1 Solving PDEs with Fourier transforms

For a function $f(x)$ that is defined for $-\infty < x < \infty$, with $f(x) \to 0$ for $|x| \to \infty$, the Fourier tranform (FT) can be defined as

$$\hat{f}(k) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx. \tag{1}$$

The all important property of the FT centres on what it does to a derivative: using an integration by parts,

$$\mathcal{F}\left\{\frac{df}{dx}\right\} = ik\hat{f}. \tag{2}$$

i.e. it turns a derivative into an algebraic factor.

The inverse transform is defined by a very similar-looking integral:

$$f(x) = \mathcal{F}^{-1}\left\{\hat{f}(k)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{f}(k)\, dk. \tag{3}$$

Here, k appears as a parameter in the integral, and the FT therefore furnishes a new function of k. We will use the hat notation for the FT.

Note that

$$\mathcal{F}^{-1}\{\mathcal{F}\{f(x)\}\} = f(x) \quad \Longrightarrow \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ik(x-\check{x})} f(\check{x})\, d\check{x}\, dk. \tag{4}$$

(using $\check{x}$ as the integration variable in $\mathcal{F}\{f(x)\}$, so as to avoid any confusion with the x that appears in the inverse transform). In other words,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-\check{x})}\, dk \equiv \delta(x-\check{x}), \tag{5}$$

where $\delta(x)$ is Dirac's delta function, which has the property,

$$\int_{-\infty}^{\infty} f(\check{x})\delta(\check{x}-x)\, d\check{x} = f(x). \tag{6}$$

It can also be helpful sometimes to observe that

$$\hat{f}(0) = \int_{-\infty}^{\infty} f(x)\, dx. \tag{7}$$

2 An example: the diffusion equation on the infinite line

As we want to solve PDEs with the FT, we need its action on the solution $u(x,t)$ and its derivatives:

$$\hat{u}(k,t) = \mathcal{F}\{u(x,t)\}, \quad \mathcal{F}\{u_t\} = \hat{u}_t, \quad \mathcal{F}\{u_x\} = ik\hat{u}, \quad \mathcal{F}\{u_{xx}\} = -k^2\hat{u}, \tag{8}$$

provided $u \to 0$ for $|x| \to \infty$ (the $t-$derivative always slips outside the spatial FT). Now let's solve the PDE problem:

$$u_t = u_{xx}, \quad u(x,0) = f(x), \quad u, f \to 0 \text{ for } |x| \to \infty. \tag{9}$$

We zap the PDE and IC with the FT:

$$\hat{u}_t = -k^2 \hat{u}, \quad \hat{u}(k,0) = \hat{f}(k). \tag{10}$$

The FT therefore transforms the PDE into an ODE. Solving that ODE then gives

$$\hat{u}(k,t) = \hat{f}(k)e^{-k^2 t} \quad \longrightarrow \quad u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2 t}\hat{f}(k)\, dk, \tag{11}$$

after using the inverse FT. The solution of the PDE is therefore reduced to two integrals: one for $\hat{f}(k)$; the other for the inverse transform.

3 Sample transform pairs

For the example, $f(x) = e^{-a|x|}$, $a > 0$, we have

$$\mathcal{F}\left\{e^{-a|x|}\right\} = \hat{f}(k) = \int_0^{\infty} e^{-ikx-ax} dx + \int_{-\infty}^{0} e^{-ikx+ax} dx = \frac{2a}{a^2+k^2} \tag{12}$$

Similarly, if $\hat{f}(k) = e^{-a|k}$,

$$f(x) = \mathcal{F}^{-1}\left\{e^{-a|k|}\right\} = \frac{a}{\pi(a^2+x^2)}. \tag{13}$$

More generally,

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx \quad \rightarrow \quad \hat{f}(z) = \int_{-\infty}^{\infty} e^{-izk} f(k)\, dk \tag{14}$$

$$\rightarrow \quad (2\pi)^{-1}\hat{f}(-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixk} f(k)\, dk \equiv \mathcal{F}^{-1}\left\{f(k)\right\}. \tag{15}$$

Hence, given one transform pair, we can immediately write down another one using the "reciprocity relation":

$$\{f(x), \hat{f}(k)\} \quad \Longrightarrow \quad \{(2\pi)^{-1}\hat{f}(-x), f(k)\}. \tag{16}$$

This clearly works for the preceding example:

$$\left\{e^{-a|x|}, \frac{2a}{a^2+k^2}\right\} \quad \Longrightarrow \quad \left\{\frac{a}{\pi(a^2+(-x)^2)}, e^{-a|k|}\right\}, \tag{17}$$

as derived explicitly already.

4 Integrating a Gaussian off the real axis

The solution of the diffusion equation given above can be be compressed further: introducing explicitly the definition of $\hat{f}(k) = \mathcal{F}\{f(x)\}$ into the inverse transform and switching the order of the two integrals gives

$$u(x,t) = \int_{-\infty}^{\infty} G(x-\check{x},t) f(\check{x})\, d\check{x} \tag{18}$$

(again taking the spatial integral in $\mathcal{F}\{f(x)\}$ to have an integration variable of $\check{x}$, so as to avoid any confusion with the x in the inverse transform), where

$$G(X,t) = \int_{-\infty}^{\infty} e^{-k^2 t+ikX}\frac{dk}{2\pi} = e^{-X^2/4t}\int_{-\infty-iw}^{\infty-iw} e^{-z^2 t}\frac{dz}{2\pi}, \quad z = k - iw, \quad w = \frac{X}{2t}. \tag{19}$$

The last change of variable in the integral for $G(X,t)$ is convenient to write the integrand as a Gaussian, for which we know that

$$\int_{-\infty}^{\infty} e^{-z^2 t}\frac{dz}{2\pi} = \frac{1}{\sqrt{4\pi t}}. \tag{20}$$

Unfortunately, however, the variable change has also displaced the integral from the real axis of the complex $z-$plane. The new path is a parallel line with $\mathrm{Im}(z) = -w$. This is not as bad as it might seem, as Cauchy's theorem comes to our rescue and tells us that the integral along the real axis in (20) is actually the same as the last integral in (19).

To see this, we first recall Cauchy's theorem: if we have an analytical function $F(z)$ (such as an exponential or Gaussian), then any integral around a closed loop $\mathcal{C}$ on the complex plane must vanish:

$$\int_{\mathcal{C}} F(z)\, dz = 0. \tag{21}$$

Conventionally, the integral around the contour $\mathcal{C}$ is performed in an anti-clockwise sense (figure 1(a)); going the wrong way incurs a switch of sign (do not try this in the car).

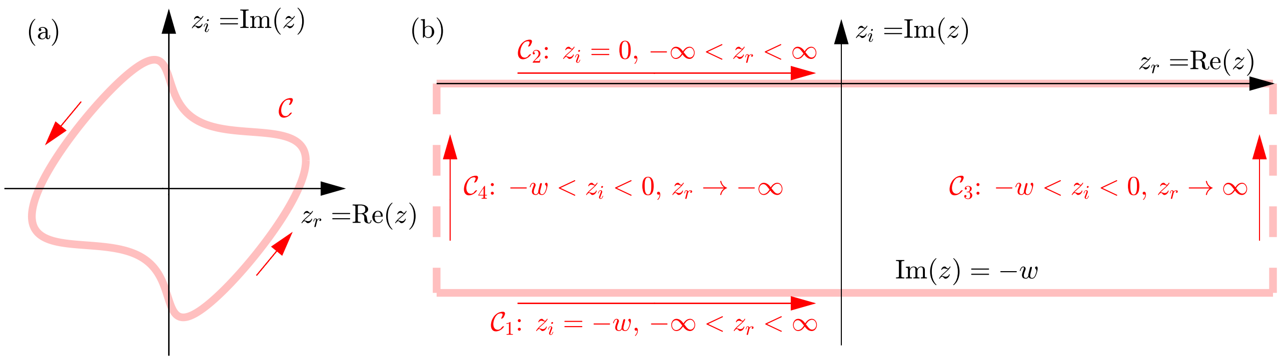

Figure 1: (a) A closed path integral on the complex $z-$plane. (b) Contour of integration for the shifted Gaussian.

We can use this theorem in the following way: we build a closed contour on the complex $z-$plane by piecing together the lines defined by

$$\begin{array}{ll} \mathcal{C}_1: & z_i = -w,\ -\infty < z_r < \infty \\ \mathcal{C}_2: & z_i = 0,\ -\infty < z_r < \infty \\ \mathcal{C}_3: & -w < z_i < 0,\ z_r \to \infty \\ \mathcal{C}_4: & -w < z_i < 0,\ < z_r \to -\infty \end{array} \tag{22}$$

and illustrated in figure 1(b). We know the integral along $\mathcal{C}_2$ from (20); that along $\mathcal{C}_1$ is the one we need. The closed contour to use with Cauchy is $\mathcal{C}_1 + \mathcal{C}_3 - \mathcal{C}_2 - \mathcal{C}_4$, where the signs refer to traversing the line in the wrong direction, in order to take an anti-clockwise path around the composite loop, but keep sensible conventions for each individual integral. Because $e^{-tz^2} \to 0$ along $\mathcal{C}_3$ and $\mathcal{C}_4$, the dashed end sections in figure 1(b) do not make any contribution. As a result, Cauchy's theorem implies that the contributions of $\mathcal{C}_1$ and $-\mathcal{C}_2$ must combine and vanish, establishing that the integral along $\mathcal{C}_1$ equals (20).

After this little excursion over the complex $z-$plane, we may now write

$$u(x,t) = \int_{-\infty}^{\infty} G(x-\check{x},t) f(\check{x})\, d\check{x}, \quad G(X,t) = \frac{e^{-X^2/4t}}{\sqrt{4\pi t}}, \tag{23}$$

which is a single integral. The solution can therefore be expressed quite generally (any initial condition will do), as suitable integral involving $f(x)$ and the "Green function" $G(X,t)$.

5 Integrating the Lorentzian

The definition of the FT and its inverse, with their complex exponential integrands, can often lead us to contour integrals. Another example is the inversion of $\hat{f}(k) = 2/(k^2+1)$, which we know is the exponential $f(x) = e^{-|x|}$. More directly,

$$\mathcal{F}^{-1}\left\{\frac{2}{k^2+1}\right\} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{ikx}dk}{(k+i)(k-i)}. \tag{24}$$

This integral is again along the real axis, a line denoted by $\mathcal{C}_L$ in figure 2, with z_r =Re$(z) \equiv k$. We can complete the contour of integration as a closed loop by adding a semicircular arc enclosing either the entire upper or lower half plane; see figure 2. When $x > 0$, the integrand $e^{ixz}/(z^2+1)$ vanishes along the upper semicircular arc $\mathcal{C}_A$, and so the integral in (24) is equal to the contour integral around the anti-clockwise path $\mathcal{C}_L + \mathcal{C}_A$. Conversely, when $x < 0$, the integrand vanishes along $\mathcal{C}'_A$, and so we may complete the *clockwise* path $\mathcal{C}_L + \mathcal{C}'_S$.

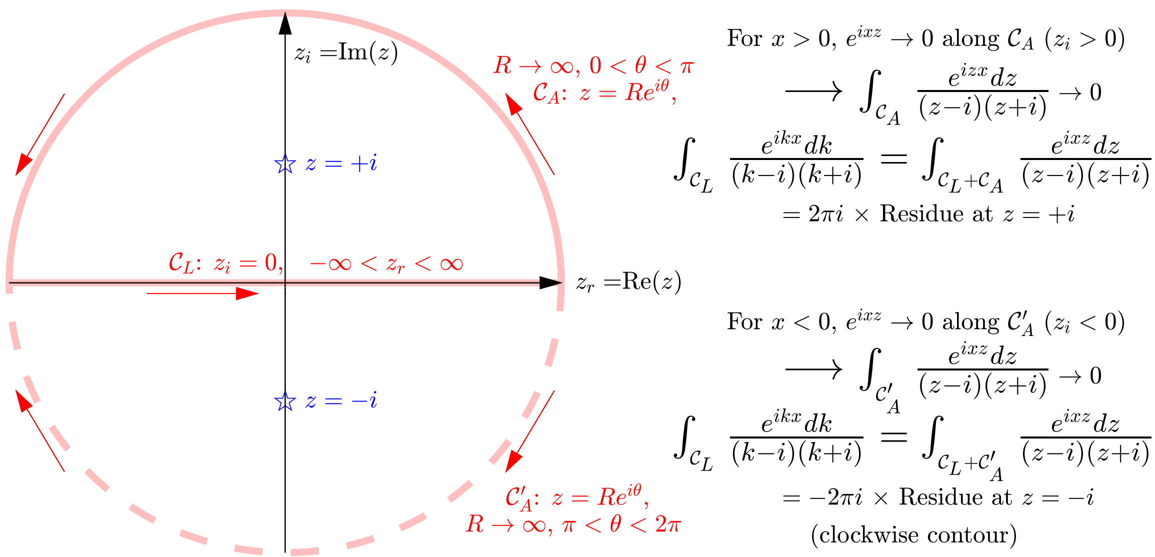

Figure 2:

The integrand in (24) is not, however, analytic anymore within either of these two closed contours: there are poles at $z = \pm i$. According to the usual rules of residue calculus, the first integral ($\mathcal{C}_L + \mathcal{C}_A$, enclosing the upper half plane, applying when $x > 0$) is therefore equal to $2\pi i$ times the residue of the pole at $z = +i$, which is $e^{-x}/(2i)$ (for a simple pole, the integrand can be written as $F(z)/(z - z_*)$, where $z_* = +i$ is the pole, and the residue is $F(z_*)$). Consequently,

$$\mathcal{F}^{-1}\left\{\frac{2}{k^2+1}\right\} = e^{-x}, \quad x > 0. \tag{25}$$

Similarly, the second integral ($\mathcal{C}_L + \mathcal{C}'_A$, enclosing the lower half plane, and relevant when $x < 0$) is evaluated as $-2\pi i$ times the residue of the pole at $z = -i$ (which is $e^{x}/(-2i)$), the extra minus sign appearing because the contour is clockwise. Hence

$$\mathcal{F}^{-1}\left\{\frac{2}{k^2+1}\right\} = e^{x}, \quad x < 0, \tag{26}$$

and we have recovered the earlier result (with $a = 1$).

6 Convolution

On zapping a ODE with the FT and solving the resulting ODE for $\hat{u}(k,t)$, we usually obtain a product of transforms. For example, in the problem above, we have the product of $\hat{f}(k)$ and $e^{-k^2 t}$. Such a product actually always corresponds to a convolution integral when transformed back to real space. To see this, let

$$f \circ g = \int_{-\infty}^{\infty} f(\check{x}) g(x - \check{x})\, d\check{x} \tag{27}$$

denote the convolution of functions $f(x)$ and $g(x)$. Zapping with the FT:

$$\mathcal{F}\{f \circ g\} = \int_{-\infty}^{\infty} e^{-ikx} \int_{-\infty}^{\infty} f(\check{x}) g(x - \check{x})\, d\check{x}\, dx = \int_{-\infty}^{\infty} e^{-ik\check{x}} f(\check{x})\, d\check{x} \int_{-\infty}^{\infty} e^{-ik\tilde{x}} g(\tilde{x})\, d\tilde{x}, \tag{28}$$

after the change of variable $x = \check{x} + \tilde{x}$. But the final pair of integrals is simply the product $\hat{f}(k)\hat{g}(k)$. In other words,

$$f \circ g = \mathcal{F}^{-1}\left\{\hat{f}(k)\hat{g}(k)\right\}. \tag{29}$$

7 More transforms

Assembling several of the preceding results, we can build the beginnings of a table of known transforms. Our excursion on the complex plane with a Gaussian actually adds to this list of results because (with $a > 0$)

$$\mathcal{F}\left\{e^{-ax^2}\right\} = \int_{-\infty}^{\infty} e^{-ax^2 - ikx} dx = e^{-k^2/4a} \int_{-\infty - ik/2a}^{\infty - ik/2a} e^{-az^2} dz = \sqrt{\frac{\pi}{a}} e^{-k^2/4a}, \tag{30}$$

from the same arguments.

There is also the particularly simple case of Dirac's delta function, which gives $\mathcal{F}\{\delta(x)\} = 1$ and more. The delta function is an example of a *generalized function*, which only makes sense when multiplied by a normal test function and then integrating. Playing around with $\delta(x)$ or $\delta(k)$, we see immediately the price of the failure of the integral in the FT or its inverse to converge: the partner function is a generalized one.

$$\begin{array}{cc}
f(x) & \hat{f}(k) \\
\delta(x-a) & e^{-ika} \\
(2\pi)^{-1} e^{iax} & \delta(k-a) \\
e^{-a|x|},\ a>0 & 2a/(a^2+k^2) \\
a/[\pi(a^2+x^2)] & e^{-a|k|},\ a>0 \\
e^{-ax^2},\ a>0 & \sqrt{\pi/a}\, e^{-k^2/4a} \\
e^{-x^2/4a}/\sqrt{4\pi a} & e^{-ak^2},\ a>0 \\
f \circ g & \hat{f}\hat{g} \\
f(x-a) & e^{-ika}\hat{f}(k) \\
e^{iax} f(x) & \hat{f}(k-a) \\
f(ax) & \hat{f}(k/a)/|a| \\
e^{-ax} H(x),\ a>0 & (a+ik)^{-1}
\end{array} \tag{31}$$

Also included in the table are the shift and scaling theorems for the FT, all of which follow from using the definition of the transform. The final entry illustrates how the use of the Heaviside step function

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases} \tag{32}$$

can be used to extend some of the functions that we can analytically transform. That said, the need to compute integrals of functions multiplied by complex exponentials can be a bit limiting when extending the table.

8 Laplace's equation for a half-plane

Consider the PDE problem,

$$u_{xx} + u_{yy} = 0, \quad u(x,0) = f(x), \quad -\infty < x < \infty, \quad 0 \le y < \infty, \quad u \to 0 \text{ for } (|x|, y) \to \infty. \tag{33}$$

We can FT the PDE and BCs in the variable x, to obtain

$$\hat{u}_{yy} = k^2 \hat{u}, \quad \hat{u}(k,0) = \hat{f}(k), \quad \hat{u}(k,y) \to 0 \text{ for } y \to \infty. \tag{34}$$

Hence

$$\hat{u}(k,y) = \hat{f}(k) e^{-|k|y}, \tag{35}$$

bearing in mind that k could have either sign. This product of transforms corresponds to a convolution integral. In fact, we know that $\hat{g}(k) = e^{-a|k|}$ corresponds to $g(y) = a/[\pi(a^2 + x^2)]$, so we can identify a with y and write

$$u(x,y) = \int_{-\infty}^{\infty} f(\check{x}) G(x - \check{x}, y)\, d\check{x}, \quad G(X,y) = \frac{y}{\pi(X^2 + y^2)}, \tag{36}$$

where $G(X,y)$ is a Green function for Laplace's equation in this geometry. For example, if we have a line of charges distributed along the strip $0 < x < 1$, so that $f(x) = 1$ there, but $f(x) = 0$ otherwise, then the potential in $y > 0$ is

$$u(x,y) = \frac{y}{\pi} \int_0^1 \frac{d\check{x}}{(\check{x} - x)^2 + y^2} = \frac{1}{\pi} \tan^{-1}\left(\frac{1-x}{y}\right) + \frac{1}{\pi} \tan^{-1}\left(\frac{x}{y}\right). \tag{37}$$

9 An inhomogeneous PDE

The FT technology can be applied easily to inhomogeneous PDEs, such as the problem

$$u_t = u_{xx} + q(x), \quad u(x,0) = f(x), \tag{38}$$

with $\{u, f, q\} \to 0$ for $x \to \pm\infty$ to assure the convergence of the FTs. Zapping with the transform gives

$$\hat{u}_t = -k^2 \hat{u} + \hat{q}, \quad \hat{u}(k,0) = \hat{f}(k), \tag{39}$$

and so

$$\hat{u}(k,t) = \hat{f}(k) e^{-k^2 t} + \frac{\hat{q}}{k^2}(1 - e^{-k^2 t}), \tag{40}$$

which leaves us with computing the inverse transform. In fact, we could write instead

$$u = \int_{-\infty}^{\infty} G(x - \check{x}, t) f(\check{x})\, d\check{x} + \int_{-\infty}^{\infty} F(x - \check{x}, t) q(\check{x})\, d\check{x}, \tag{41}$$

where $G(X, t)$ is the Green function in (23) and

$$F(X, t) = \int_0^t G(X, \tilde{t})\, d\tilde{t} \quad \left(\Longrightarrow \hat{F}(k, t) = \int_0^t \hat{G}(k, \tilde{t})\, d\tilde{t} = \int_0^t e^{-k^2 \tilde{t}}\, d\tilde{t} = \frac{1 - e^{-k^2 t}}{k^2} \right). \tag{42}$$

This analysis can be generalized to solve the problem with a time-dependent source function $q(x, t)$.

A key limitation on using the FT in this way is that it struggles with coefficients that depend on x, or nonlinear terms in u. *e.g.* both $\mathcal{F}\{f(x) \tanh x\}$ and $\mathcal{F}\{f^2\}$ cannot easily be related to $\hat{f}(k)$.

10 Higher dimensions

We can attack problems with higher spatial dimension with the FT. For example, we might solve

$$u_t = u_{xx} + u_{yy}, \quad u(x, y, 0) = f(x, y), \tag{43}$$

assuming f and u both decay for $(|x|, |y|) \to \infty$. Now we have two space dimensions, either of which can be zapped with a FT. In fact, we can perform a double zap by defining an FT in k by

$$\hat{g}(\ell) = \int_{-\infty}^{\infty} e^{-i\ell y} g(y)\, dy. \tag{44}$$

The double transform of $u(x, y, t)$ is then

$$\hat{\hat{u}}(k, \ell, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ikx - i\ell y} u(x, y, t)\, dx\, dy, \tag{45}$$

with the double hat implying a double transform. Note that we need to keep k and ℓ independent of one another as x and y are that way (indicating two transform coordinates are needed). Double zapping the PDE and IC then gives

$$\hat{\hat{u}}(k, \ell, t) = \hat{\hat{f}}(k, \ell) e^{-(k^2 + \ell^2)t}. \tag{46}$$

Finally,

$$u(x, y, t) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ikx + i\ell y} \hat{\hat{u}}(k, \ell, t)\, dk\, d\ell \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x - \check{x}, y - \check{y}, t) f(\check{x}, \check{y})\, d\check{x}\, d\check{y} \tag{47}$$

where the Green function,

$$G(X, Y, t) = \frac{e^{-(X^2 + Y^2)/4t}}{4\pi t}, \tag{48}$$

which can be established by doubling up on the analysis of §4.

In principle, one could also zap every space derivative in sight to reduce PDEs without time into algebraic problems. This eases the pathway to the transform solution, but requires more inversions. Usually, reducing the PDE to an ODE is most direct, but not always. And the FT always requires an infinite interval, and so halfplanes are inaccessible. For the latter, there are "half-range" Fourier transforms that can be helpful. We encounter these in Assignment 4. For time, which is usually defined so that $t \geq 0$, it is better to use a Laplace transform.