rho)$ such that

$$[n]f(s, \rho) = \begin{cases} [n]L(s, \rho; F/Q), & \text{if } \gcd(n, A(\rho; F/Q)) = 1 \\ 0, & \text{otherwise} \end{cases}$$

We also denote $f(s, \rho \otimes \tau^*) = f(s, \Res_{\Gal(F/Q) \times \Gal(F'/Q)}^{\Gal(F/Q)} \rho \otimes \tau^*)$.

A direct consequence of the Euler product and the definition of the conductor of Artin $L$-functions is the following lemma.

**Lemma 3.2.2.** Let $f(s, \rho)$ be the unramified Dirichlet series associated to $L(s, \rho; F/Q)$. Then

$$f(s, \rho) = L_{unram}(s, \rho; F/Q)$$

As such, Artin $L$-functions and its unramified Dirichlet series differ only by a finite Euler product which is entire and does not vanish in $\Re(s) > 0$. Hence, we will lose no generality by examining only $f(s, \rho)$ instead of the full $L(s, \rho; F/Q)$. In particular,

$$\res_{s=1} f(s, \rho) = \res_{s=1} L(s, \rho; F/Q)$$

**Remark 3.2.3:** For $L(s, \rho, F/Q)$, the eigenvalues of $\rho^*$ are the conjugate of the eigenvalues of $\rho$, as the latter are roots of unity by Remark 2.2.14. Thus, for any prime $p$,

$$[p]f(s, \rho^*) = \begin{cases} [p]L(s, \rho^*; F/Q) = \Tr(\rho^*(\sigma_p)) = \overline{\Tr(\rho(\sigma_p))} = [p]L(s, \rho^*; F/Q), & \text{if } p \nmid A(\rho; F/Q) \\ 0, & \text{otherwise} \end{cases}$$

Thus, for squarefree $n$, by multiplicativity,

$$[n]f(s, \rho^*) = [n]f(s, \rho)$$
In the following, let us also denote
\[ \sum_n^b a_n := \sum_{n \text{ squarefree}}^a a_n = \sum_{n} |\mu(n)| a_n \]
where \( \mu \) is the usual Möbius function. We also define, for \( z \geq 1 \),
\[ P(z) := \prod_{p < z} p \]
The following lemma is straightforward.

**Lemma 3.2.4.** Write
\[ f(s, \rho) = f^b(s, \rho) f^s(s, \rho) \]
where
\[ f^b(s, \rho) = \sum_{\gcd(n, P(z)) = 1} ([n] f(s, \rho)) n^{-s} = \prod_{p \geq z} (1 + ([p] f(s, \rho)) p^{-s}) \]
Then, for \( z \geq 1 \) large enough, we get that for all \( L(s, \rho; F/\mathbb{Q}) \in \mathcal{F} \), \( f^s(s, \rho) \) is entire and non-vanishing at \( \Re(s) > \frac{1}{2} \), and for every \( \varepsilon > 0 \), there exists a constant \( M(\varepsilon) > 0 \) such that
\[ |f^s(s, \rho)| \leq M(\varepsilon) \]
in \( \Re(s) > \frac{1}{2} + \varepsilon \).

We will now define the final ingredient from this section. We note that the order of the residue at \( s = 1 \) of \( f^b(s, \rho \otimes \rho^*) \) is \( \kappa(\mathcal{F}) \), and the Laurent expansion of \( f^b(s, \rho \otimes \rho^*) \) at \( s = 1 \) is
\[ f^b(s, \rho \otimes \rho^*) = \sum_{k \geq -\kappa(\mathcal{F})} c_k(\rho)(s - 1)^k \]
For \( \varphi \) a suitable test function, we note that
\[ \res_{s=0} f^b(s + 1, \rho \otimes \rho^*) \varphi(s)x^s = c_0(\rho)O(1) + \sum_{k=1}^{\kappa(\mathcal{F})} c_{-k}(\rho)O(\log^k x) \ll (\log x) \sum_{k=1}^{\kappa(\mathcal{F})} c_{-k}(\rho)O(\log^{k-1} x) \]
We therefore denote
\[ s_\varphi(\rho, x) = \sum_{k=1}^{\kappa(\mathcal{F})} c_{-k}(\rho)O(\log^{k-1} x) \]

### 3.2.2 Pseudocharacters

Kowalski and Michel adapted Selberg’s usage of pseudocharacters to their setting, which they used as a sort of counting device to help deduce their main theorem. We will adapt their results to our setting.

We begin by defining pseudocharacters.

**Definition 3.2.5.** We define a function \( \psi_\rho : \mathbb{N} \to \mathbb{R} \) associated to \( L(s, \rho; F/\mathbb{Q}) \) by
\[ \psi_\rho(n) = \begin{cases} \frac{\mu(n)n}{[n] f(s, \rho)^{1/2}} : & \text{if } \gcd(n, A(\rho; L/\mathbb{Q})) = 1 \\ 0, & \text{otherwise} \end{cases} \]
and we define a **pseudocharacter** associated to \( L(s, \rho; F/\mathbb{Q}) \) and integer \( n \) to be the function \( \psi_{\rho, n} : \mathbb{N} \to \mathbb{R} \) given by
\[ \psi_{\rho, n}(r) = \mu(r)^2 \psi_\rho(\gcd(n, r)) \]
We note that since $\psi_p(n)$ is a multiplicative arithmetic function, the $\psi_{p,n}$ are indeed the pseudocharacters as defined in [Jut78].

Following [KM02], we also define, for $0 < \delta < \frac{1}{2}$, the following subset of integers

$$R_{\delta,z}(\rho) = \{ r \in \mathbb{N}; \ r \ \text{squarefree}, \ \gcd(r, A(\rho; F/Q)P(z)) = 1, \ \text{such that} \ p | r \implies |[p]f(s, \rho)| > p^{-\delta} \} \subseteq \mathbb{N}$$

We will need a result about the abundance of integers in $R_{\delta,z}(\rho)$.

**Lemma 3.2.6.** Fix $z \geq 1$ large enough and $0 < \delta < \frac{1}{2}$. There exists a constant $M$ such that for $L(s, \rho; F/Q) \in \mathcal{F}$, we get that

$$\sum_{r \leq R} \frac{1}{|\psi_{p}(r)|} \leq M s_{\varphi}(\rho, R) \log R$$

for $R \geq 2$. Further, let $X \in \mathbb{N}$ be such that for $L(s, \rho; F/Q) \in \mathcal{F}(X)$. For $C > ma$, there exists a constant $M'(C)$ for which we get that

$$\sum_{r \leq R} \frac{1}{|\psi_{p}(r)|} \geq M'(C)s_{\varphi}(\rho, R) \log R$$

for $R > X^C$.

**Proof.** First, note that for $r$ squarefree,

$$\frac{1}{|\psi_{p}(r)|} = \frac{|[r]f(s, \rho)|^2}{r}$$

Also, for primes $p$, by Remark 3.2.3

$$|[p]f(s, \rho)|^2 = ([p]f(s, \rho))(|[p]f(s, \rho^*)|) = |[p]f(s, \rho \otimes \rho^*)|$$

Hence, since $[\cdot]f(s, \rho)$ are multiplicative arithmetic functions, we get that

$$\sum_{r \leq R} \frac{1}{|\psi_{p}(r)|} = \sum_{r \leq R} \frac{[r]f(s, \rho \otimes \rho^*)}{r}$$

Now, note that

$$\sum_{r \leq R} \frac{1}{|\psi_{p}(r)|} = \sum_{r \leq R} \frac{[r]f(s, \rho \otimes \rho^*)}{r} \leq \sum_{r \leq R} \frac{[r]f(s, \rho \otimes \rho^*)}{r}$$

Using a suitable test function $\varphi$, we estimate using the usual Mellin transform,

$$\sum_{s' < R \leq s < \sigma} \text{res}_{s'=2} \left( f^\sigma(s + 1, \rho \otimes \rho^*)\varphi(s)x^s + \frac{1}{2\pi i} \int_{(\sigma')} f^\sigma(s + 1, \rho \otimes \rho^*)\varphi(s)x^s ds \right)$$

But since $\mathcal{F}$ is Kowalski–Michiel $\mathbb{Q}$-ameanable, $L(s, \text{Conv}_Q(\rho, \rho); FF'/Q)$ has a pole only at $s = 1$ and satisfy a uniform convexity bound, and so does $f^\sigma(s, \rho \otimes \rho^*)$. In particular, the integral can be bounded independent of $L(s, \rho; F/Q)$ within $\mathcal{F}$. Therefore, with $\sigma' < 0$,

$$\sum_{r \leq R} \frac{1}{|\psi_{p}(r)|} \leq s_{\varphi}(\rho, R) \log R + O(1)$$
with the bound on the $O(1)$ factor being independent of individual $L(s, \rho; F/\mathbb{Q})$ within $\mathcal{F}$. This gives us the first result.

For the second assertion, we note that
\[
\prod_{\substack{p \nmid R_{\alpha, z}(ho) \\ p \mid A(\rho; F/\mathbb{Q})}} (1 + ||p|f(s, \rho)|^2p^{-s}) \leq \prod_{\substack{p \mid A(\rho; F/\mathbb{Q}) \\ ||p|f(s, \rho)| \leq p^{-\delta}}} (1 + p^{-2\delta - s})
\]
converges for $\text{Re}(s) > 1 - 2\delta$. So by letting
\[
F(s, \rho) = \sum_{r \in R_{\alpha, z}} ||r|f(s, \rho)|^2r^{-s}
\]
we get that
\[
f^\delta(s, \rho \otimes \rho) = F(s, \rho) \prod_{\substack{p \nmid R_{\alpha, z}(ho) \\ p \mid A(\rho; F/\mathbb{Q})}} (1 + ||p|f(s, \rho)|^2p^{-s})
\]
Now, for $\varepsilon > 0$ the uniform convexity bound and Corollary \[2.2.31\] will put a bound on $f^\delta(s, \rho \otimes \rho)$ on the line $\text{Re}(s) = 1 - 2\delta + \varepsilon$ of size of magnitude $X^{ma(2\delta - \varepsilon + \frac{1}{\text{gcd}(\rho, \tau)}}) = X^{C(2\delta - \varepsilon)}$ with implied constant depending on $\frac{(C - ma)(2\delta - \varepsilon)}{ma}$. We estimate using Mellin transform to get
\[
\sum_{r \in R} \frac{1}{|\psi(r)|} = \sum_{r \in R} \frac{[r]|f(s, \rho \otimes \rho^*)|}{r} \geq M(C)s\phi(\rho, R) \log R
\]
as long as $R > X^C$.

The following lemma of \[Jut78\] about pseudocharacters has been adapted to our setting.

**Lemma 3.2.7.** Let $L(s, \rho; F/\mathbb{Q}), L(s, \tau; F'/\mathbb{Q}) \in \mathcal{F}$, and let $r \in R_{\alpha, z}(\rho), t \in R_{\alpha, z}(\tau)$. Define $h_{r, t}^{\rho, \tau}$ to be such that
\[
\sum_{d=1}^{\infty} h_{r, t}^{\rho, \tau}(d) d^{-s} = \prod_{\substack{p \mid |r \\ p \mid t}} (1 + (\psi_p(p) - 1)p^{-s}) \prod_{\substack{p \mid t \\ p \mid r}} (1 + (\psi_p(p) - 1)p^{-s}) \prod_{p \mid \text{gcd}(r, t)} (1 + (\psi(p)\psi_r(p) - 1)p^{-s})
\]
Then, for $n \geq 1$, we get that
\[
\psi_{\rho, r}(n)\psi_{\tau, t}(n) = \mu(n)^2 \sum_{d \mid n} h_{r, t}^{\rho, \tau}(d)
\]
If $F = F'$ and $\tau \equiv \rho^*$, then
\[
\sum_{d=1}^{\infty} h_{r, t}^{\rho, \rho^*}(d) \eta_{\rho}(d) [d]f(s, \rho)|^2d^{-1} = \delta_{r, t} |\psi_{\rho}(r)|
\]
where $\delta_\cdot$ is the usual Kronecker delta function and
\[
\eta_{\rho}(d) = \prod_{p \mid d} (1 + ||p|f(s, \rho)|^2p^{-1})^{-1}
\]
Proof. We first note that
\[
\zeta(s) \sum_{d=1}^{\infty} h^{\rho,\tau}_{r,t}(d)d^{-s} = \prod_{p|r} (1 - p^{-s})^{-1} \prod_{p|t} \left( 1 + \psi_p(p) \sum_{k=1}^{\infty} (p^k)^{-s} \right) \prod_{p|\gcd(r,t)} \left( 1 + \psi_p(p) \sum_{k=1}^{\infty} (p^k)^{-s} \right) \prod_{p|\gcd(r,t)} \left( 1 + \psi_p(p) \psi_p(p) \sum_{k=1}^{\infty} (p^k)^{-s} \right) 
\]

\[
= \prod_{p|r} (1 - p^{-s})^{-1} \prod_{p|t} \left( 1 + \psi_p(p) \sum_{k=1}^{\infty} (p^k)^{-s} \right) \prod_{p|\gcd(r,t)} \left( 1 + \psi_p(p) \sum_{k=1}^{\infty} (p^k)^{-s} \right) \prod_{p|\gcd(r,t)} \left( 1 + \psi_p(p) \psi_p(p) \sum_{k=1}^{\infty} (p^k)^{-s} \right) 
\]

\[
= \sum_{n=1}^{\infty} \left( \prod_{p|\gcd(n,r)} \psi_p(p) \right) \left( \prod_{p|\gcd(n,t)} \psi_p(p) \right) n^{-s} 
\]

\[
\implies \psi_{\rho,\tau}(n) \psi_{\tau,\rho}(n) = \mu(n)^2 \prod_{p|\gcd(n,r)} \psi_p(p) \prod_{p|\gcd(n,t)} \psi_p(p) = \mu(n)^2 \left( \zeta(s) \sum_{d=1}^{\infty} \frac{1}{[d]f(s,\rho)^2} \right) = \mu(n)^2 \sum_{d|n} \frac{1}{[d]f(s,\rho)^2} 
\]

Secondly, when \( F = F' \) and \( \tau \equiv \rho^s \), notice that \( h^{\rho,\rho^s}_{r,t}(d) \) is only non-zero for square-free \( d \) and that for primes \( p \), by Remark 3.2.3
\[
\psi_{\rho^s}(p) = \begin{cases} \frac{-1}{[p]f(s,\rho)^2}, & \text{if } \gcd(n, A; L/\mathbb{Q}) = 1 \\ 0, & \text{otherwise} \end{cases} = \psi_p(p) 
\]

Thus, at \( s = 1 \),
\[
\sum_{d=1}^{\infty} \frac{1}{[d]f(s,\rho)^2} \eta_p(d) = \sum_{d=1}^{\infty} \frac{1}{[d]f(s,\rho)^2} \prod_{p|d} \left( 1 + \frac{[p]f(s,\rho)^2}{1 + [p]f(s,\rho)^2 p^{-1}} \right) d^{-1} 
\]

\[
= \prod_{p|\gcd(r,t)} \left( 1 + \frac{[p]f(s,\rho)^2}{1 + [p]f(s,\rho)^2 p^{-1}} \right) \prod_{p|\gcd(r,t)} \left( 1 + \frac{[p]f(s,\rho)^2}{1 + [p]f(s,\rho)^2 p^{-1}} \right) = \delta_{r,t} \psi_p(p) 
\]

With this, we will prove the general mean estimate of the pseudocharacters of Kowalski and Michel, which they generalised from [DK00], adapting them to our setting.

**Proposition 3.2.8.** Fix \( z \geq 1 \) large enough and \( 0 < \delta < \frac{1}{2} \), and let \( F \) be a Kowalski–Michel \( \mathbb{Q} \)-ameanable family. Let \( N \geq R^2 \). Then, there exists a function \( G_F(X, N, R) \) such that for \( \varepsilon > 0 \) satisfying
\[
N^{\frac{1}{2} - \varepsilon} > X^{d + \frac{m_a}{4}} R^{2 + 2\delta} \log R 
\]
there exists a constant \( M(\varepsilon) \) such that for \( a_n \in \mathbb{C} \),
\[
\sum_{L(s, \rho; F/Q) \in F(X)} \sum_{r \in R} \frac{1}{\psi_p(r)} \sum_{n \sim N} a_n \psi_{\rho,\tau}(n) \left( \left| \sum_{r \leq R} \frac{1}{\psi_p(r)} \right|^2 \right) \leq M(\varepsilon) NG_F(X, N, R) \sum_{n \sim N} |a_n|^2 
\]
Proof. The proof is essentially as in [KM02], which we will detail here for completeness. By considering the linear operator
\[ Q : (a_n)_{n \sim N} \mapsto \left( \sum_{n \sim N} a_n \frac{\psi_{\rho,r}(n) \langle [n]f(s, \rho) \rangle}{\sqrt{s_{\varphi}(\rho, R)|\psi_{\rho}(r)|}} \right)_{L(s,\rho,F/Q) \in F(X) \atop r \in R_{\delta,z}(\rho)} \]
of finite-dimensional Hilbert spaces, the inequality is equivalent to
\[ \|Q\|^2 \leq M(\varepsilon) N \]
By general Hilbert space theory, we get that the norm of \( T \) is the same as the norm of the conjugate of its adjoint, given by
\[ (b_{\rho,r})_{L(s,\rho,F/Q) \in F(X) \atop r \in R_{\delta,z}(\rho)} \mapsto \left( \sum_{n \sim N} \sum_{r \in R} b_{\rho,r} \frac{\psi_{\rho,r}(n) \langle [n]f(s, \rho) \rangle}{\sqrt{s_{\varphi}(\rho, R)|\psi_{\rho}(r)|}} \right)_{n \sim N} \]
Thus, the inequality is now equivalent to
\[ \sum_{n \sim N} \left| \sum_{L(s,\rho,F/Q) \in F(X) \atop r \in R_{\delta,z}(\rho)} b_{\rho,r} \frac{\psi_{\rho,r}(n) \langle [n]f(s, \rho) \rangle}{\sqrt{s_{\varphi}(\rho, R)|\psi_{\rho}(r)|}} \right|^2 \leq M(\varepsilon) N \sum_{L(s,\rho,F/Q) \in F(X) \atop r \in R_{\delta,z}(\rho)} |b_{\rho,r}|^2 \]
By expanding the sum on the left hand side, we get that
\[
\sum_{n \sim N} \left| \sum_{L(s,\rho,F/Q) \in F(X) \atop r \in R_{\delta,z}(\rho)} b_{\rho,r} \frac{\psi_{\rho,r}(n) \langle [n]f(s, \rho) \rangle}{\sqrt{s_{\varphi}(\rho, R)|\psi_{\rho}(r)|}} \right|^2 \\
\leq \sum_{n=1}^{\infty} \sum_{L(s,\rho,F/Q) \in F(X) \atop r \in R_{\delta,z}(\rho)} \sum_{t \leq R} \frac{b_{\rho,r}b_{\tau,t}}{\sqrt{s_{\varphi}(\rho, R)s_{\varphi}(\tau, R)|\psi_{\rho}(r)|\psi_{\tau}(t)|}} \psi_{\rho,r}(n)\psi_{\tau,t}(n) \langle [n]f(s, \rho) \rangle \langle [n]f(s, \tau) \rangle \phi \left( \frac{n}{N} \right) \\
= \sum_{L(s,\rho,F/Q) \in F(X) \atop L(s,\tau,F'/Q) \in F(X) \atop r \in R_{\delta,z}(\rho) \atop t \in R_{\delta,z}(\tau)} \frac{b_{\rho,r}b_{\tau,t}}{\sqrt{s_{\varphi}(\rho, R)s_{\varphi}(\tau, R)|\psi_{\rho}(r)|\psi_{\tau}(t)|}} S(\rho, \tau, r, t) \\
\]
where \( \theta : \mathbb{R} \geq 0 \to [0, 1] \) is a suitable smooth test function with compact support in \( [\frac{1}{2}, 3] \) and \( \theta(x) = 1 \) for \( x \in [1, 2] \), and
\[ S(\rho, \tau, r, t) := \sum_{n=1}^{\infty} \psi_{\rho,r}(n)\psi_{\tau,t}(n) \langle [n]f(s, \rho) \rangle \langle [n]f(s, \tau) \rangle \theta \left( \frac{n}{N} \right) \]
We note that by definition, \( \psi_{\rho,r}(n) \) and \( \psi_{\tau,t}(n) \) are zero if \( n \) squarefree or \( (n, P(z)) \neq 1 \). Along with
Lemma 3.2.7 and Remark 3.2.3, we get that

\[ S(\rho, \tau, r, t) = \sum_{n=1}^{\infty} \psi_{\rho, r}(n) \psi_{\tau, t}(n)(([n]f(s, \rho))([n]f(s, \tau))\theta \left( \frac{n}{N} \right) \]

\[ = \sum_{\gcd(n, P(z))=1}^{b} \psi_{\rho, r}(n) \psi_{\tau, t}(n)(([n]f(s, \rho))([n]f(s, \tau^*))\theta \left( \frac{n}{N} \right) \]

\[ = \sum_{\gcd(n, P(z))=1}^{b} \left( \sum_{d|n} h_{\rho, \tau^*}^{\delta_{\tau}}(d) \right) \left( \sum_{\gcd(n, d)=1}^{b} \sum_{\gcd(n, P(z))=1}^{b} [n]f(s, \rho \otimes \tau^*)\theta \left( \frac{nd}{N} \right) \right) \]

\[ \text{Let us set} \]

\[ T(x) = \sum_{\gcd(n, d)=1}^{\delta_{\tau}} [n]f(s, \rho \otimes \tau^*)\theta \left( \frac{n}{x} \right) \]

To estimate the above, we consider the Dirichlet series

\[ g_{d, z}^{\rho, \tau^*}(s) = \sum_{\gcd(n, d)=1}^{\delta_{\tau}} [n]f(s, \rho \otimes \tau^*)n^{-s} = f^0(s, \rho \otimes \tau^*) \prod_{p|d} \left( 1 + [p]f(s, \rho \otimes \tau^*)p^{-s} \right)^{-1} \]

Using Mellin transform, we get that

\[ T(x) = \sum_{\sigma' < \Re z < 3} \res_{s=z} \left( g_{d, z}^{\rho, \tau^*}(s) \tilde{\theta}(s)x^s \right) + \frac{1}{2\pi i} \int_{\sigma'} g_{d, z}^{\rho, \tau^*}(s) \tilde{\theta}(s)x^s \, ds \]

where \( \frac{1}{2} < \sigma' < 1 \). Since \( F \) is Kowalski–Michel \( \mathbb{Q} \)-ameanable, we get that the only pole of \( g_{d, z}^{\rho, \tau^*}(s) \) is at \( s = 1 \) if \( \tau \equiv \rho \) and \( F = F^* \), and no poles otherwise. In the former case,

\[ \res_{s=1} \left( g_{d, z}^{\rho, \tau^*}(s) \tilde{\psi}(s)x^s \right) = \res_{s=0} \left( f^0(s + 1, \rho \otimes \tau^*) \prod_{p|d} \left( 1 + [p]f(s, \rho \otimes \tau^*)p^{-s-1} \right)^{-1} \tilde{\theta}(s + 1)x^s \right) x \]

\[ \leq x \eta_{\rho}(d) \sum_{k=1}^{\kappa(F)} c_{-k}(\rho) O(\log^{k-1} xd) \]

where the constants depends on \( \theta \) and \( d \). We denote

\[ g_{\theta}(\rho, d, x) = \sum_{k=1}^{\kappa(F)} c_{-k}(\rho) O(\log^{k-1} xd) \]

We have therefore that

\[ T(x) = \delta_{\rho, \tau} \delta_{F, F^*} \eta_{\rho}(d)x g_{\theta}(\rho, d, x) + \frac{1}{2\pi i} \int_{\sigma'} g_{d, z}^{\rho, \tau^*}(s) \tilde{\psi}(s)x^s \, ds \]
CHAPTER 3. KOWALSKI–MICHEL AMENABLE FAMILIES

To bound the integral, we use the uniform convexity bound on $L^p$ spaces, giving us a constant $K(\varepsilon)$ such that

$$|f(s, \rho \otimes \tau^*)| \leq K(\varepsilon) \left( A(\rho \otimes \tau^*; FF'/Q) (|t| + 2)^{m^2} \right)^{\frac{1}{1+\varepsilon}}$$

By Corollary 2.2.31

$$A(\rho \otimes \tau^*; FF'/Q) \leq (A(\rho; F/Q)A(\tau^*; F'/Q))^m \leq C_{\text{cond}}^2 X^{2m}$$

so we get that a constant $K'(\varepsilon)$ such that

$$|f(s, \rho \otimes \tau^*)| \leq K'(\varepsilon) X_{\frac{m}{2}}$$

Thus, on $\text{Re}(s) = \sigma' = \frac{1}{2} + \varepsilon$, and bounding $f^2(s, \rho \otimes \tau^*)$ by Lemma 3.2.4 there is a constant $K''(\varepsilon)$ such that

$$|g_{d,z}^{\rho, \tau^*}(s)| \leq K''(\varepsilon) X_{\frac{m}{2}}$$

So, there exists a constant $K'''(\varepsilon)$ such that

$$\frac{1}{2\pi i} \int_{(\sigma')} g_{d,z}^{\rho, \tau^*}(s) \tilde{\psi}(s)s^{\frac{d}{2}} ds \leq K'''(\varepsilon) X_{\frac{m}{2}} x_{\frac{1}{2} + \varepsilon}$$

Substituting $T$, and denoting

$$g_{\theta}(\rho, N) = \max_{d \leq N} g_{\theta} \left( \rho, d, \frac{N}{d} \right)$$

we get that $S(\rho, \tau, r, t)$ is bounded by

$$\sum_{d \leq N} h_{r,t}^{\rho, \tau^*}(d) ([d]f(s, \rho))([d]f(s, \tau^*)) T \left( \frac{N}{d} \right)$$

$$= \delta_{\rho, \tau^*} \delta_{F,F'} N g_{\theta}(\rho, N) \sum_{d \leq N} h_{r,t}^{\rho, \tau^*} (d) \eta_{\rho}(d) ([d]f(s, \rho))([d]f(s, \tau^*)) d^{-1} + \text{Err}$$

where there exists a constant $C(\varepsilon)$ such that the error term is at most

$$\text{Err} \leq C(\varepsilon) N^{\frac{1}{4} + \varepsilon} X_{\frac{m}{2}} \sum_{d \leq N} \sum_{\gcd(d, P(z)) = 1} \frac{|h_{r,t}^{\rho, \tau^*} (d) ([d]f(s, \rho))([d]f(s, \tau^*))|}{\sqrt{d}}$$

$$\leq C(\varepsilon) N^{\frac{1}{4} + \varepsilon} X_{\frac{m}{2}} \sum_{d \leq N} \sum_{\gcd(d, P(z)) = 1} \frac{|h_{r,t}^{\rho, \tau^*} (d) ([d]f(s, \rho))([d]f(s, \tau^*))|}{\sqrt{d}}$$

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By definition of $h_{r,t}(d)$ and since $r \in R_{\delta,z}(\rho), t \in R_{\delta,z}(\tau)$, we get that

$$\sum_{d \leq N, \gcd(d,P(z)) = 1} \frac{|h_{r,t}^{\rho,\tau}(d)([d]f(s,\rho))([d]f(s,\tau^*))|}{\sqrt{d}}$$

$$\ll \prod_{p \mid r} \left( 1 + |\psi_p(p)| - 1 \right) \prod_{p \mid t} \left( 1 + |\psi_\tau(p)| - 1 \right) \prod_{p \mid t} \left( |\psi_\tau(p)| \right) \prod_{p \mid t} \left( |\psi_\tau(p)| \right)$$

$$\ll \prod_{p \mid r} \left( |\psi_p(p)| \right) \prod_{p \mid t} \left( |\psi_\tau(p)| \right) \prod_{p \mid t} \left( |\psi_\tau(p)| \right) \ll r^{1+\delta} t^{1+\delta} \leq R^{2+2\delta}$$

Thus, there exists a constant $C'(\varepsilon)$ such that

$$\text{Err} \ll C'(\varepsilon) R^{2+2\delta} N^{\frac{1}{2}+\varepsilon} X^\frac{ma}{2}$$

Meanwhile, the main term for $S(\rho,\tau,r,t)$ only appears when $\rho \simeq \tau$ and $F = F'$, and in this case contributes by Lemma 3.2.7

$$N_{g}(\rho,N) \sum_{d \leq N, \gcd(d,P(z)) = 1} h_{r,t}(d) \eta_\rho(d)([d]f(s,\rho))([d]f(s,\rho^*)) d^{-1} = N_{g}(\rho,N) \delta_{r,t} |\psi_p(r)|$$

Finally, returning to

$$\sum_{L \in R_{\delta,z}(\rho)} \sum_{r,t \leq R} b_{\rho,r} b_{\tau,t} \sqrt{s_\rho(\rho,R)s_\tau(\tau,R) |\psi_p(\rho)| |\psi_\tau(\tau)|} S(\rho,\tau,r,t)$$

- The error term of $S(\rho,\tau,r,t)$ contribute at most

$$C'(\varepsilon) R^{2+2\delta} N^{\frac{1}{2}+\varepsilon} X^\frac{ma}{2} \sum_{L \in R_{\delta,z}(\rho)} \sum_{r \leq R} \sum_{t \leq R} b_{\rho,r} b_{\tau,t} \sqrt{s_\rho(\rho,R)s_\tau(\tau,R) |\psi_p(\rho)| |\psi_\tau(\tau)|}$$

$$= C'(\varepsilon) R^{2+2\delta} N^{\frac{1}{2}+\varepsilon} X^\frac{ma}{2} \left( \sum_{L \in R_{\delta,z}(\rho)} \sum_{r \leq R} |b_{\rho,r}|^2 \right) \left( \sum_{L \in R_{\delta,z}(\rho)} \frac{1}{s_\rho(\rho,R)} \sum_{r \leq R} \frac{1}{|\psi_p(\rho)|} \right)$$

$$\leq C'(\varepsilon) R^{2+2\delta} N^{\frac{1}{2}+\varepsilon} X^\frac{ma}{2} \left( \sum_{L \in R_{\delta,z}(\rho)} \sum_{r \leq R} \frac{1}{s_\rho(\rho,R)} \sum_{t \leq R} \frac{1}{|\psi_\tau(\tau)|} \right)$$

where the last inequality as afforded by Cauchy-Schwarz inequality. By Lemma 3.2.6, we get that

$$\sum_{L \in R_{\delta,z}(\rho)} \frac{1}{s_\rho(\rho,R)} \sum_{r \leq R} \frac{1}{|\psi_p(\rho)|} \ll |F(X)| \log R \ll X^d \log R$$
so there is a constant $E(\varepsilon)$ such that the error term contribute at most

$$E(\varepsilon) N^{\frac{1}{2}+\varepsilon} X^{\frac{m\alpha + d}{2}} R^{2+2\delta} \log R \left( \sum_{L(s,\rho; F/\mathbb{Q}) \in \mathcal{F}(X)} \sum_{r \in R} |b_{\rho,r}|^2 \right)$$

- The main term of $S(\rho, \tau, r, t)$ contribute only when $\rho \equiv \tau$ and $r = t$, and in that case gives the contribution

$$N \sum_{L(s,\rho; F/\mathbb{Q}) \in \mathcal{F}(X)} \frac{g_\theta (\rho, N)}{s_\psi (\rho, R)} \sum_{r \in R} |b_{\rho,r}|^2$$

Since $\mathcal{F}(X)$ is finite, we can bound

$$\frac{g_\theta (\rho, N)}{s_\psi (\rho, R)} \leq G_\mathcal{F}(X, N, R)$$

for some function $G_\mathcal{F}(X, N, R)$.

Combining both analysis, we get a constant $M(\varepsilon)$ such that

$$\sum_{n \sim N} \left| \sum_{L(s,\rho; F/\mathbb{Q}) \in \mathcal{F}(X)} \sum_{r \in R \in R_{\delta}(\rho)} b_{\rho,r} \frac{\psi_{\rho,r}(n)([n] f(s, \rho))}{\sqrt{s_\psi (\rho, R)|\psi_\rho (r)|}} \right|^2$$

$$\leq \sum_{L(s,\rho; F/\mathbb{Q}) \in \mathcal{F}(X)} \sum_{r \in R \in R_{\delta}(\rho)} \frac{b_{\rho,r}b_{\tau,t}}{\sqrt{s_\psi (\rho, R)s_\psi (\tau, R)|\psi_\rho (r)|\psi_\tau (t)}} S(\rho, \tau, r, t)$$

$$\leq M(\varepsilon) \left( NG_\mathcal{F}(X, N, R) + N^{\frac{1}{2}+\varepsilon} X^{\frac{m\alpha + d}{2}} R^{2+2\delta} \log R \right) \sum_{L(s,\rho; F/\mathbb{Q}) \in \mathcal{F}(X)} \sum_{r \in R} |b_{\rho,r}|^2$$

The assumption that $N^{\frac{1}{2}+\varepsilon} > X^{d+\frac{m\alpha}{2}} R^{2+2\delta} \log R$ finally gives the result.

Fix appropriate test functions $\varphi, \theta$. Let us focus on this function $G_\mathcal{F}(X, N, R)$ that was defined in the proof above. In the case where $\kappa(\mathcal{F}) = 1$, that is pole at $s = 1$ of $f^j(s, \rho \otimes \rho^*)$ for all members $L(s, \rho \otimes \rho) \in \mathcal{F}$ is simple, then $G_\mathcal{F}(X, N, R) \ll 1$, and thus we retrieve the result of Kowalski–Michel. In fact, we conjecture that this is true in general.

**Conjecture 3.2.9.** Fix $z \geq 1$ large enough, and let $\mathcal{F}$ be a Kowalski–Michel $\mathbb{Q}$-ameanable family. Let $N \geq R^2$. Then, the function $G_\mathcal{F}(X, N, R)$ in Proposition 3.2.8 satisfy

$$G_\mathcal{F}(X, N, R) \ll 1$$

Under the veracity of this conjecture, we note that we will get the corresponding Kowalski–Michel estimate for Kowalski–Michel $\mathbb{Q}$-ameanable family. In other words, we have replaced the reliance of Strong Artin Conjecture with Conjecture 3.2.9. Unfortunately, we could not prove Conjecture 3.2.9 at this moment. Instead, we will assume Conjecture 3.2.9 henceforth.

Let us now define

$$M(\alpha, T) = \{ z \in \mathbb{C}; \text{Re}(z) \geq \alpha, |\text{Im}(z)| \leq T \}$$

and consider for each $L(s, \rho; F/\mathbb{Q}) \in \mathcal{F}$ a finite subset $J(\rho) \subseteq M(\alpha, T)$.
Definition 3.2.10. $J(\rho)$ is q-well-spaced if for all $s_0 \neq s_1 \in J_\rho$,

\[
|\text{Im}(s_0 - s_1)| \geq \frac{1}{\log q}
\]

Then, proceeding as in [Mon06, §7], with Proposition 3.2.8 replacing the role of Theorem 2.5 in [Mon06], we get the following corollary of Proposition 3.2.8.

Corollary 3.2.11. For each $L(s, \rho; F/\mathbb{Q}) \in \mathcal{F}(X)$, let $J(\rho)$ be a $X$-well-spaced finite subset of $M(\alpha, T)$. Fix $z \geq 1$ large enough and $0 < \delta < \frac{1}{2}$. Suppose also that $\mathcal{F}$ is Kowalski–Michel $\mathbb{Q}$-ameanable family, and assume Conjecture 3.2.9. Let $N \geq R^2$, and suppose there exists $\varepsilon > 0$ such that

\[
N^{\frac{1}{2} - \varepsilon} > X^{d + \frac{ma}{2}} R^{2 + 2\delta} \log R
\]

Then, there exists constants $M(\varepsilon)$ and $B > 0$ such that for $a_n \in \mathbb{C}$,

\[
\sum_{L(s, \rho; F/\mathbb{Q}) \in \mathcal{F}(X)} \frac{1}{s_\varphi(\rho, R)} \sum_{s_0 \in J(\rho)} \sum_{r \in R_{\delta, s}(\rho)} \left| \frac{1}{|s_\varphi(\rho, R)|} \sum_{n \sim N} a_n \psi_{\rho, r}(n)([n]f(s, \rho))n^{-s_0} \right|^2 \leq M(\varepsilon) T^B \log(NX) \left( 1 + \frac{\log 2N}{\log 2R} \right)^2 \sum_{n \sim N} |a_n|^2 n^{1 - 2\alpha}
\]

3.2.3 A Zero-Detector

For each $L(s, \rho; F/\mathbb{Q}) \in \mathcal{F}$, we denote the set of zeroes of $L(s, \rho; F/\mathbb{Q})$ in $M(\alpha, T)$ by

\[
Z_\rho(\alpha, T) := \{ s \in M(\alpha, T); \ L(s, \rho; F/\mathbb{Q}) = 0 \}
\]

Now, suppose we are given for each $L(s, \rho; F/\mathbb{Q}) \in \mathcal{F}(X)$, a finite subset $J_\rho(\alpha, T) \subseteq Z_\rho(\alpha, T)$ which is $X$-well-spaced. Then, we wish to bound the size of the set

\[
J_X(\alpha, T) := \bigcup_{L(s, \rho; F/\mathbb{Q}) \in \mathcal{F}(X)} J_\rho(\alpha, T)
\]

We then have the following lemma regarding the size of the set $J_X(\alpha, T)$.

Lemma 3.2.12. Let $R > X^C$, where $C > ma$. Then, we have that there exists a constant $M(C)$ such that

\[
|J_X(\alpha, T)| \log R \leq M(C) \sum_{L(s, \rho; F/\mathbb{Q}) \in \mathcal{F}(X)} \frac{1}{s_\varphi(\rho, R)} \sum_{s_0 \in J(\rho, T)} \sum_{r \in R_{\delta, s}(\rho)} \left| \frac{1}{|s_\varphi(\rho, R)|} \psi_{\rho, r}(n)([n]f(s, \rho))n^{-s_0} \right|^2 
\]

\[
= M(C) \sum_{L(s, \rho; F/\mathbb{Q}) \in \mathcal{F}(X)} \sum_{r \in R_{\delta, s}(\rho)} \frac{|J_\rho(\alpha, T)|}{s_\varphi(\rho, R)|\psi_{\rho, r}(n)|}
\]

Proof. By Lemma 3.2.6, we get that there exists a constant $M(C)$ such that

\[
\sum_{L(s, \rho; F/\mathbb{Q}) \in \mathcal{F}(X)} \frac{1}{s_\varphi(\rho, R)} \sum_{s_0 \in J_\rho(\alpha, T)} \sum_{r \in R_{\delta, s}(\rho)} \left| \frac{1}{|s_\varphi(\rho, R)|} |\psi_{\rho, r}(n)| \right| \geq M(C) \sum_{L(s, \rho; F/\mathbb{Q}) \in \mathcal{F}(X)} \frac{1}{s_\varphi(\rho, R)} \sum_{s_0 \in J_\rho(\alpha, T)} s_\varphi(\rho, R) \log R
\]

\[
= M(C) |J_X(\alpha, T)| \log R
\]
Now, we wish to leverage Corollary 3.2.11 to bound the sum in the previous lemma, which will in turn help us bound \(|J_X(\alpha, T)|\). We will therefore need a sort of function which can “detect” zeroes of \(L(s, \rho; F/\mathbb{Q})\).

We begin by defining the following.

**Definition 3.2.13.** Let \(1 \leq z_1 < z_2\). We define

\[
\lambda_{z_1, z_2}(d) = \begin{cases} 
\mu(d), & \text{if } d \leq z_1 \\
\mu(d) \frac{\log \left( \frac{d}{z_2} \right)}{\log \left( \frac{z_1}{z_2} \right)}, & \text{if } z_1 < d \leq z_2 \\
0, & \text{otherwise}
\end{cases}
\]

We also denote

\[
\Delta_{z_1, z_2}(n) = \sum_{d \mid n} \lambda_{z_1, z_2}(d)
\]

We quote the following result of Graham in [Gra81].

**Lemma 3.2.14.** For \(\frac{1}{2} \leq \alpha < 1\), we get that there exists a constant \(C\) such that

\[
\sum_{z_1 < n \leq x} \Delta_{z_1, z_2}(n)^2 n^{1-2\alpha} \leq C \frac{\log \left( \frac{x}{z_1} \right)}{\log \left( \frac{z_2}{z_1} \right)} x^{2-2\alpha}
\]

We now define a condition on a pair of parameters.

**Definition 3.2.15.** Let \(X \in \mathbb{N}, 0 < \delta < \frac{1}{2}, z, R, x, T \geq 1\) and \(1 \leq z_1 < z_2\). We say that a pair of real numbers \((\varepsilon, \alpha)\), with \(\varepsilon > 0\) and \(\frac{1}{2} \leq \alpha < 1\), is \((X, \delta, z, R, z_1, z_2, x, T)\)-suitable if they satisfy the conditions

\[
x \gg \left( z_2 X^{\frac{1}{2}} T^m R^{1+4\delta} \right)^{\frac{1}{2\alpha-1} + \varepsilon} \\
\log x \leq \log XT \\
(\log X)^{\frac{1}{2}} \leq \log R \leq \frac{1}{2} \log x
\]

We also define, given \((X, \delta, z, R, z_1, z_2, x, T)\), a zero-detector, which helps us in bounding each \(J_{\rho}(\alpha, T)\).

**Definition 3.2.16.** Let \(X \in \mathbb{N}, 0 < \delta < \frac{1}{2}, z, R, x, T \geq 1\) and \(1 \leq z_1 < z_2\). For \(L(s, \rho; F/\mathbb{Q}) \in \mathcal{F}(X)\) and \(r \in R_{\delta,z}(\rho)\) with \(r \leq R\), we define a \((X, \delta, z, R, z_1, z_2, x, T)\)-zero-detector as the function

\[
z_{(X, \delta, z, R, z_1, z_2, x, T)}(s, \rho, r) = \sum_{\text{gcd}(n, P(z)) = 1} \Delta_{z_1, z_2}(n) \psi_{\rho, r}(n) ([n] f(s, \rho)) e^{-\frac{n}{x}} n^{-s}
\]

We now wish to give a lower bound to these zero-detector at the zeroes of \(L(s, \rho; F/\mathbb{Q})\). This involves bounds on the Dirichlet series

\[
\sum_{\text{gcd}(n, P(z)) = 1} \Delta_{z_1, z_2}(n) \psi_{\rho, r}(n) ([n] f(s, \rho)) n^{-s}
\]

The following lemma gives a factorisation of the Dirichlet series.
Lemma 3.2.17. We get the factorisation
\[
\sum_{\gcd(n,P(z)) = 1}^{b} \Delta_{z_1,z_2}(n)\psi_{r,r}(n)\left([n]f(s,\rho)\right)n^{-s} = f^p(s,\rho)M^{z_1,z_2}_r(s,\rho)
\]

where
\[
M^{z_1,z_2}_r(s,\rho) = \sum_{\gcd(n,P(z)) = 1} \prod_{p|gcd(r,mz)} (1 + \psi_{p}(p)\left([p]f(s,\rho)\right)p^{-s}) \prod_{p|n} (1 + ([p]f(s,\rho)p^{-s})) \lambda_{z_1,z_2}(n)\psi_{p,r}(n)\left([n]f(s,\rho)\right)n^{-s}
\]

Proof. This is rather straightforward: By mutlilplicative of \(\psi_{p,r}\) and \([\cdot]f(s,\rho)\), we get that
\[
\sum_{\gcd(n,P(z)) = 1}^{b} \Delta_{z_1,z_2}(n)\psi_{r,r}(n)\left([n]f(s,\rho)\right)n^{-s}
\]

\[
= \sum_{\gcd(n,P(z)) = 1}^{b} \left(\sum_{d|n} \lambda_{z_1,z_2}(d)\right)\psi_{r,r}(n)\left([n]f(s,\rho)\right)n^{-s}
\]

\[
= \sum_{\gcd(d,P(z)) = 1}^{b} \lambda_{z_1,z_2}(d)\psi_{r,r}(d)\left([d]f(s,\rho)\right)d^{-s} \sum_{\gcd(n,dP(z)) = 1}^{b} \psi_{r,r}(n)\left([n]f(s,\rho)\right)n^{-s}
\]

\[
= \sum_{\gcd(d,P(z)) = 1}^{b} \lambda_{z_1,z_2}(d)\psi_{r,r}(d)\left([d]f(s,\rho)\right)d^{-s} \prod_{p|d\rho} (1 + \psi_{p,r}(p)\left([p]f(s,\rho)\right)p^{-s}) \prod_{p|d\rho} (1 + ([p]f(s,\rho)p^{-s})
\]

\((p \mid r \implies p \geq z)\)

\[
= f^p(s,\rho) \sum_{\gcd(d,P(z)) = 1}^{b} \prod_{p|gcd(r,mz)} (1 + \psi_{p}(p)\left([p]f(s,\rho)\right)p^{-s}) \prod_{p|d\rho} (1 + ([p]f(s,\rho)p^{-s}) \lambda_{z_1,z_2}(d)\psi_{r,r}(d)\left([d]f(s,\rho)\right)d^{-s}
\]

\[
= f^p(s,\rho)M^{z_1,z_2}_r(s,\rho)
\]

The following lemma will provide a bound on \(M^{z_1,z_2}_r(s,\rho)\).

Lemma 3.2.18. Fix \(z \geq 1\) large enough and \(0 < \delta < \frac{1}{2}\). Let \(L(s,\rho; F/Q) = \mathcal{F}(X), r \in R_{\delta,z}(\rho)\) and \(1 \leq z_1 < z_2\). Then, for \(\varepsilon > 0\), there exists a constant \(C(\varepsilon)\) such that for all \(s \in \mathbb{C}\) with \(0 < \Re(s) \leq 1\),

\[|M^{z_1,z_2}_r(s,\rho)| \leq C(\varepsilon)r^{1+2\delta-\Re(s)+\varepsilon}z_2^{-1-\Re(s)+\varepsilon}
\]

Proof. First, we note that for \(n\) not squarefree,

\[\psi_{p,r}(n) = 0\]

Since for \(n \leq z_2\),

\[|\lambda_{z_1,z_2}(n)| \leq 1\]

and since for \(n > z_2\), \(\lambda_{z_1,z_2}(n) = 0\), we get that

\[
M^{z_1,z_2}_r(s,\rho) = \sum_{\gcd(n,P(z)) = 1}^{b} \prod_{p|gcd(r,mz)} (1 + \psi_{p}(p)\left([p]f(s,\rho)\right)p^{-s}) \prod_{p|n} (1 + ([p]f(s,\rho)p^{-s}) \lambda_{z_1,z_2}(n)\psi_{r,r}(n)\left([n]f(s,\rho)\right)n^{-s}
\]

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We now analyse the bound of each piece of the summand. For that, we recall the familiar arithmetic functions

- Note that for each prime \( p \),

\[
|1 + \psi_p(p)([p]f(s, \rho))p^{-\tau}| \leq 1 + \frac{p^{1-\Re(s)}}{|[p]f(s, \rho)|}
\]

Since \( r \in R_{\delta, z}(\rho) \), for all \( p \mid \frac{r}{\gcd(r, n)} \),

\[
p \mid r \implies \frac{1}{|[p]f(s, \rho)|} < p^\delta
\]

Thus, we get that

\[
\prod_{p \mid \gcd(r, n)} |1 + \psi_p(p)([p]f(s, \rho))p^{-\tau}| \leq \prod_{p \mid |r_n} (1 + p^{1+\delta-\Re(s)}) = \sum_{d \mid \gcd(r, n)} d^{1+\delta-\Re(s)} \leq \tau(r)r^{1+\delta-\Re(s)}
\]

- Recall that for each (unramified) prime \( p \), \( |[p]f(s, \rho)| \) is a sum of roots of unity by Remark 2.2.14. Hence \( |[p]f(s, \rho)| \leq \dim \rho \). Thus, for \( z \) large enough, say \( z \geq (2 \dim \rho)^{1/\Re(s)} \), we get that for \( p \geq z \),

\[
p^{\Re(s)} \geq 2 \dim \rho \implies \frac{p^{\Re(s)} - \dim \rho}{p^{\Re(s)}} \geq \frac{1}{2}
\]

\[
\implies |1 - |[p]f(s, \rho)|p^{-\Re(s)}| \geq 1 - |[p]f(s, \rho)|p^{-\Re(s)} \geq 1 - \frac{\dim \rho}{p^{\Re(s)}} = \frac{p^{\Re(s)} - \dim \rho}{p^{\Re(s)}} \geq \frac{1}{2}
\]

Hence, we get that

\[
\frac{1}{\prod_{p \mid |r_n} (1 + ([p]f(s, \rho))p^{-\tau})} \leq \prod_{p \mid |r_n} 2 = \tau(r)n = \tau(r)\left(\frac{n}{\gcd(r, n)}\right)
\]

- Now, note that if \( n \) is squarefree, then

\[
|\psi_{p, r}(n)([n]f(s, \rho))| = |\psi_p(\gcd(r, n))(n)f(s, \rho))|
\]

\[
= \frac{\gcd(r, n)}{[\gcd(r, n)]f(s, \rho)}\left|\frac{n}{\gcd(r, n)}\right|f(s, \rho) \leq (\dim \rho)^{\sigma(\frac{n}{\gcd(r, n)})}\frac{\gcd(r, n)}{[\gcd(r, n)]f(s, \rho)}
\]

where the last inequality is since for each (unramified) prime \( p \), \( |[p]f(s, \rho)| \) is a sum of roots of unity by Remark 2.2.14. Since \( r \in R_{\delta, z}(\rho) \), for all \( p \mid \gcd(r, n) \),

\[
p \mid r \implies \frac{1}{|[p]f(s, \rho)|} < p^\delta
\]
Thus, we get for $n$ squarefree that

$$ |\psi_{\rho,r}(n)[n]f(s,\rho)| \leq (\dim \rho)^{\frac{\gcd(n)}{\gcd(r,n)}} \frac{\gcd(r,n)}{|[\gcd(r,n)]f(s,\rho)|} $$

$$ \leq (\dim \rho)^{\frac{\gcd(n)}{\gcd(r,n)}} \gcd(r,n)^{1+\delta} \leq (\dim \rho)^{\frac{\gcd(n)}{\gcd(r,n)}} r^\delta \gcd(r,n) $$

Combining all these and using the fact that $\tau(n) \ll_n n^\varepsilon$, we get that there exists a constant $C(\varepsilon)$ such that

$$ |M_r^{z_1,z_2}(s,\rho)| \leq \sum_{1 \leq n \leq z_2} \frac{|\prod_{p | \gcd(n,P(z))} (1 + \psi_{\rho,p}([p]f(s,\rho))p^{-s})|}{\prod_{p | r} (1 + ([p]f(s,\rho))p^{-s})} |\psi_{\rho,r}(n)[n]f(s,\rho)| n^{-\Re(s)} $$

$$ \leq C(\varepsilon) \sum_{1 \leq n \leq z_2} n^{1-2\delta-\Re(s)+\varepsilon} n^{-\Re(s)+\varepsilon} $$

The result follows from the fact that

$$ \sum_{1 \leq n \leq z_2} n^{1-2\delta-\Re(s)+\varepsilon} \ll z_2^{1-\Re(s)+\varepsilon} $$

With this, we can prove the following lemma about these zero-detector, which explains the naming choice.

**Lemma 3.2.19.** Let $X \in \mathbb{N}$, $0 < \delta < \frac{1}{2}$, $z, R, x, T \geq 1$ and $1 \leq z_1 < z_2$. Let $F$ be Kowalski–Michel $\mathbb{Q}$-ameanable. Then, for each $(X, \delta, z, R, z_1, z_2, x, T)$-suitable pair $(\varepsilon, \alpha)$, there is a constant $C(\varepsilon)$ such that for $L(s,\rho;F/\mathbb{Q}) \in F(X)$ and $r \in R_{\delta,z}(\rho)$ with $r \leq R$, for $s_0 \in M(\alpha,T)$ with

$$ L(s_0,\rho;F/\mathbb{Q}) = 0 $$

we get that

$$ C(\varepsilon) \leq z(X,\delta,z,R,z_1,z_2,x,T)(s_0,\rho,r) $$

**Proof.** Let $(\varepsilon, \alpha)$ be $(X, \delta, z, R, z_1, z_2, x, T)$-suitable. First, we note that $\Gamma(s)$ is the Mellin transform of $e^{-s}$. Thus, using Mellin transform, we get

$$ e^{-\frac{\pi}{2}} + \sum_{\gcd(n,P(z)) = 1} n^{\sigma} \psi_{\rho,r}(n)[n]f(s,\rho) n^{-\sigma} e^{-\frac{n}{2}} = \frac{1}{2\pi i} \int_{(3)} f^\sigma(s + s_0,\rho) M_r^{z_1,z_2}(s + s_0,\rho) \Gamma(s) x^s \, ds $$

Let $\sigma = \frac{1}{2} - \alpha + \varepsilon$, where $\varepsilon > 0$, so that $\sigma + \alpha > \frac{1}{2}$. Since $\Gamma(s)$ has a simple pole only at $s = 0$, which is cancelled out by the zero of $f^\sigma(s + s_0,\rho)$ at $s = 0$, and $f^\sigma(s,\rho)$ has no poles due to $F$ being Kowalski–Michel $\mathbb{Q}$-ameanable, we can shift the contour to the line $\Re(s) = \sigma$, giving

$$ e^{-\frac{\pi}{2}} + \sum_{\gcd(n,P(z)) = 1} n^{\sigma} \psi_{\rho,r}(n)[n]f(s,\rho) n^{-\sigma} e^{-\frac{n}{2}} = \frac{1}{2\pi i} \int_{(\sigma)} f^\sigma(s + s_0,\rho) M_r^{z_1,z_2}(s + s_0,\rho) \Gamma(s) x^s \, ds $$

Since $(\varepsilon, \alpha)$ is $(X, \delta, z, R, z_1, z_2, x, T)$-suitable, using the uniform convexity bound of the family, along with Lemma 3.2.4 applied on $f^\sigma(s + s_0,\rho)$ on the line $\sigma + \Re(s) \geq \sigma + \alpha = \frac{1}{2} + \varepsilon$, and using the bound in Lemma 3.2.18 on $M_r^{z_1,z_2}$, we get that the integral is bounded by a constant $C(\varepsilon)$. The result then follows. \qed
A consequence of the lemma, along with Corollary 3.2.11 is the following.

**Corollary 3.2.20.** Let \( F \) be Kowalski–Michel \( Q \)-ameanable, and assume Conjecture 3.2.9. Fix \( \varepsilon \geq 1 \) large enough and \( 0 < \delta < \frac{1}{2} \). For each \( L(s, \rho; F, Q) \in F(X) \), let \( J_{\rho}(\alpha, T) \subseteq Z_\rho(\alpha, T) \) be a \( X \)-well-spaced finite set. Then, there exists a constant \( C(\varepsilon) \) such that if \( (\varepsilon, \alpha) \) is \( (X, \delta, z, R, z_1, z_2, x, T) \)-suitable, with

\[
x^{1-\varepsilon} > X^{d+\frac{ma}{2}} R^{2+2\delta} \log R
\]

and there exists a constant \( k \) such that

\[
\log \left( \frac{x}{z_1} \right) \leq k \log \left( \frac{z_2}{z_1} \right)
\]

we have that

\[
\sum_{L(s, \rho; F, Q) \in F(X)} \sum_{r \in R_{\alpha, z}(\rho)} \frac{|J_{\rho}(\alpha, T)|}{s_{\phi}(\rho, R)|\psi_{\phi}(r)|} \leq C(\varepsilon) T^B (\log X) x^{2(1-\alpha)}
\]

**Proof.** By Lemma 3.2.19 we get a constant \( C'(\varepsilon) \) such that

\[
\sum_{L(s, \rho; F, Q) \in F(X)} \sum_{r \in R_{\alpha, z}(\rho)} \frac{|J_{\rho}(\alpha, T)|}{s_{\phi}(\rho, R)|\psi_{\phi}(r)|} \leq C'(\varepsilon) \sum_{L(s, \rho; F, Q) \in F(X)} \frac{1}{s_{\phi}(\rho, R)} \sum_{s_{0} \in J_{\rho}(\alpha, T)} \sum_{r \in R_{\alpha, z}(\rho)} \frac{|z(x, \delta, z, R, z_1, z_2, x, T)(s_0, \rho, r)|^2}{|\psi_{\phi}(r)|}
\]

By Corollary 3.2.11 and since \( (\varepsilon, \alpha) \) is \( (X, \delta, z, R, z_1, z_2, x, T) \)-suitable, there exists a constant \( C''(\varepsilon) \) such that the right hand side is bounded by

\[
C''(\varepsilon) T^B \log X \sum_{n > x} |\Delta_{z_1, z_2}(n)| e^{-\frac{n}{2}} n^{1-2\alpha} \leq C''(\varepsilon) T^B \log X \sum_{z_1 < n < x} \Delta_{z_1, z_2}(n)^2 n^{1-2\alpha}
\]

where the second inequality is due to the fact that \( |e^{-\frac{n}{2}}| \leq 1 \) and \( \Delta_{z_1, z_2}(n) = 0 \) for \( 1 < n < z_1 \) by definition of \( \Delta_{z_1, z_2}(n) \). Finally, by Lemma 3.2.14 along with the fact that the assumption implies that

\[
\log \left( \frac{x}{z_1} \right) \leq k
\]

we get a constant \( C(\varepsilon) \) such that

\[
\sum_{L(s, \rho; F, Q) \in F(X)} \sum_{r \in R_{\alpha, z}(\rho)} \frac{|J_{\rho}(\alpha, T)|}{s_{\phi}(\rho, R)|\psi_{\phi}(r)|} \leq C(\varepsilon) T^B (\log X) x^{2-2\alpha}
\]

\[\square\]

### 3.3 Zero-Region of Kowalski–Michel \( Q \)-Ameanable Families

Using what we developed in the previous section, we can now proceed as in [KM02] to give a Kowalski–Michel estimate of the zero-region of Kowalski–Michel \( Q \)-ameanable families, which is itself based on [Jut78].
Consider $M(\alpha, T)$, where $\frac{3}{4} \leq \alpha \leq 1$ and $T \geq 2$. For each $k \in \mathbb{Z}$, we denote

$$M_k(\alpha, T) = \left\{ s \in M(\alpha, T); \text{Re}(s) \geq \alpha, \frac{k}{\log X} \leq \text{Im}(s) < \frac{k + 1}{\log X} \right\}$$

We partition $M(\alpha, T)$ as

$$M(\alpha, T) = \bigcup_{k=[-T\log X]}^{[T\log X]} M_k(\alpha, T)$$

We will state the following density lemma of Linnik.

**Lemma 3.3.1.** Let $t \in \mathbb{R}$. The number of zeroes of $L(s, \rho; K/F)$ in the square

$$\left\{ s \in \mathbb{C}; \alpha \leq \text{Re}(s) \leq 1, \left| \text{Im}(s) - t \right| \leq \frac{1}{2}(1 - \alpha) \right\}$$

is

$$\ll (1 - \alpha) \log(q(|t| + 2)) + 1$$

With the result of Lemma 3.2.12 and Corollary 3.2.20, we essentially have the result of [KM02] for Kowalski–Michel $\mathbb{Q}$-ameanable families. All it remains now is to choose suitable parameters to invoke those results, which we will demonstrate how below.

**Theorem 3.3.2.** Let $F$ be Kowalski–Michel $\mathbb{Q}$-ameanable, and assume Conjecture 3.2.9. Let $\alpha \geq \frac{3}{4}$ and $T \geq 2$. Then, there exists a constant $B > 0$, depending on the family parameters, such that for all $c > 10ma + 4d$, there exists a constant $M(c)$ such that

$$N_{F,\alpha,T}(X) \ll M(c)T^B X^{c\frac{1-\alpha}{2\alpha-1}}$$

for all $X \in \mathbb{N}$.

**Proof.** Let

$$R = X^{ma}, \quad z_1 = X, \quad z_2 = X^\omega$$

where $\delta > 0$ and $\omega > 0$ are such that

$$4ma + 2d > \frac{1}{2} + \omega + 4ma\delta$$

For $c > 5ma + 2d$, let $\varepsilon > 0$ be such that $(\varepsilon, \alpha)$ is $(X, \delta, z, R, z_1, z_2, x, T)$-suitable with

$$X^{\left(\frac{1}{2} - \varepsilon\right)\frac{c}{2\alpha-1}} > X^{d + \frac{ma}{2} R^{2 + 2\delta} \log R}$$

By letting $x = X^{\frac{c}{2\alpha-1}}$, the result will then follow by Corollary 3.2.20, Lemma 3.2.12 and Lemma 3.3.1. \qed

**Remark 3.3.3:** As a concluding remark, we will note the difference between Theorem 3.3.2 and [KM02], specifically the difference between our $c$ and the $c$ in [KM02]. In reality, we believe that our $c$ is the correct lower bound even in [KM02] using their arguments, which we mirrored here, based on our understanding of their argument. In effect, there should be no difference in the result between our setting and that of Kowalski–Michel, under the assumption of Conjecture 3.2.9.
Episode IV

A NEW FAMILY

In this chapter, we describe and study a particular family of Artin $L$-functions. These Artin $L$-functions will be associated to a special type of representation, which we will describe and study as well. In particular, we will prove variants of Aramata–Brauer Theorem for these representations. This in turn will allow us to show that the family is Kowalski–Michel ameanable, and hence we can apply results of the previous chapters regarding the zero-region of the family.

4.1 $\nu$-REGULAR REPRESENTATIONS

In this section, we will describe and study a special representation of a group. For this section, we will set the following: Let $G$ be a finite non-trivial group, and let $\hat{G}$ be the collection of all irreducible representations of $G$. Let $\text{reg}_G$ be the regular representation of $G$, i.e.

$$\text{reg}_G = \bigoplus_{\rho \in \hat{G}} (\dim \rho) \rho$$

4.1.1 SOCLE OF A GROUP

We begin by reviewing some basic group theoretic facts that we will need in the succeeding sections. For more in-depth discussions on minimal normal subgroups and socle of a group, we refer the interested readers to [Rot12].

We start with a definition.

**Definition 4.1.1.** A normal subgroup $M \trianglelefteq G$ of $G$ is a minimal normal subgroup of $G$ if $M$ is non-trivial and the only normal subgroups of $G$ contained in $N$ is $\{e\}$ or $N$. We denote the set of all minimal normal subgroups of $G$ by $\mathcal{M}(G)$. We also denote the set of all abelian minimal normal subgroups of $G$ by $\mathcal{A}(G)$, and the set of all non-abelian minimal normal subgroups of $G$ by $\mathcal{T}(G)$. In particular,

$$\mathcal{M}(G) = \mathcal{A}(G) \cup \mathcal{T}(G)$$

We will give a few examples.

**Example 4.1.1.** Consider the group $\mathfrak{S}_n$, where $n \geq 5$. Then, the alternating group $A_n \trianglelefteq \mathfrak{S}_n$ is the only non-trivial normal subgroup of $\mathfrak{S}_n$, and thus $A_n$ is the unique minimal normal subgroup of $\mathfrak{S}_n$.

We recall the following properties of minimal normal subgroups.

**Proposition 4.1.2.** Let $M \trianglelefteq G$ be a minimal normal subgroup of $G$, and let $N \trianglelefteq G$ be a normal subgroup of $G$. Then, either $M \leq N$ or $\langle M, N \rangle = M \times N$.

**Proposition 4.1.3.** For $T \in \mathcal{T}(G)$ distinct and $N \trianglelefteq G$ such that $T \not\leq N$, we have that the only non-trivial normal subgroups of $G$ contained in $\langle T, N \rangle = T \times N$ are of form $\{e\} \times M$, $T \times \{e\}$ or $T \times M$, where $M \leq N$ is non-trivial and normal in $G$. 54
Proof. Let \( K \leq G \) be non-trivial and such that \( K \leq T \times N \). There are two cases.

Case 1: Suppose \( K \cap T = T \). Since \( K \cap N \) is a normal subgroup of \( G \) which does not contain \( T \), by Proposition 4.1.2, we get that
\[
(K \cap N) \times T = \langle K \cap N, T \rangle \leq K
\]
If there exists \((n, t) \in K \setminus (K \cap N) \times T\), then in particular \( n \notin K \cap N \). Now, \((e, t^{-1}) \in (K \cap N) \times T \subseteq K\), and that
\[
(n, e) = (e, t^{-1})(n, t) \in K \implies n \in K \cap N
\]
which is a contradiction. Hence, \( K = (K \cap N) \times T \).

Case 2: Suppose \( K \cap T = \{e\} \). Then, by Proposition 4.1.2, \( K \) centralises \( T \). Since \( T \) is non-abelian, and in fact \( Z(T) = \{e\} \), every element of \( K \) should be of the form \((n, e)\). Thus, \( K \subseteq N \).

The result follows.

Minimal normal subgroups are particularly nice, as demonstrated by the following proposition.

**Proposition 4.1.4.** A minimal normal subgroup of \( G \) is either simple or a direct product of isomorphic simple groups.

We can now define the following.

**Definition 4.1.5.** A socle of \( G \) is the subgroup \( \text{Soc}(G) \) generated by all the minimal normal subgroups of \( G \), i.e.
\[
\text{Soc}(G) = \langle \mathcal{M}(G) \rangle
\]

Here are some examples, corresponding to the examples given earlier.

**Example 4.1.2.** Consider the group \( \mathfrak{S}_n \), where \( n \geq 5 \). Since \( A_n \) is the unique minimal normal subgroup of \( \mathfrak{S}_n \), we get that \( \text{Soc}(\mathfrak{S}_n) = A_n \).

The following proposition shows the decomposition of the socle of a finite group.

**Proposition 4.1.6.** There exists distinct minimal normal subgroups \( M_1, \ldots, M_k \in \mathcal{M}(G) \) of \( G \) such that
\[
\text{Soc}(G) = M_1 \times \cdots \times M_k
\]
Moreover, we get that
\[
\mathcal{T}(G) \subseteq \{M_1, \ldots, M_k\}
\]

**Proof.** Consider the poset
\[
P = \left\{ S \subseteq \mathcal{M}(G); \langle S \rangle = \prod_{M \in S} M \right\}
\]
with \( S_1 \leq S_2 \) in \( P \) if \( S_1 \subseteq S_2 \). Since \( G \) is finite, there exists a subset
\[
S = \{M_1, \ldots, M_k\} \subseteq \mathcal{M}(G)
\]
such that it is maximal in \( P \). Then, for all minimal normal subgroup \( M \) of \( G \), by Proposition 4.1.2 and by maximality of \( S \) we get that \( M \subseteq \langle S \rangle \). Thus, we get that
\[
\text{Soc}(G) = \langle S \rangle = M_1 \times \cdots \times M_k
\]
For the last assertion, suppose \( M \in \mathcal{M}(G) \) is such that \( M \notin S \). Then, since \( M \) centralises each element of \( S \) by Proposition 4.1.2, we get that \( M \subseteq Z(\text{Soc}(G)) \), which implies that \( M \) is abelian. Hence, every non-abelian minimal normal subgroup of \( G \) must be contained in \( S \).
As an immediate consequence, we get the following corollary.

**Corollary 4.1.7.** Soc$(G)$ is a direct product of simple groups.

In light of Proposition 4.1.6, we write

$$\text{Soc}(G) = \text{Ab}(\text{Soc}(G)) \times T(\text{Soc}(G))$$

where $\text{Ab}(\text{Soc}(G)) = \text{Soc}(G) / T(\text{Soc}(G))$ and

$$T(\text{Soc}(G)) = \prod_{T \in \mathcal{T}(G)} T$$

We note that $\text{Ab}(\text{Soc}(G))$ is abelian. Now, by iterated applications of Proposition 4.1.3, we get the following.

**Corollary 4.1.8.** The normal subgroups of $G$ contained in $\text{Soc}(G)$ is of the form

$$A \times \left( \prod_{T \in S} T \right)$$

where $A$ is a normal subgroup of $G$ contained $\text{Ab}(\text{Soc}(G))$ and $S \subseteq \mathcal{T}(G)$.

### 4.1.2 Faithful and Relatively Faithful Representations

We will now introduce the concept of relatively faithful representations, which will become important to us in the later part of this chapter.

The following is a basic definition in representation theory.

**Definition 4.1.9.** A representation $\rho : G \to \text{GL}(V)$ of $G$ is **faithful** if $\ker(\rho) = \{e\}$. We denote

$$\hat{G}^\nu = \left\{ \rho \in \hat{G} ; \ker(\rho) = \{e\} \right\}$$

We note here that $\hat{G}^\nu$ may be empty.

**Example 4.1.3.** For any group, recall that the regular representation of the group is faithful. This is a major example of a faithful representation.

**Example 4.1.4.** For any simple group $G$, recall that every non-trivial irreducible representation of $G$ is faithful. Thus, $\hat{G}^\nu = \hat{G} \backslash \{1_G\}$.

**Example 4.1.5.** Consider the following subgroup of order 18 of $\mathfrak{S}_6$

$$G = \langle (123), (456), (23)(56) \rangle$$

One can check that $G$ has no faithful irreducible representation, i.e. $\hat{G}^\nu = \emptyset$.

Faithful representations are less studied in classical representation theory, unlike say irreducible representations. For our purposes, for representation $\rho : G \to \text{GL}(V)$ of $G$, we get the diagram

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where, as before, the map
\[ \pi_N : G \to G/N \]
is the usual quotient map. Thus, since Artin L-function satisfies inflation, we view the Artin L-function associated to \( \rho \) as “originating” from \( \tilde{\rho} \), a faithful representation of \( G/\ker(\rho) \), and hence we descend down to the level where \( \rho \) is faithful. To be more precise, we have the following proposition.

**Proposition 4.1.10.** We have that
\[
\hat{G} = \bigsqcup_{N \leq G} \left\{ \rho \circ \pi_N; \rho \in \hat{G}/N^\nu \right\}
\]

**Proof.** For any irreducible representation \( \rho : G \to \text{GL}(V) \) of \( G \), we get the diagram
\[
1 \to \ker(\rho) \to G \xrightarrow{\pi_{\ker(\rho)}} G/\ker(\rho) \to 1
\]
\[
\downarrow \rho \quad \downarrow \tilde{\rho} \\
\text{GL}(V)
\]
where now, \( \tilde{\rho} \) is clearly faithful. Now, if \( W \leq V \) is \( G/\ker(\rho) \)-invariant, then for all \( g \in G \),
\[
\rho(g)W = \tilde{\rho} \circ \pi_{\ker(\rho)}(g)W = \tilde{\rho}(g \ker(\rho))W \subseteq W
\]
and by irreducibility of \( \rho \), we get that \( W \) is either \( \{0\} \) or \( V \). Thus, \( \tilde{\rho} \in \hat{G}/\ker(\rho) \).

Conversely, for an irreducible faithful representation \( \rho : G/N \to \text{GL}(V) \) of \( G \), then for \( W \leq V \) which is \( G \)-invariant under \( \rho \circ \pi_N \), for all \( gN \in G/N \)
\[
\rho(gN)W = \rho \circ \pi_N(gW) \subseteq W
\]
and by irreducibility of \( \rho \), we get that \( W \) is either \( \{0\} \) or \( V \). Thus, \( \rho \circ \pi_N \in \hat{G} \). The result follows. \( \square \)

Here, we collect a few cute facts regarding faithful representations.

**Proposition 4.1.11.** Let \( \rho \in \hat{G}^\nu \), then \( Z(G) \) is cyclic.

**Proof.** For \( z \in Z(G) \), we get that \( \rho(z) : V \to V \) such that \( \rho(z) \neq 0 \). Since \( Z(G) \) is abelian, \( \rho(z) \) is a(n endo)morphism of irreducible representations, so by Schur’s Lemma, we get that there exists \( c_z \in \mathbb{C}^\times \) such that \( \rho(z) = c_z I \). Consider the group homomorphism \( \varphi : Z(G) \to \mathbb{C}^\times \) given by
\[
\varphi(z) = c_z
\]
Since \( \rho \) is faithful, we get that \( \varphi \) is a injective group homomorphism, i.e. \( Z(G) \) is isomorphic to a finite abelian subgroup of \( \mathbb{C}^\times \) via \( \varphi \). Hence, we get that \( Z(G) \) is cyclic. \( \square \)
Theorem 4.1.12. Let $\rho : G \to \text{GL}(V)$ be a faithful representation of $G$. Then, for $\tau \in \hat{G}$, where $\tau : G \to \text{GL}(W)$, we get that $W$ is a subrepresentation of $\text{Sym}^n(V)$.

Proof.

We now introduce the notion of relatively faithful representations.

Definition 4.1.13. Let $N \leq G$ be a subgroup. A representation $\rho : N \to \text{GL}(V)$ of $N$ is $G$-relatively faithful if $\ker(\rho)$ is core-free in $G$, i.e.

$$\bigcap_{g \in G} \ker(\rho)^g = \{e\}$$

We denote

$$\hat{N}^\nu(G) = \left\{ \rho \in \hat{N} ; \rho \text{ is } G\text{-relatively faithful} \right\}$$

We note that for representation $\rho : G \to \text{GL}(V)$ of $G$, since $\ker(\rho)$ is normal, the normal core of $\ker(\rho)$ is

$$\bigcap_{g \in G} \ker(\rho)^g = \ker(\rho)$$

Hence, $\rho$ is $G$-relatively faithful if and only if $\rho$ is faithful. We see, therefore, that the notion of $G$-relative faithfulness is an extension of the notion of faithfulness.

The following proposition is immediate.

Proposition 4.1.14. For any $M \in \mathcal{M}(G)$, we get that

$$\hat{M}^\nu(G) = \hat{M} \setminus \{1_M\}$$

Proof. For any $\rho \in \hat{M}$, note that the normal core in $G$

$$\bigcap_{g \in G} \ker(\rho)^g \leq M$$

of $\ker(\rho)$ is normal in $G$. Since $M$ is a minimal normal subgroup, this implies that the normal core of $\ker(\rho)$ is either $\{e\}$ or $M$. The only irreducible representation of $M$ with the normal core in $G$ of its kernel being $M$ is the trivial one, so all other irreducible representations of $M$ is therefore $G$-relatively faithful.

We also prove the following result, which explains the motivation for the naming choice.

Proposition 4.1.15. Let $H \leq G$, and let $\rho \in \hat{H}^\nu(G)$. Then, the representation $\text{Ind}_H^G \rho$ of $G$ is faithful.

Proof. This is immediate by recalling that the kernel of $\text{Ind}_H^G \rho$ is

$$\bigcap_{g \in G} \ker(\rho)^g$$

We will also need the following result.

Proposition 4.1.16. For $T \in \mathcal{T}(G)$ distinct and $N \leq G$ such that $T \nleq N$, we get that

$$\left(\overline{N \times T}\right)^\nu(G) = \left\{ \rho \otimes \tau ; \rho_1 \in \hat{N}^\nu(G), \rho_2 \in \hat{T}^\nu(G) \right\}$$
Every representation in \( T \times N \) is of the form \( \rho \otimes \tau \), where \( \rho \in \hat{N} \) and \( \tau \in \hat{T} \). Recall that a subgroup is core-free in \( G \) if and only if it does not contain any non-trivial normal subgroup of \( G \). By Proposition \[4.1.3\] we get that ker(\( \rho_1 \otimes \rho_2 \)) is core-free in \( G \) if and only if it does not contain either \( T \) or the normal subgroup of \( G \) contained in \( N \). Now, for any normal group \( N' \) of \( G \) contained in \( N \),

\[
N' \subseteq \ker(\rho \otimes \tau) \iff \forall n \in N', I = \rho \otimes \tau(n) = \rho(n) \iff \ker(\rho \otimes \tau) = \ker(\rho)
\]

Also,

\[
T \subseteq \ker(\rho \otimes \tau) \iff \forall t \in T, I = \rho \otimes \tau(t) = \tau(t) \iff \ker(\rho \otimes \tau) = \ker(\tau)
\]

Hence, ker(\( \rho_1 \otimes \rho_2 \)) is core-free in \( G \) if and only if both ker(\( \rho \)) and ker(\( \tau \)) are core-free in \( G \).

As a direct consequence, we get the following result.

**Corollary 4.1.17.** We get in particular that

\[
\text{Soc}(G)^{\nu}(G) = \left\{ \rho \otimes \left( \bigotimes_{T \in T(G)} \tau_T \right); \rho \in \text{Ab}(\text{Soc}(G))^\nu(G), \tau_T \in \hat{\nu}(G) \right\}
\]

**Proof.** This is just an iteration application of Proposition \[4.1.16\] \( \blacksquare \)

### 4.1.3 \( \nu \)-Regular and Relative \( \nu \)-Regular Representations

We will now introduce a new representation, which will mirror regular representation for faithful and relatively faithful irreducible representations.

We start with a series of definitions.

**Definition 4.1.18.** For finite group \( G \) such that \( \hat{G}^{\nu} \neq \emptyset \), the \( \nu \)-regular representation of \( G \) is the representation

\[
\text{reg}_{G}^{\nu} = \bigoplus_{\rho \in \hat{G}^{\nu}} (\dim \rho)\rho
\]

**Definition 4.1.19.** Let \( H \leq G \) be a subgroup such that \( \hat{H}^{\nu}(G) \neq \emptyset \). The \( G \)-relative \( \nu \)-regular representation of \( H \) is the representation

\[
\text{reg}_{H}^{\nu}(G) = \bigoplus_{\rho \in \hat{H}^{\nu}(G)} (\dim \rho)\rho
\]

We provide a few examples.

**Example 4.1.6.** For any simple group \( G \), recall that \( \hat{G}^{\nu} = \hat{G} \setminus \{1_G\} \), and so we get that

\[
\text{reg}_{G}^{\nu} = \text{reg}_{G} \oplus 1_G
\]

**Example 4.1.7.** Consider \( \mathfrak{S}_4 \). One can compute and find that

\[
\hat{\mathfrak{S}_4}^{\nu} = \{ \rho, \rho \otimes \text{sgn} \}
\]

where \( \rho \) is the standard representation of \( \mathfrak{S}_4 \) and \( \text{sgn} \) is the sign representation. Both faithful irreducible representations of \( \mathfrak{S}_4 \) are of degree 3. Thus

\[
\text{reg}_{\mathfrak{S}_4}^{\nu} = (3\rho) \oplus (3\rho \otimes \text{sgn})
\]
Consider also the socle of $\mathcal{G}_4$, $\text{Soc}(\mathcal{G}_4) \cong \left( \mathbb{Z}/2\mathbb{Z} \right)^2$. Since $\text{Soc}(\mathcal{G}_4)$ is an abelian group which is not cyclic, we get that $\widehat{\text{Soc}(\mathcal{G}_4)} = \emptyset$. However, one can compute and find that

\[
\widehat{\text{Soc}(\mathcal{G}_4)}(\mathcal{G}_4) = \text{Soc}(\mathcal{G}_4) \setminus \{1_{\text{Soc}(\mathcal{G}_4)}\}
\]

\[
\implies \text{reg}^\nu_{\text{Soc}(\mathcal{G}_4)}(\mathcal{G}_4) = \rho_1 \oplus \rho_2 \oplus \rho_3
\]

where $\rho_i$ are the three non-trivial irreducible representations of $\text{Soc}(\mathcal{G}_4)$.

By Proposition 4.1.10, we get a decomposition of regular representations to $\nu$-regular representations, as demonstrated below.

**Proposition 4.1.20.** We get that

\[
\text{reg}_G = \bigoplus_{N \leq G} \text{reg}^\nu_G / N \circ \pi_N
\]

A very direct consequence of the definition and Corollary 4.1.17 is the following proposition.

**Proposition 4.1.21.** We get that

\[
\text{reg}^\nu_{\text{Soc}(G)}(G) = \text{reg}^\nu_{\text{Ab}(\text{Soc}(G))}(G) \otimes \text{reg}^\nu_{\text{T}_1}(G) \otimes \cdots \otimes \text{reg}^\nu_{\text{T}_m}(G)
\]

We prove the following lemma that relates the $\nu$-regular representation of $G$ with the $G$-relative $\nu$-regular representation of $\text{Soc}(G)$.

**Lemma 4.1.22.** We have that

\[
\text{reg}^\nu_G = \text{Ind}^G_{\text{Soc}(G)} \text{reg}^\nu_{\text{Soc}(G)}(G)
\]

**Proof.** First, for $\rho \in \widehat{\text{Soc}(G)} \setminus \widehat{\text{Soc}(G)}(G)$, note that

\[
\ker \text{Ind}^G_{\text{Soc}(G)} \rho = \bigcap_{\gamma \in G} \ker(\rho)^\gamma \ni \{e\}
\]

and so $\langle \chi, \text{Ind}^G_{\text{Soc}(G)} \rho \rangle = 0$ for $\chi \in \hat{G}^\nu$. Since $r_G = \text{Ind}^G_{\text{Soc}(G)} \tau_{\text{Soc}(G)}$, we get that

\[
\dim \chi = \langle \chi, \text{reg}_G \rangle = \langle \chi, \text{Ind}^G_{\text{Soc}(G)} \text{reg}_{\text{Soc}(G)} \rangle = \langle \chi, \text{Ind}^G_{\text{Soc}(G)} \text{reg}_{\text{Soc}(G)}(G) \rangle + \sum_{\rho \in \text{Soc}(G) \setminus \widehat{\text{Soc}(G)}(G)} \dim \rho \langle \chi, \text{Ind}^G_{\text{Soc}(G)} \rho \rangle = \langle \chi, \text{Ind}^G_{\text{Soc}(G)} \text{reg}_{\text{Soc}(G)}(G) \rangle
\]

Let $\rho \in \overline{\text{Soc}(G)}$. Suppose $\chi \in \hat{G} \setminus \hat{G}^\nu$ is such that $\langle \chi, \text{Ind}^G_{\text{Soc}(G)} \rho \rangle \neq 0$, and let $N \subseteq \ker \chi$ be a minimal normal subgroup of $G$. By Frobenius Reciprocity,

\[
\langle \text{Res}^G_{\text{Soc}(G)} \chi, \rho \rangle = \langle \chi, \text{Ind}^G_{\text{Soc}(G)} \rho \rangle \neq 0
\]

and this implies that $N \subseteq \ker \rho$. Thus, $\ker \rho$ is not core-free in $G$, and hence by definition, $\rho \notin \overline{\text{Soc}(G)}(G)$. Thus, for $\rho \in \overline{\text{Soc}(G)}(G)$, we get that $\langle \chi, \text{Ind}^G_{\text{Soc}(G)} \rho \rangle = 0$ for all $\chi \in \hat{G} \setminus \hat{G}^\nu$. This implies that

\[
\langle \chi, \text{Ind}^G_{\text{Soc}(G)} \text{reg}_{\text{Soc}(G)}(G) \rangle \neq 0 \implies \chi \in \hat{G}^\nu
\]

and we get that $\text{reg}^\nu_G = \text{Ind}^G_{\text{Soc}(G)} \text{reg}_{\text{Soc}(G)}(G)$. 

\[ \square \]
Here is an example showing Lemma 4.1.22 in action.

**Example 4.1.8.** Consider $S_4$. We already showed in Example 4.1.7 that

$$\text{reg}_{\text{Soc}(S_4)}^\nu(S_4) = \rho_1 \oplus \rho_2 \oplus \rho_3$$

where $\rho_i$ are the three non-trivial irreducible representations of $\text{Soc}(S_4) \cong (\mathbb{Z}/2\mathbb{Z})^2$. By computing (using GAP, for example), one can also see that for $i = 1, 2, 3$,

$$\text{Ind}_{\text{Soc}(S_4)}^\nu\rho_i = \rho \oplus (\rho \otimes \text{sgn})$$

where $\rho$ is the standard representation of $S_4$ and $\text{sgn}$ is the sign representation. Thus, we have that

$$\text{Ind}_{\text{Soc}(S_4)}^\nu\text{reg}_{\text{Soc}(S_4)}^\nu(S_4) = 3\rho \oplus 3(\rho \otimes \text{sgn}) = \text{reg}_{\text{Soc}(S_4)}^\nu$$

as predicted by Lemma 4.1.22.

The following property is needed for studying the convolution over $F$ of two $\nu$-regular representations.

**Proposition 4.1.23.** We get that

$$(\text{reg}_G^\nu)^* = \text{reg}_G^\nu$$

**Proof.** Recall that a (finite-dimensional) representation of $\rho : G \to \text{GL}(V)$ is irreducible if and only if its dual is. Further,

$$g \in \ker(\rho) \iff \rho(g) = I \iff \forall \phi \in V^*, \rho^*(g)f(v) = f(\rho(g)^{-1}v) = f(v), \forall v \in V \iff \rho^*(g) = I \iff g \in \ker(\rho^*)$$

The result now follows by definition of $\nu$-regular representation. 

4.1.4 **Variations on a Theorem of Aramata–Brauer**

In this section, we consider the Artin $L$-functions associated to $\nu$-regular representations of $G$. Let us set our notation: Let $K/F$ be a Galois extension of number fields. We denote the following.

**Definition 4.1.24.** We denote the Artin $L$-function associated to $\nu$-regular representation $\text{reg}_{\text{Gal}(K/F)}^\nu$ as

$$\zeta_{K/F}^\nu(s) = L(s, \text{reg}_{\text{Gal}(K/F)}^\nu; K/F)$$

We also denote the Artin $L$-function associated to $\text{Conv}_F(\text{reg}_{\text{Gal}(K/F)}^\nu, \text{reg}_{\text{Gal}(K'/F')}^\nu)$ as

$$(\zeta_{K/F}^\nu, \zeta_{K'/F'}^\nu)(s) = L(s, \text{Conv}_F(\text{reg}_{\text{Gal}(K/F)}^\nu, \text{reg}_{\text{Gal}(K'/F')}^\nu); KK'/F)$$

We now prove an extension of Aramata–Brauer Theorem, for $\zeta_{K/F}^\nu(s)$.

**Theorem 4.1.25.** Let $K/F$ be a Galois extension. The Artin $L$-function $\zeta_{K/F}^\nu(s)$ is entire.
Proof. By Lemma 4.1.22, we get that
\[ |G|_{\text{reg}} = |G : \text{Soc}(G)||\text{Soc}(G)||\text{Ind}_{\text{Soc}(G)}^G \text{reg}_{\text{Soc}(G)}(G) \]
\[ = |G : \text{Soc}(G)| |A| \text{Ind}_{\text{Soc}(G)}^G \text{reg}_{A}(G) \otimes |T_1| \text{reg}_{T_1}(G) \otimes \cdots \otimes |T_m| \text{reg}_{T_m}(G) \]
By Proposition 4.1.14 and by Lemma 2.3.5 each \(|T_i| \text{reg}_{T_i}(G)\) can be written as
\[ |T_i| \text{reg}_{T_i}(G) = |T_i| (\text{reg}_{T_i} \otimes 1_{T_i}) = \bigoplus_{C_i} \bigoplus_{\psi_C \in C_i \setminus 1_{C_i}} m_{C_i, \psi_C} \text{Ind}_{C_i}^{T_i} \psi_{C_i} \]
where the sum is taken over all cyclic subgroups of \(T_i\) and each \(m_{C_i, \psi_C} \in \mathbb{N}\). Thus, we get that \(|G|_{\text{reg}}\) is a positive integer combination of representations of the form
\[ \text{Ind}_{A \times C_1 \times \cdots \times C_m}^G \chi \otimes \psi_{C_1} \otimes \cdots \otimes \psi_{C_m} \]
where \(\chi \otimes \psi_{C_1} \otimes \cdots \otimes \psi_{C_m}\) is a non-trivial 1-dimensional representation of \(A \times C_1 \times \cdots \times C_m\). The result now follows easily similar to the original Aramata–Brauer theorem. \(\Box\)

Remark 4.1.26: Note that for each term \(\text{Ind}_{A \times C_1 \times \cdots \times C_m}^G \chi \otimes \psi_{C_1} \otimes \cdots \otimes \psi_{C_m}\) the term \(\text{Ind}_{A \times C_1 \times \cdots \times C_m}^G \chi^{-1} \otimes \psi_{C_1}^{-1} \otimes \cdots \otimes \psi_{C_m}^{-1}\) also appears as a summand.

We also prove an extension of Aramata–Brauer Theorem, for \((\chi_{K/F}, \chi_{L/F})(s)\).

**Theorem 4.1.27.** Let \(K, K'/F\) be Galois extensions such that \(\text{Gal}(K/F) \cong G\) and \(\text{Gal}(K'/F) \cong G\). Then,
\[ (\chi_{K/F}, \chi_{K'/F})(s) \]
is entire if and only if \(K \neq K'\). If \(K = K'\), then it has a pole only at \(s = 1\).

**Proof.** First, by Proposition 4.1.3, we get that
\[ \text{Conv}_F(\text{reg}_{\text{Gal}(K/F)}, \text{reg}_{\text{Gal}(K'/F)}) = \text{Res}_{\text{Gal}(K/K') \times \text{Gal}(K'/F)}^F \text{reg}_{\text{Gal}(K/F)} \otimes \text{reg}_{\text{Gal}(K'/F)} \]
By Theorem 4.1.25 consider each summand of \(|G|_{\text{reg}}^F \otimes \text{reg}_{\text{Gal}(K'/F)}^F\) as
\[ \left( \text{Ind}_{S}^\text{Gal}(K/F) \chi \right) \otimes \left( \text{Ind}_{R}^\text{Gal}(K'/F) \rho \right) = \text{Ind}_{S \times R}^{\text{Gal}(K/K') \times \text{Gal}(K'/F)} \chi \otimes \rho \]
where \(\chi\) and \(\rho\) are non-trivial 1-dimensional representations on subgroup \(S \subseteq \text{Soc}(\text{Gal}(K/F))\) and \(R \subseteq \text{Soc}(\text{Gal}(K'/F))\) respectively. By Mackey Decomposition Theorem, we get that
\[ \text{Res}_{\text{Gal}(K/K') \times \text{Gal}(K'/F)}^F \text{Ind}_{S \times R}^{\text{Gal}(K/K') \times \text{Gal}(K'/F)} \chi \otimes \rho = \bigoplus_{\text{Gal}(K/K') \sigma(S \times R)} \text{Ind}_{S \times R}^{\text{Gal}(K/K') \sigma(S \times R)} \text{Res}_{\text{Gal}(K/K') \sigma(S \times R)}^{S \times R} \left( \chi \otimes \rho \right)^\sigma \]

- Suppose that \(K \neq K'\). Now, for all \(\sigma = (\sigma_1, \sigma_2) \in \text{Gal}(K/F) \times \text{Gal}(K'/F)\)
\[ \text{Gal}(K/K') \cap (S \times R)^\sigma = \{ \phi = (\phi_1, \phi_2) \in (S \times R)^\sigma; \phi_1|_{K \cap K'} = \phi_2|_{K \cap K'} \} \]
Let \(N_K \leq \text{Gal}(K/F)\) be associated to \(K \cap K'\). Since \(K \neq K'\), \(N_K \neq \{e\}\), so they must contain \(T_K\) a minimal normal subgroups of \(\text{Gal}(K/F)\). In particular, every \(\phi \in T_K\) is such that \(\phi|_{K \cap K'}\) is identity, and thus
\[ (S \cap T_K)^{\sigma_1} \times \{e\} = ((S \cap T_K) \times \{e\})^{\sigma} \subseteq \text{Gal}(K/K') \cap (S \times R)^\sigma \]
and by the definition of $\chi$, the $(\chi \otimes \rho)^\sigma$ is non-trivial on $(S \cap T)_{\mathbb{K}}^\sigma \times \{e\}$, and thus

$$\text{Res}_{\text{Gal}(K/K') \cap (S \times R)^\sigma} (\chi \otimes \rho)^\sigma$$

is a non-trivial 1-dimensional representation. It then follows that each summand of

$$|G|^2 \text{Conv}_F (\text{reg}_F^\nu \text{Gal}(K/F), \text{reg}_F^\nu \text{Gal}(K'/F)) = \text{Res}_{\text{Gal}(K/F) \times \text{Gal}(K'/F)} |G|^2 \text{reg}_F^\nu \text{Gal}(K/F) \otimes \text{reg}_F^\nu \text{Gal}(K'/F)$$

is a direct sum of induced non-trivial 1-dimensional representations, and thus it follows that

$$(\zeta_{K/F}^\nu, \zeta_{K'/F}^\nu)(s)^{|G|^2}$$

is entire. Since $(\zeta_{K/F}^\nu, \zeta_{K'/F}^\nu)(s)$ is meromorphic, we get that it is therefore entire.

- Suppose that $K = K'$. Consider the case where $S = R$. Now, for $\sigma = e$,

$$\text{Gal}(K/F) \cap (S \times S) = \{ (\phi, \phi); \phi \in S \}$$

By Remark 4.1.26 we can choose $\rho = \chi^{-1}$, in which case since $\chi \otimes \chi^{-1}$ is 1-dimensional, for all $\phi \in S$

$$\chi(\phi) \otimes \chi^{-1}(\phi) = I$$

This implies that

$$\text{Res}_{\text{Gal}(K/F) \cap (S \times S)} \chi \otimes \chi^{-1}$$

is therefore trivial. It then follows that $(\zeta_{K/F}^\nu, \zeta_{K'/F}^\nu)(s)$ must have a pole at $s = 1$.

\section{4.2 Family of Artin $L$-Functions Associated to $\nu$-Regular Representation}

We now turn our attention to defining a new family of Artin $L$-functions which is amenable to the analytical methods of Kowalski–Michel, as established in the previous chapter.

We set the following notation: Let $G$ be a finite group, and let $F$ be a number field.

\subsection{4.2.1 A New Family and Its Kowalski–Michel Ameanability}

We will now introduce a new family of Artin $L$-functions.

\begin{definition}
Let $G$ be a finite group, and let $F$ be a number field. Let $\mathcal{F}_{F,G}^\nu$ be defined as

$$\mathcal{F}_{F,G}^\nu = \{ \mathcal{F}_{F,G}^\nu(X); X \in \mathbb{N} \}$$

where for each $X \in \mathbb{N}$,

$$\mathcal{F}_{F,G}^\nu(X) = \{ \zeta_{K/F}^\nu(s); K/F \text{ Galois, Gal}(K/F) \cong G, N^F_Q(\Delta_{K/F}) < X \}$$

We prove that $\mathcal{F}_{F,G}^\nu$ is indeed a family of Artin $L$-functions.

\begin{lemma}
$\mathcal{F}_{F,G}^\nu$ is a family of Artin $L$-function of degree $\dim \text{reg}_G^\nu$.
\end{lemma}

\begin{proof}
We prove that $\mathcal{F}_{F,G}^\nu$ satisfies the conditions of being a family.
\end{proof}
For any $\zeta_{K/F}(s) \in \mathcal{F}_{F,G}(X)$, by Proposition 4.1.20 additivity of Artin conductors and the Conductor-Discriminant Formula we get that the conductor of $\zeta_{K/F}(s)$ is

$$A(\text{reg}_{\text{Gal}(K/F)} \nu K/F; K/F) = |\Delta_{F/Q}|^{\dim \text{reg}_{\text{Gal}(K/F)} \nu Q} \left( f(\text{reg}_{\text{Gal}(K/F)} \nu Q, K/F) \right) \leq |\Delta_{F/Q}|^{\dim \text{reg}_{\text{Gal}(K/F)} \nu Q} \left( f(\text{reg}_{G} \nu Q, K/F) \right) \leq |\Delta_{F/Q}|^{\dim \text{reg}_{\text{Gal}(K/F)} \nu Q} X$$

For every $X \in \mathbb{N}$, we get that

$$|\mathcal{F}_{F,G}(X)| \leq N_{F,G}(X)$$

where $N_{F,G}(X)$ is the number of Galois extensions of $F$ of degree $n$ with absolute discriminant at most $X$, i.e.

$$N_{F,G}(X) := \left| \left\{ K/F \text{ Galois; } [K : F] = n, N_{Q}^F(\Delta_{K/F}) < X \right\} \right|$$

By a well-known result of Schmidt [Sch95], we get that

$$|\mathcal{F}_{F,G}(X)| \leq N_{F,G}(X) \leq N_{F,G}(X) := \left| \left\{ K/F ; [K : F] = n, N_{Q}^F(\Delta_{K/F}) < X \right\} \right| \leq C(F, |G|) X^{\frac{|G|+2}{4}}$$

where $C(F, n)$ is a constant depending on $F$ and $n$.

With the constants $a = 1$, $C_{\text{cond}} = |\Delta_{F/Q}|^{\dim \text{reg}_{\text{Gal}(K/F)} \nu Q}$, $d = \frac{|G|+2}{4}$ and $C_{\text{order}} = C(F, |G|)$, this makes $\mathcal{F}_{F,G}$ a family of Artin $L$-functions of degree $\dim \text{reg}_{\text{Gal}(K/F)} \nu Q$.

More importantly for us, we will prove that $\mathcal{F}_{F,G}$ is Kowalski–Michel ameanable, and thus is ameanable to the results in the previous chapter.

**Theorem 4.2.3.** $\mathcal{F}_{F,G}$ is Kowalski–Michel $F$-ameanable, with (one possible) $F$-family parameter

$$\left( \dim \text{reg}_{\text{Gal}(K/F)} \nu Q, F, a = 1, C_{\text{cond}} = |\Delta_{F/Q}|^{\dim \text{reg}_{\text{Gal}(K/F)} \nu Q}, d = \frac{|G|+2}{4}, C_{\text{order}} = C(F, |G|), C_{\text{convex}}, C'_{\text{convex}} \right)$$

**Proof.** Every member of $\zeta_{K/F} \nu F_{F,G}$ is entire by Theorem 4.1.25 and satisfy the same uniform convexity bound by Lemma 2.5.2, as they are all Artin $L$-functions of the same degree and associated to the same representation. Further, by Theorem 4.1.27 $\mathcal{F}_{F,G}$ satisfy the convolution property, and that each $(\zeta_{K/F}, \zeta_{G/F})(s)$ satisfy the same uniform convexity bound again by Lemma 2.5.2, as they are all Artin $L$-functions of the same degree. \qed

In light of Lemma 3.1.6, even more is true.

**Corollary 4.2.4.** For $\mathfrak{A} \subseteq F$ with $F/\mathfrak{A}$ being Galois, we have that $\mathcal{F}_{F,G}$ is Kowalski–Michel $\mathfrak{A}$-ameanable, with $\mathfrak{A}$-family parameter

$$\left( \dim \text{reg}_{\text{Gal}(K/F)} \nu Q, \mathfrak{A}, a = 1, C_{\text{cond}} = |\Delta_{F/Q}|^{\dim \text{reg}_{\text{Gal}(K/F)} \nu Q}, d = \frac{|G|+2}{4}, C_{\text{order}} = C(F, |G|), C_{\text{convex}}, (C'_{\text{convex}})^{[F: \mathfrak{A}]} \right)$$

To conclude, we relate the new family to the family of Dedekind zeta-functions. We now define the following family.

**Definition 4.2.5.** Let

$$\mathcal{F}_{F,G}' = \left\{ \mathcal{F}_{F,G}(X); X \in \mathbb{N} \right\}$$

where for each $X \in \mathbb{N}$,

$$\mathcal{F}_{F,G}(X) = \left\{ \frac{\zeta_K(s)}{\zeta_F(s)} ; K/F \text{ Galois, } \text{Gal}(K/F) = G, N_{Q}^F(\Delta_{K/F}) < X \right\}$$
The following proposition is straightforward.

**Proposition 4.2.6.** $\mathcal{F}^\nu_{F,G}$ is a family of Artin $L$-functions of degree $|G| - 1$.

**Proof.** We prove that $\mathcal{F}^\nu_{F,G}$ satisfies the conditions of being a family.

- For any $\frac{\zeta_F(s)}{\zeta_F(s)} \in \mathcal{F}^\nu_{F,G}(X)$, by additivity of conductors and the Conductor-Discriminant Formula we get that the conductor of $\zeta^\nu_{K/F}(s)$ is

$$A(\text{reg}_{\text{Gal}(K/F)} \ominus 1_{\text{Gal}(K/F)}; K/F) \leq A(\text{reg}_{\text{Gal}(K/F)}; K/F) = |\Delta_{F/Q}|^{|G|} N^F_Q(f(\text{reg}_G; K/F)) \leq |\Delta_{F/Q}|^{|G|} X$$

- For every $X \in \mathbb{N}$, we get by a well-known result of Schmidt that

$$|\mathcal{F}^\nu_{F,G}(X)| \leq N^\text{Gal}_{F,G}(X) \leq N_{F,G}(X) := \left| \{ K/F; [K:F] = n, N^F_{\mathbb{Q}}(\Delta_{K/F}) < X \} \right| \leq C(F, |G|) X^{|G|+2}$$

where $C(F, n)$ is a constant depending on $F$ and $n$.

With the constants $a = 1$, $C_{\text{cond}} = |\Delta_{F/Q}|^{|G|}$, $d = \frac{|G|+2}{4}$ and $C_{\text{order}} = C(F, |G|)$, this makes $\mathcal{F}^\nu_{F,G}$ a family of Artin $L$-functions of degree $|G| - 1$.

We denote the $a$ and $d$ that makes $\mathcal{F}^\nu_{F,G}$ a family by $a_{F,G}$ and $d_{F,G}$ respectively.

We now factorise $\zeta_F(s)$ courtesy of Proposition 4.1.20.

**Proposition 4.2.7.** We get that

$$\zeta_F(s) = \prod_{N \leq G} \zeta^\nu_{K/N \supseteq F}(s) \prod_{N \leq G} \zeta^\nu_{K/N \supseteq F}(s)$$

In light of Proposition 4.2.7, we can think of each $\frac{\zeta_F(s)}{\zeta_F(s)} \in \mathcal{F}^\nu_{F,G}$ as

$$\left( \zeta^\nu_{K/N \supseteq F}(s) = L(s, \text{reg}_{G/N}; K^N/F) \in \mathcal{F}^\nu_{F,G/N} \right)_{N \leq G}$$

### 4.2.2 Zero-Free Region For Subfamilies of Dedekind Zeta-Functions

We now wish to leverage Kowalski–Michel zero-region estimate to subfamilies of Dedekind zeta-functions. For a subfamily $\mathcal{H}$ of $\mathcal{F}^\nu_{F,G}$, we denote the $a$ and $d$ that makes $\mathcal{F}^\nu_{F,G}$ a family by $a_{F,G}(\mathcal{H})$ and $d_{F,G}(\mathcal{H})$ respectively. We will assume Conjecture 3.2.9. Let us define the following.

**Definition 4.2.8.** Let $\mathcal{H}$ be a subfamily of $\mathcal{F}^\nu_{F,G}$. We define

$$\mathcal{F}^\nu_{F,G/N}(\mathcal{H}) = \left\{ \mathcal{F}^\nu_{F,G/N}(\mathcal{H})(X); X \in \mathbb{N} \right\}$$

where

$$\mathcal{F}^\nu_{F,G/N}(\mathcal{H})(X) = \left\{ \frac{\zeta_F(s)}{\zeta_F(s)} \in \mathcal{F}^\nu_{F,G/N}(X); \frac{\zeta_F(s)}{\zeta_F(s)} \in \mathcal{H}(X) \right\}$$

It is clear that $\mathcal{F}^\nu_{F,G/N}(\mathcal{H})$ is a subfamily of $\mathcal{F}^\nu_{F,G/N}$, so by Proposition 3.1.5, $\mathcal{F}^\nu_{F,G/N}(\mathcal{H})$ is Kowalski–Michel $F$-ameanable.

We will describe the following condition of Pierce–Turnage-Butterbaugh–Wood on subfamilies of $\mathcal{F}^\nu_{F,G}$, which they denoted as inequality (6.6) of Condition 6.4 in [PTBW17].
Definition 4.2.9. Let $\mathcal{H}$ be a subfamily of $\mathcal{F}_{F,G}'$. We say $\mathcal{H}$ is sterile if for each normal $N < G$ (note that $N \neq G$), there exists $0 \leq \tau_N < d_{F,G}(\mathcal{H})$ and constant $M_N$, depending on the family parameters of $\mathcal{F}_{F,G}'$, such that for all $X \in \mathbb{N}$, for $L(s, \text{reg}^\nu_{G/N}; K'/F) \in \mathcal{F}_{F,G}'(X)$

$$\left\{ \left. \frac{\zeta_K(s)}{\zeta_F(s)} \in \mathcal{H}(X) \mid \zeta_{\mathcal{K}/F}(s) = L(s, \text{reg}^\nu_{G/N}; K'/F) \right\} \leq M_N X^{\tau_N}$$

In this case, the contamination index of $\mathcal{H}$ is

$$\tau = \max_{N < G} \tau_N$$

We will remark that the sterility condition only depends on members of $\mathcal{F}_{F,G}'(\mathcal{H})$, rather than the full $\mathcal{F}_{F,G}'$, to satisfy the required inequality.

One way to think of the sterility condition is the following: When trying to bring Kowalski–Michel zero-region estimate for each $\mathcal{F}_{F,G}'$ up to the level of $\mathcal{H}$, we needed to ensure that each “bad” Artin $L$-functions, the ones which are not zero-free, do not “contaminate” too many members $\mathcal{H}$. Propogation of such “contamination” is effectively controlled by the inequality condition in the definition of sterility.

Then, proceeding as in [PTBW17], we get the average zero-free region result of Pierce–Turnage-Butterbaugh–Wood which is unconditional, in the case where $F/Q$ is Galois.

Theorem 4.2.10. Let $F/Q$ be Galois, and assume Conjecture 3.2.6. Let $\mathcal{H}$ be a sterile subfamily of $\mathcal{F}_{F,G}'$ with contamination index $\tau = \max_{N < G} \tau_N$. Then, for any $0 < \Delta < 1 - \frac{\tau}{d_{F,G}(\mathcal{H})}$ and $\eta < \frac{1}{4}$, there exists $B$ (depending on the Kowalski–Michel family parameter of $\mathcal{F}_{F,G}'(\mathcal{H})$ for each $N < G$), a constant $M(G, a_{F,G}(\mathcal{H}), d_{F,G}(\mathcal{H}), \Delta, \tau)$ and $0 < \delta \leq \frac{1}{4}$, given by

$$\delta = \frac{\varepsilon}{20m[F : \mathbb{Q}]a_{F,G}(\mathcal{H}) + 8d_{F,G}(\mathcal{H}) + 4\varepsilon}$$

with $m = \max_{N < G} \text{dim} \text{reg}^\nu_{G/N}$ and

$$\Delta = 1 - \frac{\tau}{d_{F,G}(\mathcal{H})} - \frac{\varepsilon}{2d_{F,G}(\mathcal{H})}$$

such that for all $X \in \mathbb{N}$, at most $M(G, a_{F,G}(\mathcal{H}), d_{F,G}(\mathcal{H}), \Delta, \tau) X^{(1 - (1 - \eta)\Delta)d_{F,G}(\mathcal{H})}$ members of $\mathcal{H}(X)$ that can have a zero in the region

$$[1 - \delta, 1] \times \left[ -X^{\eta \Delta d_{F,G}(\mathcal{H})}, X^{\eta \Delta d_{F,G}(\mathcal{H})} \right]$$

Proof. Let $\varepsilon > 0$ be such that

$$\Delta = 1 - \frac{\tau}{d_{F,G}(\mathcal{H})} - \frac{\varepsilon}{2d_{F,G}(\mathcal{H})}$$

As $F/Q$ is Galois, for each $N < G$, by Corollary 4.2.4, $\mathcal{F}_{F,G}'(\mathcal{H})$ is Kowalski–Michel $\mathbb{Q}$-ameanable, so let

$$(n_N, \mathbb{Q}, a_N, C_{\text{cond},N}, d, C_{\text{order},N}, C_{\text{convex},N}, C'_{\text{convex},N})$$

be the Kowalski–Michel $\mathbb{Q}$-family parameters of $\mathcal{F}_{F,G}'(\mathcal{H})$. We define

$$c_N = 10n_N[F : \mathbb{Q}]a_{F,G}(\mathcal{H}) + 4d_{F,G}(\mathcal{H}) + \varepsilon$$
Note that we can choose the \( a_N \) and \( d_N \) such that \( a_{F,G}(\mathcal{H}) = \max_{N < G} a_N \) and \( d_{F,G} = \max_{N < G} d_N \), so for simplicity we will assume that. Thus

\[
c_N = 10n_N[F : \mathbb{Q}]a_{F,G}(\mathcal{H}) + 4d_{F,G}(\mathcal{H}) + \varepsilon > 10n_Na_N + 4d_N
\]

Let us define, for each \( N < G \),

\[
\alpha_N = \frac{c_N + (1 - \Delta)d_{F,G}(\mathcal{H}) - \tau}{c_N + 2((1 - \Delta)d_{F,G}(\mathcal{H}) - \tau)} = \frac{c_N + \frac{\varepsilon}{2}}{c_N + \varepsilon} = 1 - \frac{\varepsilon}{2c_N + 2\varepsilon} \geq \frac{3}{4}
\]

Note that by simple calculation,

\[
c_N \frac{1 - \alpha_N}{2\alpha_N - 1} = \frac{\varepsilon}{2} = (1 - \Delta)d_{F,G}(\mathcal{H}) - \tau
\]

By Theorem 3.3.2 we get that there exists a constant \( B > 0 \), depending on the family parameters, there exists a constant \( M_N \) (which depends on \( n_N, a_{F,G}(\mathcal{H}), d_{F,G}(\mathcal{H}), \Delta \) and \( \tau \)) such that for all \( T_N \geq 2 \),

\[
N_{\mathcal{F},G,F,G}^{\nu}(\mathcal{H},\alpha_N,T_N)(X) \leq M_NT_BX^{c_N\frac{1 - \alpha_N}{2\alpha_N - 1}} = M_NT_BX^{(1 - \Delta)d_{F,G}(\mathcal{H}) - \tau}
\]

We finally set \( T_N = X^{\frac{\eta\Delta d_{F,G}(\mathcal{H})}{B_N}} \), and thus we get that

\[
N_{\mathcal{F},G,F,G}^{\nu}(\mathcal{H},\alpha_N,T_N)(X) \leq M_NX^{\eta\Delta d_{F,G}(\mathcal{H})+(1-\Delta)d-\tau} = M_NX^{(1-\eta\Delta)d_{F,G}(\mathcal{H})-\tau}
\]

Let

\[
\alpha = \max_{N < G} \alpha_N = 1 - \frac{\varepsilon}{2[F : \mathbb{Q}]\max_{N < G} c_N + 2\varepsilon} = 1 - \frac{\varepsilon}{20[F : \mathbb{Q}]\max_{N < G} n_Na_{F,G}(\mathcal{H}) + 8d_{F,G}(\mathcal{H}) + 4\varepsilon}
\]

which is therefore dependent on \( G, a_{F,G}(\mathcal{H}), d_{F,G}(\mathcal{H}), \Delta \) and \( \tau \). Further, set

\[
T = \min_{N < G} T_N = X^{\frac{\eta\Delta d_{F,G}(\mathcal{H})}{B_N}} \quad \text{and} \quad M = \max_{N < G} M_N
\]

Now, combining all the above and by sterility of \( \mathcal{H} \), we get that

\[
N_{\mathcal{H},\alpha,T}(X) \leq \sum_{N < G} X^{T_N}N_{\mathcal{F},G,F,G}^{\nu}(\mathcal{H},\alpha_N,T_N)(X) \leq MX^{(1 - (1 - \eta)\Delta)d_{F,G}(\mathcal{H})} \{[ N < G]\}
\]

By noting that \( M \) is therefore dependent on \( G, a_{F,G}(\mathcal{H}), d_{F,G}(\mathcal{H}), \Delta \) and \( \tau \), we get that there exists at most \( MX^{(1 - (1 - \eta)\Delta)d_{F,G}(\mathcal{H})} \) members of \( \mathcal{H}(X) \) that can have a zero in the region

\[
[\alpha, 1] \times \left[ -X^{\frac{\eta\Delta d_{F,G}(\mathcal{H})}{B}}, X^{\frac{\eta\Delta d_{F,G}(\mathcal{H})}{B}} \right]
\]

where \( B = \max_{N < G} B_N \). Finally, we note here the dependency of \( B \) on the Kowalski–Michel family parameter of \( \mathcal{F}_{F,G}^{\nu}(\mathcal{H}) \) for each \( N < G \).

We wish to point out that for general number fields \( F \), given a Kowalski–Michel type zero-region estimate for Kowalski–Michel \( F \)-ameanable families, one can proceed exactly as in the proof of Theorem 4.2.10 to get a similar unconditional average zero-free region of members of sterile subfamilies of \( \mathcal{F}_{F,G}^{\nu} \).

To conclude, let us define the following.
Definition 4.2.11. Let \(0 < \delta < \frac{1}{2}\). A degree \(n\) \(G\)-extension of \(F/K\), i.e. extension \(K/F\) with \([K : F] = n\) and \(\text{Gal}(K^F/F) = G\), is a \(\delta\)-exceptional field if \(\frac{\zeta_{K^F/F}(s)}{\zeta_F(s)}\) has a zero in the region

\[
[1 - \delta, 1] \times \left[ -\left(\log \left| \Delta_{K^F/F} \right| \right)^{\frac{2}{3}}, \left(\log \left| \Delta_{K^F/F} \right| \right)^{\frac{2}{3}} \right]
\]

We note that under the veracity of Generalised Riemann Hypothesis, there exists no \(\delta\)-exceptional field for any \(\delta < \frac{1}{2}\). We also define the following.

Definition 4.2.12. For

\[
Z \subseteq \left\{ K/F; [K : F] = n, \text{Gal}(K^F/F) = G \right\}
\]

we denote

\[
\mathcal{H}(Z) = \{ \mathcal{H}(Z)(X); X \in \mathbb{N} \}
\]

where

\[
\mathcal{H}(Z)(X) = \left\{ \frac{\zeta_{K^F/F}(s)}{\zeta_F(s)}; K/F \in Z, \left| \Delta_{K^F/F} \right| < X \right\}
\]

with each \(\mathcal{H}(Z)(X)\) a multi-set.

A consequence of Theorem 4.2.10 is the following.

Corollary 4.2.13. Let \(F/\mathbb{Q}\) be Galois, and assume Conjecture 3.2.9. Let

\[
Z \subseteq \left\{ K/F; [K : F] = n, \text{Gal}(K^F/F) = G \right\}
\]

If \(\mathcal{H}(Z)\) is a sterile subfamily of \(\mathcal{F}_{F, G}^\nu\) with contamination index \(\tau\), then for any \(0 < \varepsilon < \frac{1}{2}\) with

\[
\Delta = 1 - \frac{\tau}{d_{F,G}(\mathcal{H}(Z))} - \frac{\varepsilon}{2d_{F,G}(\mathcal{H}(Z))} > 0
\]

there exists a constant \(D(Z, \tau, \varepsilon)\) such that for all \(X \in \mathbb{N}\), there are at most \(D(Z, \tau, \varepsilon)X^{\tau + \varepsilon}\) \(\delta\)-exceptional fields in

\[
\left\{ K/F \in Z; \left| \Delta_{K^F/F} \right| < X \right\}
\]

where

\[
\delta = \frac{\varepsilon}{20m[F : \mathbb{Q}]a_{F,G}(\mathcal{H}(Z)) + 8d_{F,G}(\mathcal{H}(Z)) + 4\varepsilon}
\]

with \(m = \max_{N \leq G} \dim \text{reg}_{G/N}^\nu\).

Proof. Define

\[
\Delta = 1 - \frac{\tau}{d_{F,G}(\mathcal{H}(Z))} - \frac{\varepsilon}{2d_{F,G}(\mathcal{H}(Z))} > 0
\]

and let \(\eta = \frac{\varepsilon}{2d_{F,G}(\mathcal{H}(Z))}\). Note that

\[
(1 - (1 - \eta)\Delta)d_{F,G}(\mathcal{H}(Z)) = \tau + \frac{\varepsilon}{2}(1 + \Delta) \leq \tau + \varepsilon
\]

Then, by Theorem 4.2.10 and noting that the Kowalski–Michel family parameter of \(\mathcal{F}_{F,G/N}^\nu(\mathcal{H}(Z))\) depends on \(Z\), we get that there exists a \(B\) depending on \(Z\) and \(M(Z, \tau, \varepsilon)\) such that for all \(X \in \mathbb{N}\), at
most $M(Z, \tau, \varepsilon)X^{\tau+\varepsilon}$ fields $K/F \in \mathbb{Z}_n^\varphi(F, G)$ with $|\Delta_{K^p/F}| < X$ have the property that $\frac{\zeta_{K^p(s)}}{\zeta_{F}(s)}$ has a zero in $[1-\delta, 1] \times \left[-X^{\frac{\varepsilon}{2\delta}}, X^{\frac{\varepsilon}{2\delta}}\right]$

Let $D'(Z, \tau, \varepsilon)$ be a constant such that for all $y \geq D'(Z, \tau, \varepsilon)$, we have that

$$(\log y)^{\frac{\delta}{2}} \leq y^{\frac{\varepsilon}{2\delta}}$$

Then, the only possible $\delta$-exceptional field that is not eliminated via the count above are the ones whose absolute discriminant is less than $D'(Z, \tau, \varepsilon)$. Now,

$$_{K' \text{ fields}}$$

$$\frac{\{ K/F \in \mathbb{Z}_n^\varphi(F, G); \ |\Delta_{K^p/F}| < D'(Z, \tau, \varepsilon)\}}{\{ \mathcal{H}(Z)(D'(Z, \tau, \varepsilon)) \}} \leq KD'(Z, \tau, \varepsilon)^{d_{F,G}(\mathcal{H}(Z))}$$

for some constant $K$ since $\mathcal{H}(Z)$ is a family of Artin $L$-functions. We therefore get the result. 

\[\square\]

### 4.2.3 Sterile Subfamilies of $\mathcal{F}_{\mathbb{Q}, G}$

Although we developed no additional result towards the study of sterile subfamilies in this thesis, we will present what is known so far for completeness.

We will now focus on the case of $F = \mathbb{Q}$, and assume Conjecture 3.2.9. In this case, we can convert the sterility condition to a field-counting condition. To make this transition rigorously, we cite the following lemma of Pierce–Turnage-Butterbaugh–Wood, which they refined from a result of Klüners and Nicolae in [KN16].

**Lemma 4.2.14.** Let $\chi$ be a character of $G \leq S_n$, a fixed transitive subgroup. For any Galois extensions $K/\mathbb{Q}$, $K'/\mathbb{Q}$ with $\text{Gal}(K/\mathbb{Q}) \cong \text{Gal}(K'/\mathbb{Q}) \cong G$, we have that

$L(s, \chi; K/\mathbb{Q}) = L(s, \chi; K'/\mathbb{Q})$

if and only if $K^\ker(\chi) = (K')^\ker(\chi)$.

For a proof, we refer the interested readers to [PTBW17] for the reduction to the case $\chi$ is faithful, and to [KN16] for the proof in that case. We will, however, stress that the lemma critically relies on the fact that we are working over $\mathbb{Q}$. Klüners and Nicolae did prove a relative version of the lemma, which Pierce–Turnage-Butterbaugh–Wood have remarked that it may be possible to obtain an analogous version of the lemma for certain cases of $F \neq \mathbb{Q}$ and certain choices of $G$.

With this lemma, the following proposition is straightforward.

**Proposition 4.2.15.** Let $\mathcal{H}$ be a subfamily of $\mathcal{F'}_{\mathbb{Q}, G}$. Then, $\mathcal{H}$ is sterile if for each normal $N \triangleleft G$ (note that $N \neq G$), there exists $0 \leq \tau_N < d_{F,G}$ and constant $M_N$, depending on the family parameters of $\mathcal{F'}_{F,G,N}$, such that for all $X \in \mathbb{N}$, for any Galois extension $K'/\mathbb{Q}$ with $\text{Gal}(K'/\mathbb{Q}) \cong G/N$ and $|\Delta_{K'/\mathbb{Q}}| < X$,

$$\left| \left\{ \left\{ \frac{\zeta_{K}(s)}{\zeta_{Q}(s)} \in \mathcal{H}(X); \ K^N = K' \right\} \right\} \right| \leq M_N X^{\tau_N}$$

In this case, the contamination index of $\mathcal{H}$ is

$$\tau = \max_{N \triangleleft G} \tau_N$$
We will now briefly summarise the result of Pierce–Turnage-Butterbaugh–Wood which gives a list of sterile subfamilies of $\mathcal{F}_{Q,G}'$, as field-counting is beyond the scope of this thesis. We will define the following.

\textbf{Definition 4.2.16.} Let $n \in \mathbb{N}$, and let $G \subseteq \mathfrak{S}_n$ be a transitive subgroup. Let

$$\mathcal{I} = \bigcup_{i=1}^{k} C_i$$

where each $C_i$ is a conjugacy class of $G$. We say that a degree $n$ $G$-extension of $Q$, $K/Q$, i.e. extension $K/Q$ with $[K : Q] = n$ and $\text{Gal}(\overline{K}^Q/Q) = G$, has \textbf{tame-ramification type} $\mathcal{I}$ if for all rational prime $p$ which is tamely-ramified in $K$, the inertia group of primes of $\mathcal{O}_K$ lying over $p$ is generated by an element of $\mathcal{I}$. We denote

$$Z_n^{\mathcal{I}}(Q, G) = \left\{ K/Q; [K : Q] = n, \text{Gal}(\overline{K}^Q/Q) = G, K/Q \text{ has tame-ramification type } \mathcal{I} \right\}$$

Note that the terminology “tame-ramification type” is not a common terminology used in the literature, and it is defined here only for ease of reference.

\textbf{Definition 4.2.17.} Let $n \in \mathbb{N}$, and let

$$\mathcal{I} = \bigcup_{i=1}^{k} C_i$$

where each $C_i$ is a conjugacy class of $G$. We define a subfamily $\mathcal{H}_{n}^{\mathcal{I}}(Q, G)$ of $\mathcal{F}'_{Q,G}$ to be

$$\mathcal{H}_{n}^{\mathcal{I}}(Q, G) = \{ \mathcal{H}_{n}^{\mathcal{I}}(Q, G)(X); X \in \mathbb{N} \}$$

where

$$\mathcal{H}_{n}^{\mathcal{I}}(Q, G)(X) = \left\{ \frac{\zeta_{K,Q}(s)}{\zeta_{Q}(s)}, K/Q \in Z_n^{\mathcal{I}}(Q, G), \left| \Delta_{K/Q} \right| < X \right\}$$

As $a_{Q,G} \leq 1$, we can choose $a_{Q,G}(\mathcal{H}_{n}^{\mathcal{I}}(Q, G)) = 1$.

The field-counting result of Pierce–Turnage-Butterbaugh–Wood can be summarised as follows.

\textbf{Proposition 4.2.18.} $\mathcal{H}_{n}^{\mathcal{I}}(Q, G)$ is a sterile subfamily of $\mathcal{F}'_{Q,G}$ if $G, \mathcal{I}$ is one of the following:

- $G$ simple and any $\mathcal{I}$, with contamination index $\tau = 0$.
- $G$ is cyclic and $\mathcal{I}$ the set of all generators of $G$, with contamination index $\tau = 0$.
- $G = \mathfrak{S}_n$ with $3 \leq n \leq 5$ and $\mathcal{I}$ is the conjugacy class of transpositions, with contamination index $\tau = \frac{1}{3}$ when $n = 3$, $\tau = \frac{1}{2}$ when $n = 4$ and $\tau < 1$ for $n = 5$.
- $G = D_{2p}$, a dihedral group of order $2p$, with $p$ an odd prime and $\mathcal{I}$ is the conjugacy class of $(2p) \cdots (\frac{p+1}{2}, \frac{p+3}{2})$, with contamination index $\tau = \frac{1}{p-1}$.
- $G = A_4$ and $\mathcal{I}$ is the union of the conjugacy class of $(123)$ and $(132)$, with contamination index $\tau = 0.2784$...
We refer interested readers to [PTBW17]. Instead, we conclude with the following corollary, with the additional observation that in the case listed in Proposition 4.2.18, as long as

\[ \varepsilon < \frac{d_{Q,G}(\mathcal{H}_n^\varphi(Q, G))}{4} \]

we have that

\[ \Delta = 1 - \frac{\tau}{d_{Q,G}(\mathcal{H}_n^\varphi(Q, G))} - \frac{\varepsilon}{2d_{Q,G}(\mathcal{H}_n^\varphi(Q, G))} > 0 \]

**Corollary 4.2.19.** For \( G, \mathcal{I} \) being one of those listed in Proposition 4.2.18 with the corresponding contamination index \( \tau \), we have that for any \( 0 < \varepsilon < \min\left(\frac{1}{2}, \frac{d_{Q,G}(\mathcal{H}_n^\varphi(Q, G))}{4}\right) \), there exists a constant \( D(n, G, \mathcal{I}, \tau, \varepsilon) \) such that for all \( X \in \mathbb{N} \), there are at most \( D(n, G, \mathcal{I}, \tau, \varepsilon)X^{\tau + \varepsilon} \) \( \delta \)-exceptional fields in

\[ \left\{ K/Q \in \mathbb{Z}_n^\varphi(Q, G); \left| \Delta_{\mathcal{K}^\varphi/Q} \right| < X \right\} \]

where

\[ \delta = \frac{\varepsilon}{20m + 8d_{Q,G}(\mathcal{H}_n^\varphi(Q, G)) + 4\varepsilon} \]

with \( m = \max_{N < G} \dim \text{reg}_{G/N}^\nu \).

We note that the corollary differs slightly from the result in [PTBW17], since we consider fields according to the absolute discriminant \( \left| \Delta_{\mathcal{K}^\varphi/Q} \right| \) of the Galois closure of \( \mathcal{K} \) rather than the absolute discriminant \( \left| \Delta_{K/Q} \right| \) of \( K \) itself. We remark that we can obtain the result in [PTBW17] by tweaking

\[ \mathcal{H}(Z)(X) = \left\{ \frac{\zeta_{\mathcal{K}^\varphi}(s)}{\zeta_F(s)}; K/F \in Z, \left| \Delta_{K/F} \right| < X \right\} \]

and making the appropriate tweaks for each result as required.
Chapter 5

**AN APPLICATION: EFFECTIVE CHEBOTAREV DENSTY THEOREM**

In this chapter, we will give a brief overview of the work of Pierce, Turnage-Butterbaugh and Wood on a new effective Chebotarev density result. This will demonstrate an application of our result on the average zero-free region of Kowalski–Michel \( \mathbb{Q} \)-ameanable families developed in the previous chapter. Readers are encouraged to refer to [PTBW17] for a more detailed account of their work.

### 5.1 Previously Known Chebotarev Density Results

Let \( F \) be a number field, and consider a normal extension \( K/F \) with Galois group \( \text{Gal}(K/F) \). For \( C \subseteq \text{Gal}(K/F) \) a conjugacy class, consider the set

\[
P(C, K/F) = \{ p \triangleleft \mathcal{O}_F; p \text{ is unramified in } K, \sigma_p = C \}
\]

Chebotarev density result, first proved by Chebotarev in [Tsc26], describes the density of the set \( P(C, K/F) \).

**Chebotarev Density Theorem.** Let \( F \) be a number field, and consider a normal extension \( K/F \) with Galois group \( \text{Gal}(K/F) \). Define

\[
P(F) = \{ p \triangleleft \mathcal{O}_F \}
\]

For \( C \subseteq \text{Gal}(K/F) \) a conjugacy class, the set

\[
P(C, K/F) = \{ p \triangleleft \mathcal{O}_F; p \text{ is unramified in } K, \sigma_p = C \}
\]

has density

\[
\lim_{x \to \infty} \frac{\left| \{ p \in P(C, K/F); N^F_{\mathbb{Q}}(p) \leq x \} \right|}{\left| \{ p \in P(F); N^F_{\mathbb{Q}}(p) \leq x \} \right|} = \frac{|C|}{|\text{Gal}(K/F)|}
\]

Chebotarev Density Theorem is a celebrated result in analytic number theory, being the principal tool in deducing many analytic property of subset of primes. It is also a generalisation of Dirichlet’s theorem on primes in arithmetic progressions.

We define the following prime counting function in \( \mathcal{O}_F \), given by

\[
\pi(x, C, K/F) = \left| \{ p \in P(C, K/F); N^F_{\mathbb{Q}}(p) \leq x \} \right|
\]

One can rephrase the Chebotarev Density Theorem as

\[
\pi(x, C, K/F) \sim \frac{|C|}{|\text{Gal}(K/F)|} \text{Li}(x) \sim \frac{|C|}{|\text{Gal}(K/F)|} \frac{x}{\log x}
\]

where we denote

\[
\text{Li}(x) = \int_2^x \frac{1}{\log t} \, dt
\]
Chebotarev’s original result, while powerful, gave no information regarding the error terms of \( \pi(x, C, K/F) \), which is deficient in deducing more delicate results in prime counting.

We are interested in an effective version of Chebotarev Density Theorem, i.e. one with more explicit result regarding the error terms of \( \pi(x, C, K/F) \). Lagarias and Odlyzko proved two versions of effective Chebotarev Density Theorems in [LO77], which we will quote here. The first version is a version which is conditional on the Generalised Riemann Hypothesis, which has been refined further by Serre in [Ser81].

**Theorem 5.1.1.** Let \( F \) be a number field, and consider a normal extension \( K/F \) with Galois group \( \text{Gal}(K/F) \). Let the absolute discriminant of \( K \) be \( |\Delta_{K/Q}| \). If the Generalised Riemann Hypothesis holds for \( \zeta_K(s) \), then there exists a constant \( C \) such that for \( C \) belonging to a conjugacy class, and for \( x \geq 2 \), we have that

\[
\left| \pi(x, C, K/F) - \frac{|C|}{|\text{Gal}(K/F)|} \text{Li}(x) \right| \leq C \frac{|C|}{|\text{Gal}(K/F)|} x^{\frac{3}{2}} \log(|\Delta_{K/Q}| x^{[K:Q]})
\]

This version of effective Chebotarev density result has been useful in many applications, but is unfortunately dependent on the veracity of Generalised Riemann Hypothesis. However, Lagarias and Odlyzko proved an unconditional version, which we state below.

**Theorem 5.1.2.** Let \( F \) be a number field, and consider a normal extension \( K/F \) with Galois group \( \text{Gal}(K/F) \). Let the absolute discriminant of \( K \) be \( |\Delta_{K/Q}| \). If \( [K : Q] > 1 \), then \( \zeta_K(s) \) has at most one zero \( \beta_0 \) in the region

\[
\left\{ s \in \mathbb{C}; \Re(s) \geq 1 - \frac{1}{4 \log |\Delta_{K/Q}|}, \Im(s) \leq \frac{1}{4 \log |\Delta_{K/Q}|} \right\}
\]

The exceptional zero \( \beta_0 \), if it exists, is real and simple. Further, there exists effectively computable constants \( C_1 \) and \( C_2 \) such that for all \( x \geq e^{10 |K:Q| \log |\Delta_{K/Q}|^2} \),

\[
\left| \pi(x, C, K/F) - \frac{|C|}{|\text{Gal}(K/F)|} \text{Li}(x) \right| \leq \frac{|C|}{|\text{Gal}(K/F)|} \text{Li}(x^{\beta_0}) + C_1 x e^{-C_2 \left( \frac{\log x}{|\Delta_{K/Q}|} \right)^{\frac{1}{2}}}
\]

The \( x^{\beta_0} \) term is present only when \( \beta_0 \) exists.

However, this version is not useful in a lot of application, simply because of the requirement on \( x \) is too large and restrictive.

### 5.2 Effective Chebotarev Density Theorem of Pierce–Turnage-Butterbaugh–Wood

In [PTBW17], Pierce, Turnage-Butterbaugh and Wood obtained a new effective Chebotarev density result which removes the term corresponding to the exceptional zero and to hold for \( x \) as small as \( |\Delta_{K/Q}|^{\frac{1}{8}} \) for any fixed \( \delta > 0 \). Here, we will give a brief sketch of the proof of their result. For a detailed account of the proof, refer to [PTBW17].
5.2.1 Statement of Effective Chebotarev Density Theorem of Pierce–Turnage-Butterbaugh–Wood

The idea is to reach a compromise between the two versions of effective Chebotarev density result of Lagarias–Odlyzko. Specifically, we wish to prove an effective Chebotarev density result which is conditional on a weaker zero-free condition on the Dedekind zeta-function $\zeta_K(s)$, in the hopes of trading a more powerful result. In particular, instead of a zero-free condition on $\zeta_K(s)$ in the form of Generalised Riemann Hypothesis, we instead have a zero-free condition on $\frac{\zeta_K(s)}{\zeta_F(s)}$, which is a much weaker condition that is accommodating to phenomenon like the possible existence of an exceptional zero of $\zeta_F(s)$ in its standard zero-free region.

Due to the essential weakening of the Generalised Riemann Hypothesis condition, one expects to pay a cost for such a trade. In light of our result in the previous chapter and the assumptions in [PTBW17], the cost manifests as a result which is true for “almost all” fields instead of a result for all fields.

We first recall the following zero-free region results for $\zeta_F(s)$ for a number field $F$. C.f. [IK04] for more details and proof of the following.

**Lemma 5.2.1.** Let $F/\mathbb{Q}$ be a number field with absolute discriminant $|\Delta_{F/\mathbb{Q}}|$. There exists an absolute constant $c_F > 0$ such that $\zeta_F(s)$ has no zeroes in the region

$$\left\{ s \in \mathbb{C}; \text{Re}(s) \geq 1 - \frac{c_F}{|F : \mathbb{Q}|^2 \log \left( |\Delta_{F/\mathbb{Q}}| \right)} \right\}$$

except for a possibly simple real zero $\beta_0^{(F)} < 1$.

In the case where $F = \mathbb{Q}$, we have a stronger result, due to Vinogradov [Vin58] and Korobov [Kor58], which is the current best known result in this direction.

**Lemma 5.2.2.** There exists an absolute constant $c_Q > 0$ such that $\zeta_Q(s)$ has no zeroes in the region

$$\left\{ s \in \mathbb{C}; \text{Re}(s) \geq 1 - \frac{c_Q}{(\log (|t| + 2))^{\frac{7}{2}} (\log \log (|t| + 3))^{\frac{1}{2}}} \right\}$$

With the above zero-free region on $\zeta_F(s)$, we can state the effective Chebotarev Density Theorem of Pierce–Turnage-Butterbaugh–Wood. We first state it for general number field $F$.

**Theorem 5.2.3.** Let $F/\mathbb{Q}$ be a number field with absolute discriminant $|\Delta_{F/\mathbb{Q}}|$. Let $c_F$ be such that $\zeta_F(s)$ is zero-free in the region

$$\left\{ s \in \mathbb{C}; \text{Re}(s) \geq 1 - \frac{c_F}{|F : \mathbb{Q}|^2 \log \left( |\Delta_{F/\mathbb{Q}}| \right)} \right\}$$

and let $\beta_0^{(F)} < 1$ be the exceptional real zero of $\zeta_F(s)$ in this region if it exists. Let $A \geq 2$ and let $0 < \delta \leq \frac{1}{2A}$. Let $G$ be a transitive subgroup of $S_n$, where $n \in \mathbb{N}$ with $[F : \mathbb{Q}]|G| \geq 2$. Then, there exists effectively computable absolute constants $C_1, C_2, C_3, C_4$, and a constant

$$D(\delta, A, |G|, c_F, \beta_0^{(F)}, [F : \mathbb{Q}], C_1, C_2, C_3)$$

such that for any Galois extension $K/F$ with $\text{Gal}(K/F) \cong G$, $|\Delta_{K/\mathbb{Q}}| > D(\delta, A, |G|, c_F, \beta_0^{(F)}, [F : \mathbb{Q}], C_1, C_2, C_3)$ and $\frac{\zeta_K(s)}{\zeta_F(s)}$ being zero-free in the region

$$[1 - \delta, 1] \times \left[ - (\log |\Delta_{K/\mathbb{Q}}|)^{\frac{7}{2}}, (\log |\Delta_{K/\mathbb{Q}}|)^{\frac{1}{2}} \right]$$
there exists constants
\[ k_i = k_i(\delta, A, |G|, c_F, \beta_0^{(F)}, |\Delta_{F/Q}|, [F : Q], [K : Q], C_3, C_4) \]
for \( i = 1, 2, 3 \) such that for any conjugacy class \( \mathcal{C} \subseteq G \), we have for all \( x \geq k_1 e^{k_2 (\log \log |\Delta_{K/Q}|)^2} \),
\[ \left| \frac{\pi(x, \mathcal{C}, K/F)}{|G| \operatorname{Li}(x)} - \frac{|\mathcal{C}|}{|G|} \right| \leq \frac{|\mathcal{C}|}{|G|} \frac{x}{(\log x)^A} \]
We further state the result specialised to the case of \( F = \mathbb{Q} \).

**Theorem 5.2.4.** Let \( c_Q \) be such that \( \zeta(s) \) is zero-free in the region
\[
\left\{ s \in \mathbb{C}; \operatorname{Re}(s) \geq 1 - \frac{c_Q}{(\log (|t| + 2))^{\frac{2}{3}} (\log \log (|t| + 3))^{\frac{1}{3}}} \right\}
\]
Let \( A \geq 2 \) and let \( 0 < \delta \leq \frac{1}{2A} \). Let \( G \) be a transitive subgroup of \( S_n \), where \( n \in \mathbb{N} \) with \( |G| \geq 2 \). Then, there exists effectively computable absolute constants \( C_1, C_2, C_3, C_4, \) and a constant
\[ D(\delta, A, |G|, c_Q, C_1, C_2, C_3) \]
such that for any Galois extension \( K/\mathbb{Q} \) with \( \operatorname{Gal}(K/\mathbb{Q}) \cong G \), \( |\Delta_{K/Q}| > D(\delta, A, |G|, c_Q, C_1, C_2, C_3) \) and \( \frac{\zeta_K(s)}{\zeta(s)} \) being zero-free in the region
\[ [1 - \delta, 1] \times \left[ -\left( \log |\Delta_{K/Q}| \right)^{\frac{2}{3}}, \left( \log |\Delta_{K/Q}| \right)^{\frac{1}{3}} \right] \]
there exists constants
\[ k_i = k_i(\delta, A, |G|, c_Q, [K : Q], C_3, C_4) \]
for \( i = 1, 2, 3 \) such that for any conjugacy class \( \mathcal{C} \subseteq G \), we have for all \( x \geq k_1 e^{k_2 (\log \log |\Delta_{K/Q}|)^2} (\log \log \log |\Delta_{K/Q}|)^{\frac{1}{3}} \),
\[ \left| \frac{\pi(x, \mathcal{C}, K/Q)}{|G| \operatorname{Li}(x)} - \frac{|\mathcal{C}|}{|G|} \right| \leq \frac{|\mathcal{C}|}{|G|} \frac{x}{(\log x)^A} \]

### 5.2.2 Sketch of the Proof

The proof of both versions of the effective Chebotarev Density Theorem of Pierce–Turnage-Butterbaugh–Wood, Theorem 5.2.3 and Theorem 5.2.4, can be done simultaneously with the appropriate constants, and so we will proceed by focusing on Theorem 5.2.3 with appropriate changes noted for Theorem 5.2.4.

First, we will deduce explicitly the zero-free region of \( \zeta_K(s) \) based on the assumption of Theorem 5.2.3. Let \( \delta_0(\delta, \beta_0^{(F)}) \leq \delta \) be such that \( 1 - \delta_0(\delta, \beta_0^{(F)}) > \beta_0^{(F)} \), with the understanding that \( \delta_0(\delta) = \delta \) if \( \beta_0^{(F)} \) do not exists for \( F \). We now consider the region
\[ [1 - \delta_0(\delta, \beta_0^{(F)}), 1] \times \left[ -\left( \log |\Delta_{K/Q}| \right)^{\frac{2}{3}}, \left( \log |\Delta_{K/Q}| \right)^{\frac{1}{3}} \right] \]
which is still a zero-free region of \( \frac{\zeta_K(s)}{\zeta(s)} \) by assumption of the theorem, and note that \( \frac{\zeta_K(s)}{\zeta(s)} \) is entire by Aramata–Brauer Theorem. By considering the intersection of this region with the standard zero-free region of \( \zeta(s) \) afforded by Lemma 5.2.1, we get that \( \zeta(s) \) is zero-free in the region
\[
\{ s \in \mathbb{C}; \operatorname{Re}(s) \geq 1 - T_F(s) \}
\]
where we define
\[
T_F(s) = \begin{cases} 
\delta_0(\delta, \beta_0^{(F)}), & \text{if } |\text{Im}(s)| < T_0 \\
\frac{e_F}{(F:Q)^2 \log(|\Delta_F/Q||\text{Im}(s)|^{s+3}|F:Q|)}, & \text{if } T_0 < |\text{Im}(s)| < (\log |\Delta_K/Q|)^{\frac{3}{2}} \text{ and } F \neq Q \\
\frac{e_Q}{(\log(|t|+2))^{\frac{3}{2}} (\log \log(|t|+3))^{\frac{3}{2}}}, & \text{if } T_0 < |\text{Im}(s)| < (\log |\Delta_K/Q|)^{\frac{3}{2}} \text{ and } F = Q
\end{cases}
\]
and $T_0$ is the height where the standard zero-free region of $\zeta_F(s)$ intersects the line $\text{Re}(s) = 1 - \delta_0(\delta, \beta_0^{(F)})$. For simplicity, we assume that $T_0 \leq (\log |\Delta_K/Q|)^{\frac{3}{2}}$. A tedious computation shows that it is sufficient instead to assume $|\Delta_K/Q| \geq D_0(c_F, \delta)$
\[
D_0(c_F, \delta) = \begin{cases} 
e{F}, & \text{if } F \neq Q \\
e{(\exp(\exp(c_Q)))^{\frac{3}{2}}}, & \text{if } F = Q
\end{cases}
\]

The remaining part of the proof can be roughly divided into two major parts, which we sketch below.

**Proof for Large $x$**

We now wish to leverage Theorem 5.1.2. Note that
\[
|\Delta_K/Q| \geq e^{4\delta_0(\delta, \beta_0^{(F)})} \implies 1 - \delta_0(\delta, \beta_0^{(F)}) < 1 - \frac{1}{4 \log |\Delta_K/Q|}
\]
Then, for Galois extensions $K/F$ with $|\Delta_K/Q| \geq \max \left( e^{4\delta_0(\delta, \beta_0^{(F)})}, 4 \right)$, we have that the region
\[
\left\{ s \in \mathbb{C}; \text{Re}(s) \geq 1 - \frac{1}{4 \log |\Delta_K/Q|}, \text{Im}(s) \leq \frac{1}{4 \log |\Delta_K/Q|} \right\}
\]
contains no exceptional zero of $\zeta_K(s)$. Thus, by Theorem 5.1.2, there exists effectively computable constants $C_1$ and $C_2$ such that for all $x \geq e^{10|K:Q|(\log |\Delta_K/Q|)^2}$,
\[
\left| \pi(x, \mathcal{C}, K/F) - \frac{\left| \mathcal{C} \right|}{|G|} \text{Li}(x) \right| \leq C_1xe^{-C_2(\frac{\log x}{\log |G|})}^{\frac{3}{2}}
\]
It now remains to exhibit a constant $D'$ such that for $|\Delta_K/Q| \geq D'$, we get for $x \geq e^{10|K:Q|(\log |\Delta_K/Q|)^2}$ that
\[
C_1xe^{-C_2(\frac{\log x}{\log |G|})}^{\frac{3}{2}} \leq \frac{|\mathcal{C}|}{|G|} \frac{x}{(\log x)^A}
\]
Pierce–Turnage-Butterbaugh–Wood verified and calculated in Appendix C of [PTBW17] the conditions on $D'$, which depends on $A, |F : Q|, C_1, C_2$, and so we define $D' = D'(A, |F : Q|, C_1, C_2)$.

Therefore, by letting
\[
D_1(\delta, A, \beta_0^{(F)}, |F : Q|, C_1, C_2) = \max \left( 4, e^{4\delta_0(\delta, \beta_0^{(F)})}, D'(A, |F : Q|, C_1, C_2) \right)
\]
we get that for Galois extensions $K/F$ with $|\Delta_K/Q| \geq D_1(\delta, A, \beta_0^{(F)}, |F : Q|, C_1, C_2)$, for all $x \geq e^{10|K:Q|(\log |\Delta_K/Q|)^2}$,
\[
\left| \pi(x, \mathcal{C}, K/F) - \frac{\left| \mathcal{C} \right|}{|G|} \text{Li}(x) \right| \leq \frac{|\mathcal{C}|}{|G|} \frac{x}{(\log x)^A}
\]
This concludes the proof of Theorem 5.2.3 for $x \geq e^{10|K:Q|(\log |\Delta_K/Q|)^2}$.
Proof for Small $x$

The main difficulty of Theorem 5.2.3 lies in the proof of the result for the remaining region of $x$. Pierce–Turnage-Butterbaugh–Wood first define an auxiliary weighted prime-counting function

$$\psi(x, C, K/F) = \sum_{p \leq \mathcal{O}_K} \sum_{m \in \mathbb{N}} \log(N_{Q}^F p)$$

where $(\sigma_p)^m$ denotes the conjugacy class of $\sigma_p^m$ for any $Q < \mathcal{O}_K$ lying over $p$. Note that this is well-defined, since $p$ is unramified in $\mathcal{O}_K$.

Pierce–Turnage-Butterbaugh–Wood then proved the following auxiliary proposition.

**Proposition 5.2.5.** Let $F/\mathbb{Q}$ be a number field with absolute discriminant $|\Delta_{F/\mathbb{Q}}|$. Let $c_F$ be such that $\zeta_F(s)$ is zero-free in the region

$$\left\{ s \in \mathbb{C}; \, \text{Re}(s) \geq 1 - \frac{c_F}{[F : \mathbb{Q}]^2 \log \left( |\Delta_{F/\mathbb{Q}}| (|\text{Im}(s)| + 3)|F:Q| \right)} \right\}$$

and let $\beta_0^{(F)} < 1$ be the exceptional real zero of $\zeta_F(s)$ in this region if it exists. Let $A \geq 2$ and let $0 < \delta \leq \frac{1}{2A}$. Let $G$ be a transitive subgroup of $S_n$, where $n \in \mathbb{N}$ with $[F : \mathbb{Q}]|G| \geq 2$. Then, there exists effectively computable absolute constants $C_3, C_4$ such that for any $0 < c_0 \leq 1$, there exists a parameter $D_2(c_0, \delta, A, |G|, [F : \mathbb{Q}], C_3)$ such that for any Galois extension $K/\mathbb{Q}$ with

$$|\Delta_{K/\mathbb{Q}}| \geq \max \left( 4, e^{\frac{1}{4\delta_0^{(F)}}}, D_0(c_F, \delta), D_2(c_0, \delta, A, |G|, [F : \mathbb{Q}], C_3) \right)$$

and $\frac{\zeta_K(s)}{\zeta_F(s)}$ being zero-free in the region

$$[1 - \delta, 1] \times \left[ -\left( \log |\Delta_{K/\mathbb{Q}}| \right)^{\frac{3}{2}}, \left( \log |\Delta_{K/\mathbb{Q}}| \right)^{\frac{3}{2}} \right]$$

there exists constants

$$k_i = k_i(c_0, \delta, A, |G|, c_F, \beta_0^{(F)}, |\Delta_{F/\mathbb{Q}}|, [F : \mathbb{Q}], [K : \mathbb{Q}], C_3, C_4)$$

for $i = 1, 2, 3$ such that for any conjugacy class $C \subseteq G$, for all

$$k_i e^{k_i \log \log |\Delta_{K/\mathbb{Q}}|^{k_3}} \leq x \leq e^{10[K : \mathbb{Q}]|\log |\Delta_{K/\mathbb{Q}}| |^2}$$

we have

$$\left| \psi(x, C, K/F) - \frac{|C|}{|G|} x \right| \leq c_0 \frac{|C|}{|G|} \frac{x}{(\log x)^{A-1}}$$

The following is the version specialised to the case of $F = \mathbb{Q}$.

**Proposition 5.2.6.** Let $c_Q$ be such that $\zeta_Q(s)$ is zero-free in the region

$$\left\{ s \in \mathbb{C}; \, \text{Re}(s) \geq 1 - \frac{c_Q}{(\log (|t| + 2))^{\frac{3}{2}} (\log \log (|t| + 3))^{\frac{1}{2}}} \right\}$$


Let \( A \geq 2 \) and let \( 0 < \delta \leq \frac{1}{2A} \). Let \( G \) be a transitive subgroup of \( \mathfrak{S}_n \), where \( n \in \mathbb{N} \) with \( |G| \geq 2 \). Then, there exists effectively computable absolute constants \( C_3, C_4 \) such that for any \( 0 < c_0 \leq 1 \), there exists a parameter \( D_2(c_0, \delta, A, |G|, C_3) \) such that for any Galois extension \( K/\mathbb{Q} \) with

\[
|\Delta_{K/\mathbb{Q}}| \geq \max \left( 4, e^{\frac{3}{4}}, D_0(c_F, \delta), D_2(c_0, \delta, A, |G|, C_3) \right)
\]

and \( \frac{\zeta_K(s)}{\zeta_{\mathbb{Q}}(s)} \) being zero-free in the region

\[
[1 - \delta, 1] \times \left[ -\left( \log |\Delta_{K/\mathbb{Q}}| \right)^\frac{3}{2}, (\log |\Delta_{K/\mathbb{Q}}|)^\frac{3}{2} \right]
\]

there exists constants

\[
k_i' = k_i'(c_0, \delta, A, |G|, c_{\mathbb{Q}}, [K: \mathbb{Q}], C_3, C_4)
\]

for \( i = 1, 2, 3 \) such that for any conjugacy class \( C \subseteq G \), for all

\[
k_1'e_k^2(\log \log |\Delta_{K/\mathbb{Q}}|)\frac{3}{2}(\log \log \log |\Delta_{K/\mathbb{Q}}|^2)^{\frac{1}{2}} \leq x \leq e^{10[K:\mathbb{Q}](\log |\Delta_{K/\mathbb{Q}}|)^2}
\]

we have

\[
\left| \psi(x, C, K/F) - \frac{|C|}{|G|} x \right| \leq c_0 \frac{|C|}{|G|} \frac{x}{(\log x)^{A-1}}
\]

We will not provide a proof of this, instead refering interested readers to [PTBW17]. In particular, \( D_2(c_0, \delta, A, |G|, [K: \mathbb{Q}], C_3) \) is described in Lemma 5.7 of [PTBW17], while the \( k_i' \) in both propositions are described in their proof of the propositions. On the other hand, \( C_3 \) is derived from an application of Theorem 7.1 of [LO77], and \( C_4 \) is derived from bounding the \( S(x, T) \) term that appears in the theorem.

Pierce–Turnage-Butterbaugh–Wood then define the function

\[
\theta(x, C, K/F) = \sum_{\substack{p \in \mathfrak{O}_K \setminus \mathfrak{O}_K^\times \mathfrak{p} \text{ unramified in } C_K \setminus \mathfrak{O}_K \ni \mathfrak{p} \cap \mathfrak{C} = C}} \log (N_{\mathbb{Q}}^F \mathfrak{p})
\]

The goal now is to approximate this sum by \( \pi(x, C, K/F) \). Pierce–Turnage-Butterbaugh–Wood computed that

\[
|\psi(x, C, K/F) - \theta(x, C, K/F)| \leq \frac{3}{2} \log 2 \frac{[K: \mathbb{Q}]x^{\frac{3}{2}} \log x}{x}
\]

Under the assumption of Proposition 5.2.5, the \( x \) in the region

\[
k_1'e_k^2(\log \log |\Delta_{K/\mathbb{Q}}|)\frac{3}{2}(\log \log \log |\Delta_{K/\mathbb{Q}}|^2)^{\frac{1}{2}} \leq x \leq e^{10[K:\mathbb{Q}](\log |\Delta_{K/\mathbb{Q}}|)^2}
\]

satisfies the inequality

\[
\frac{3}{2} \log 2 \frac{[K: \mathbb{Q}]x^{\frac{3}{2}} \log x}{x} \leq c_0 \frac{|C|}{|G|} \frac{x}{(\log x)^{A-1}}
\]

Similarly in the case \( F = \mathbb{Q} \), the assumption of Proposition 5.2.6 forces \( x \) in the region

\[
k_1'e_k^2(\log \log |\Delta_{K/\mathbb{Q}}|)\frac{3}{2}(\log \log \log |\Delta_{K/\mathbb{Q}}|^2)^{\frac{1}{2}} \leq x \leq e^{10[K:\mathbb{Q}](\log |\Delta_{K/\mathbb{Q}}|)^2}
\]

to satisfy the same inequality

\[
\frac{3}{2} \log 2 \frac{[K: \mathbb{Q}]x^{\frac{3}{2}} \log x}{x} \leq c_0 \frac{|C|}{|G|} \frac{x}{(\log x)^{A-1}}
\]
This allows us to conclude that
\[ |\theta(x, C, K/F) - \frac{|C|}{|G|} x| \leq 2c_0 \frac{|C|}{|G|} \frac{x}{(\log x)^{A-1}} \]
under the same assumptions of Proposition 5.2.5 and Proposition 5.2.6 respectively.

For simplicity, we let
\[
x_0(F) = \begin{cases} 
k_1' e^{k_2'(\log \log |\Delta_{K'/Q}|)^2}, & \text{if } F \neq Q \\
 k_1' e^{k_2'(\log \log |\Delta_{K'/Q}|)^{2/3}} (\log \log \log |\Delta_{K'/Q}|)^{1/3}, & \text{if } F = Q 
\end{cases}
\]
with the appropriate \( k_i' \). By partial summation, we get that for \( x_0(F) \leq x \leq e^{10[K:Q](\log |\Delta_{K/Q}|)^2} \),
\[
\pi(x, C, K/F) = \int_{x_0(F)}^x \frac{\theta(t, C, K/F)}{t \log^2 t} \, dt + \frac{\theta(x, C, K/F)}{\log x}
= \int_{x_0(F)}^x \frac{\theta(t, C, K/F)}{t \log^2 t} \, dt + \int_{x_0(F)}^x \frac{\theta(t, C, K/F)}{t \log^2 t} \, dt + \frac{\theta(x, C, K/F)}{\log x}
\]
where \( \lambda \) is the minimum among \( N^F_Q(p) \) running over \( p \leq \mathcal{O}_F \). With the trivial bound of \( \theta(t, C, K/F) \leq [K : Q] t \log t \) for \( \lambda \leq t \leq x_0(F) \), we have that
\[
\int_{x_0(F)}^x \frac{\theta(t, C, K/F)}{t \log^2 t} \, dt \leq [K : Q]\text{Li}(x_0(F))
\]
Integration by parts give that
\[
\int_{x_0(F)}^x \frac{1}{t \log^2 t} \, dt + \frac{x}{\log x} = \int_{x_0(F)}^x \frac{\partial}{\partial t} \left( -\frac{1}{\log t} \right) \, dt + \frac{x}{\log x} = \text{Li}(x) - \left( \text{Li}(x_0(F)) - \frac{x_0(F)}{\log x} \right)
\]
In light of the our approximation of \( \theta(x, C, K/F) \), we have that in summary that
\[
\pi(x, C, K/F) = \frac{|C|}{|G|} \text{Li}(x) + E
\]
where
\[
|E| \leq [K : Q]\text{Li}(x_0(F)) + \left| \text{Li}(x_0(F)) - \frac{x_0(F)}{\log x} \right| + 2c_0 \frac{|C|}{|G|} \int_{x_0(F)}^x \frac{1}{(\log t)^{A+1}} \, dt + 2c_0 \frac{|C|}{|G|} \frac{x}{(\log x)^A}
\leq ([K : Q] + 2)\text{Li}(x_0(F)) + 2c_0 \frac{|C|}{|G|} \left( \int_{x_0(F)}^{\sqrt{x}} \frac{1}{(\log t)^{A+1}} \, dt + \int_{\sqrt{x}}^x \frac{1}{(\log t)^{A+1}} \, dt \right) + 2c_0 \frac{|C|}{|G|} \frac{x}{(\log x)^A}
\leq ([K : Q] + 2)\text{Li}(x_0(F)) + 2c_0 \frac{|C|}{|G|} \left( x^{\frac{3}{2}} + \int_{\sqrt{x}}^x \frac{2^{A+1}}{(\log x)^{A+1}} \, dt \right) + 2c_0 \frac{|C|}{|G|} \frac{x}{(\log x)^A}
\leq ([K : Q] + 2)\text{Li}(x_0(F)) + 2c_0 \frac{|C|}{|G|} \left( x^{\frac{3}{2}} + \frac{2^{A+1}x}{(\log x)^{A+1}} \right) + 2c_0 \frac{|C|}{|G|} \frac{x}{(\log x)^A}
\leq ([K : Q] + 2)\text{Li}(x_0(F)) + 2c_0 \frac{|C|}{|G|} x^{\frac{3}{2}} + (2^{A+2} + 2)c_0 \frac{|C|}{|G|} \frac{x}{(\log x)^A}
\]
\]
By noting that
\[ x_0(F) \leq x \leq e^{10[K:Q](\log|\Delta_{K/Q}|)^2} \implies x^{\frac{1}{2}} \leq \frac{x}{(\log x)^A} \]
we have that
\[ |E| \leq \left( [K : \mathbb{Q}] + 2 \right) \text{Li}(x_0(F)) + (2^{A+2} + 4)c_0 \frac{|C|}{|G|} \frac{x}{(\log x)^A} \]
Lastly, by enlarging the respective \( k_i' \) to the appropriate \( k_i \), we can get that in the range \( x_0(F) \leq x \leq e^{10[K:Q](\log|\Delta_{K/Q}|)^2} \),
\[ ([K : \mathbb{Q}] + 2) \text{Li}(x_0(F)) \leq (2^{A+2} + 4)c_0 \frac{|C|}{|G|} \frac{x}{(\log x)^A} \]
\[ \implies |E| \leq (2^{A+3} + 8)c_0 \frac{|C|}{|G|} \frac{x}{(\log x)^A} \]
In conclusion, by setting \( c_0 = \frac{1}{2^{A+2}+8} \) and combining with the case of large \( x \), we now have that
\[ \left| \pi(x; C, K/F) - \frac{|C|}{|G|} \text{Li}(x) \right| \leq \frac{|C|}{|G|} \frac{x}{(\log x)^A} \]
as desired in the required range of \( x \). For completion, we will record the resulting \( k_i \) and threshold \( D \) for the absolute discriminant below:
\[ k_1 = (2^{A+3} + 8) \left( \frac{72C_3C_4(2^{A+3} + 8)10^{A-1}|G|[K : \mathbb{Q}]^A}{\delta_0(\delta, \beta_0^{(F)})} \right)^{\frac{1}{\delta_0(\delta, \beta_0^{(F)})}} \]
\[ k_2 = \max \left( \frac{2A}{\delta_0(\delta, \beta_0^{(F)})}, \frac{4A[F : \mathbb{Q}]^3}{c_F \delta} \right) + 2A \]
\[ k_3 = \frac{6(12C_3(2^{A+3} + 8)) \epsilon^{\frac{1}{\delta_0(\delta, \beta_0^{(F)})}} (12C_3C_4(2^{A+3} + 8))^{\frac{1}{\delta_0(\delta, \beta_0^{(F)})}} |\Delta_{F/Q}|[K : \mathbb{Q}]}{\epsilon^{\frac{1}{\delta_0(\delta, \beta_0^{(F)})}}} \]
\[ D = \max \left( D_0(c_F, \delta), D_1(\delta, A, \beta_0^{(F)}, [F : \mathbb{Q}], C_1, C_2), D_2(c_0, \delta, A, |G|, [F : \mathbb{Q}], C_3) \right) \]

### 5.3 Removal of Strong Artin Conjecture

As a consequence of what we developed in the previous chapter, we will now demonstrate the following theorem, which specialises to a strengthening of Theorem 7.1 in [PTBW17].

**Theorem 5.3.1.** Let \( F/\mathbb{Q} \) be Galois, and assume Conjecture 3.2.9. Let
\[ Z \subseteq \left\{ K/F; [K : F] = n, \text{Gal}(K^F/F) = G \right\} \]
Suppose \( \mathcal{H}(Z) \) is a sterile subfamily of \( \mathcal{F}_{F,G} \) with contamination index \( \tau \). Let \( A \geq 2 \). Then, for every \( 0 < \varepsilon < \frac{1}{2} \) with
\[ \Delta = 1 - \frac{\tau}{d_{F,G}(\mathcal{H}(Z))} - \frac{\varepsilon}{2d_{F,G}(\mathcal{H}(Z))} > 0 \]
such that
\[ \delta = \frac{\varepsilon}{20m[F : \mathbb{Q}]a_{F,G}(\mathcal{H}) + 8d_{F,G}(\mathcal{H}) + 4\varepsilon} \leq \frac{1}{2A} \]
there exists effectively computable absolute constants $C_1, C_2, C_3, C_4$, a constant

$$D = D(Z, \tau, \varepsilon, A, |G|, c_F, \beta_0^{(F)}, [F : \mathbb{Q}], C_1, C_2, C_3)$$

such that for $X \in \mathbb{N}$, all but $DX^{\tau + \varepsilon}$ fields $K/F$ in

$$\left\{ K/F \in Z; |\Delta_{K/F}^F| < X \right\}$$

has that there exists constants

$$k_i = k_i(\delta, A, |G|, c_F, \beta_0^{(F)}, |\Delta_{F/Q}|, [F : \mathbb{Q}], [\overline{K}_F : \mathbb{Q}], C_3, C_4)$$

for $i = 1, 2, 3$ with for any conjugacy class $C \subseteq G$, we have for all $x \geq k_1 e^{k_2 (\log \log |\Delta_{K/F}^F|)^2}$,

$$\left| \pi(x, C, K_F/F) - \frac{|C|}{|G|} \text{Li}(x) \right| \leq \frac{|C|}{|G|} \frac{x}{(\log x)^{A}}$$

Proof. By Theorem 5.2.3 we get effectively computable absolute constants $C_1, C_2, C_3, C_4$, and a constant

$$D(\delta, A, |G|, c_F, \beta_0^{(F)}, [F : \mathbb{Q}], C_1, C_2, C_3)$$

such that for any $K/F \in Z$ with $\text{Gal}(K/F) \cong G$, $|\Delta_{K/F}^F| > D(\delta, A, |G|, c_F, \beta_0^{(F)}, [F : \mathbb{Q}], C_1, C_2, C_3)$ which is not $\delta$-exceptional, there exists constants

$$k_i = k_i(\delta, A, |G|, c_F, \beta_0^{(F)}, |\Delta_{F/Q}|, [F : \mathbb{Q}], [\overline{K}_F : \mathbb{Q}], C_3, C_4)$$

for $i = 1, 2, 3$ such that for any conjugacy class $C \subseteq G$, we have for all $x \geq k_1 e^{k_2 (\log \log |\Delta_{K/F}^F|)^2}$,

$$\left| \pi(x, C, K_F/F) - \frac{|C|}{|G|} \text{Li}(x) \right| \leq \frac{|C|}{|G|} \frac{x}{(\log x)^{A}}$$

We proceed to count fields which either are $\delta$-exceptional or with $|\Delta_{K/F}^F| \leq D(\delta, A, |G|, c_F, \beta_0^{(F)}, [F : \mathbb{Q}], C_1, C_2, C_3)$ as in the proof of Corollary 4.2.13 with using

$$D = \max(D'(Z, \tau, \varepsilon), D(\delta, A, |G|, c_F, \beta_0^{(F)}, [F : \mathbb{Q}], C_1, C_2, C_3))$$

and obtain the result. \qed

Specialising to $F = \mathbb{Q}$, and noting that in the case listed in Proposition 4.2.18 as long as

$$\varepsilon < \frac{d_{Q,G}(\mathcal{H}_n^f(\mathbb{Q}, G))}{4}$$

we have that

$$\Delta = 1 - \frac{\tau}{d_{Q,G}(\mathcal{H}_n^f(\mathbb{Q}, G))} - \frac{\varepsilon}{2d_{Q,G}(\mathcal{H}_n^f(\mathbb{Q}, G))} > 0$$

we get the following.
Corollary 5.3.2. Let $A \geq 2$, and assume Conjecture 3.2.9. For $G, \mathcal{F}$ being one of those listed in Proposition 4.2.18 with the corresponding contamination index $\tau$, we have that for any $0 < \varepsilon < \min\left(\frac{1}{2}, \frac{d_{2G}(H_{n}(\mathbb{Q}, G))}{4}\right)$ with

$$\delta = \frac{\varepsilon}{20m + 8d_{G}(H_{n}(\mathbb{Q}, G)) + 4\varepsilon} \leq \frac{1}{2A}$$

there exists effectively computable absolute constants $C_1, C_2, C_3, C_4$, a constant $D = D(n, G, \mathcal{F}, \tau, \varepsilon, A, c_{\mathbb{Q}}, C_1, C_2, C_3)$ such that for $X \in \mathbb{N}$, all but $DX^{\tau + \varepsilon}$ fields $K/F$ in

$$\left\{ K/F \in Z_{n}^{e}(\mathbb{Q}, G); \left| \Delta_{K^{\mathbb{Q}}/\mathbb{Q}} \right| < X \right\}$$

has that there exists constants

$$k_i = k_i(\delta, A, |G|, c_{\mathbb{Q}}, [K^{\mathbb{Q}} : \mathbb{Q}], C_3, C_4)$$

for $i = 1, 2, 3$ with for any conjugacy class $C \subseteq G$, we have for all $x \geq k_{1}e^{k_{2}(\log \log |\Delta_{K^{\mathbb{Q}}/\mathbb{Q}}|)^{\frac{3}{2}}(\log \log |\Delta_{K^{\mathbb{Q}}/\mathbb{Q}}|)^{\frac{1}{2}}}$,

$$\left| \pi(x, C, K^{\mathbb{Q}}/\mathbb{Q}) - \frac{|C|}{|G|} \text{Li}(x) \right| \leq \frac{|C|}{|G|} \frac{x}{(\log x)^A}$$

We note that the corollary differs from Theorem 7.1 of [PTBW17] in that it does not require the Strong Artin Conjecture for any case considered. We also consider fields according to the absolute discriminant $|\Delta_{K^{\mathbb{Q}}/\mathbb{Q}}|$ of the Galois closure of $K$ rather than the absolute discriminant $|\Delta_{K/\mathbb{Q}}|$ of $K$ itself. Again, we remark that we can obtain the result in [PTBW17] by tweaking

$$\mathcal{H}(Z)(X) = \left\{ \frac{\zeta_{K^{\mathbb{Q}}}(s)}{\zeta_{F}(s)}; K/F \in Z, \left| \Delta_{K/F} \right| < X \right\}$$

and making the appropriate tweaks for each result as required.
Chapter 6

CONCLUSION

We will highlight here a few open questions that we have yet to resolve in this thesis.

The main open problem is the analytical issue with non-simple poles of \(L\)-functions in our setting, encapsulated in Conjecture 3.2.9. Note that since the possible poles of Rankin-Selberg of cuspidal automorphic \(L\)-functions are simple, this issue does not manifest itself in [KM02]. The solution to this conjecture will truly make the result of Pierce–Turnage-Butterbaugh–Wood unconditional.

The other major open is finding the relation between \(\text{Conv}_K\) and \(\text{Conv}_R\) in the case where \(R/K\) is not Galois. A solution towards this will possibly allow us to prove that \(\mathcal{F}_{F,G}^\nu\) is Kowalski–Michel \(\mathbb{Q}\)-ameanable for any number field \(F\), resulting in a Kowalski–Michel estimate of zero-regions of \(\mathcal{F}_{F,G}^\nu\). This will allow us to remove the condition of \(F/\mathbb{Q}\) being Galois in all subsequent results we proved in this thesis. Towards this end, we suspect that \(\nu\)-regular representation can be written as a direct sum of induced representations, which will allow us to prove a relation we seek. In particular, we conjecture the following.

**Conjecture 6.0.3.** For any extension \(R/K\), there exists a representation \(\rho\) of \(\text{Gal}(\overline{K}/\overline{R})\) such that

\[
\text{reg}_{\text{Gal}(K/F)}^R \circ \pi_{\text{Gal}(\overline{K}/\overline{R})} = \text{Ind}^\text{Gal}(\overline{K}/\overline{R})_{\text{Gal}(K/F)} \rho
\]

The following Hasse diagram gives a picture of the conjecture.

![Hasse Diagram](image)

Figure 6.0.1: Hasse Diagram involved in Conjecture 6.0.3

We also wish to point out the paper of Thorner and Zaman [TZ18], which improves upon the result of Kowalski and Michel. We did not yet consider how our work might be affected, but we suspect that with a suitable tweak of our methods in this thesis, we might be able to combine with the methods of Thorner and Zaman to produce an improved result on the zero-region estimate of families of Artin \(L\)-functions.
Bibliography


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BIBLIOGRAPHY


