Snow day notes

1. Using the WKB method, provide an approximation for the eigenvalue, λ, of the problem

$$y'' + \lambda^2 x^2 y \sin^2 x = 0, \qquad 0 \le x \le \frac{\pi}{2}, \qquad y(0) = y(\pi/2) = 0 \ .$$

The WKB approximation to $y'' + f(x)y = 0$, $f(x) = \omega^2 > 0$, is

$$y \sim \frac{1}{\sqrt{\omega}}(a\cos\theta + b\sin\theta), \quad \theta = \int_x^{\pi/2} \omega(x')\, dx', \quad f(x_*) = 0,$$

i.e., after applying the boundary condition at $x = \pi/2$,

$$y \sim \frac{b}{\sqrt{\omega}} \sin\left[\lambda \int_x^{\pi/2} x \sin x \ dx\right] = -\frac{b}{\sqrt{\lambda x \sin x}} \sin[\lambda(1 - \sin x + x\cos x)]$$

The naive application of WKB would then demand that

$$\lambda = n\pi, \quad n = 1, 2, ...,$$

on using $y(0) = 0$.

We can arrive at a better WKB solution by noting that $f(x)$ is no longer large for $x \to 0$. Here, a local problem applies,

$$y'' + \lambda^2 x^{p-2} y = 0,$$

with $p = 6$. The solution that satisfies $y(0) = 0$ is $y(x) = A\sqrt{x}\ J_{1/p}(z)$, where $J_\nu(z)$ is a Bessel function of order ν and $z = 2\lambda x^{p/2}/p$. The large argument form of the Bessel function implies that

$$y \sim A\sqrt{x}\sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right) = \frac{A}{x}\sqrt{\frac{6}{\pi\lambda}} \cos\left(\frac{1}{3}\lambda x^3 - \frac{1}{3}\pi\right)$$

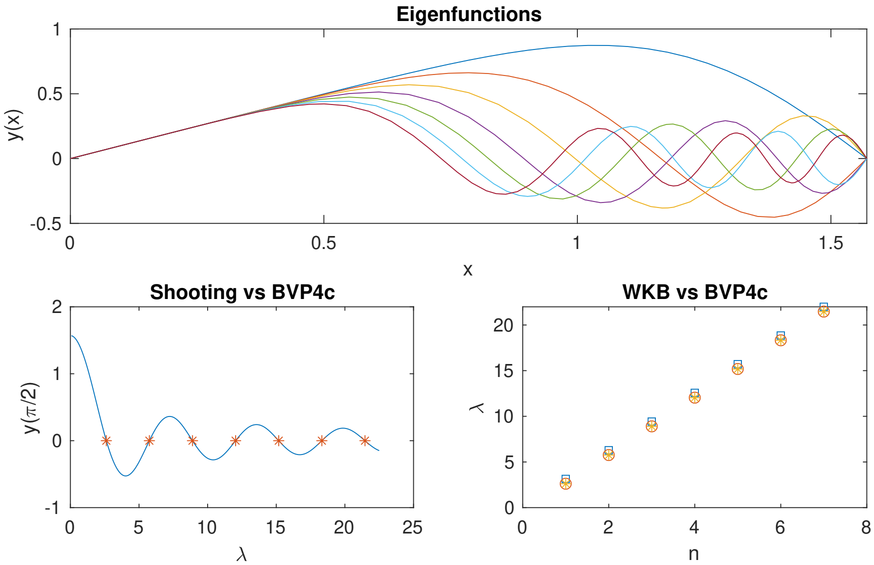
Our WKB solution, on the other hand, for $x \to 0$ is

$$y \sim -\frac{b}{x\sqrt{\lambda}} \sin\left[\lambda\left(1 - \frac{1}{3}x^3\right)\right] \equiv -\frac{b}{x\sqrt{\lambda}} \cos\left(\frac{1}{3}\lambda x^3 - \lambda - \frac{1}{2}\pi\right),$$

which implies that

$$\lambda = n\pi - \frac{1}{6}\pi, \quad n = 1, 2, ...$$

This gives an excellent approximation to even the smallest eigenvalue. See m550bvp6.m.



2. The integral considered in the last lecture is

$$I(k) = \int_0^1 \frac{dx}{(1-kx^3)^{3/2}} = \sum_{j=0}^{\infty} \binom{-\frac{3}{2}}{j} \frac{(-1)^j k^j}{3j+1} \sim 1 + \frac{3}{15}k + \frac{15}{56}k^2 + \frac{7}{32}k^3 + ...$$

Near $k = 1$, we can derive $I \sim \frac{2}{3}(1-k)^{-1/2}$. The code m550x.m computes the integral numerically and compares the result with the 4-term series. Also plotted are the improved series exploiting either multiplicative or additive extraction:

$$I \sim \frac{g_0 + g_1 k + g_2 k^2 + g_3 k^3}{\sqrt{1-k}}$$

and

$$I \sim \frac{2}{3\sqrt{1-k}} + g_0 + g_1 k + g_2 k^2 + g_3 k^3,$$

respectively. Both are big improvements over the range $[0, 1)$ for k, especially the latter. For a more quantitative comparison, the code m550x.m quotes the various values at $k = 0.9$.