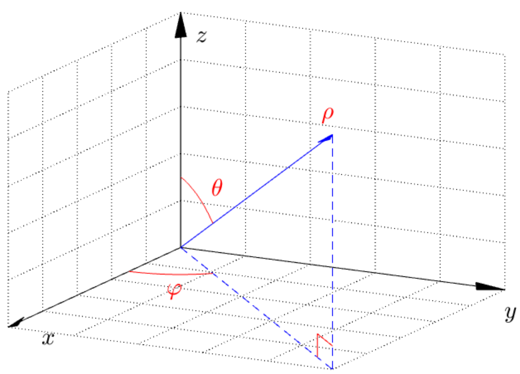
Now we move onto PDEs expressed in spherical coordinates (ρ, θ, φ). This will lead us to more Sturm-Liouville problems, to Legendre's differential equation and polynomials, and then to spherical harmonics. For this task, let's solve Laplace's equation inside the unit sphere. *i.e.* find $u(\rho, \theta, \varphi)$ satisfying

$$\nabla^2 u = \frac{1}{\rho^2}(\rho^2 u_\rho)_\rho + \frac{1}{\rho^2 \sin\theta}(\sin\theta\, u_\theta)_\theta + \frac{1}{\rho^2 \sin^2\theta} u_{\varphi\varphi} = 0, \tag{1}$$

$$u \text{ regular for } \rho \to 0,\ \theta \to 0,\ \theta \to \pi, \quad u \text{ is } 2\pi - \text{periodic in } \varphi, \quad u(1, \theta, \varphi) = f(\theta, \varphi). \tag{2}$$



The connection between the spherical polar coordinates and our usual Cartesian coordinates is

$$\begin{aligned} x &= \rho \sin\theta \cos\varphi, \\ y &= \rho \sin\theta \sin\varphi, \\ z &= \rho \cos\theta; \end{aligned}$$

ρ is distance from the centre;
θ is angle from the pole (related to latitude, but $\theta = 0$ and π correspond to the poles here);
φ is the angle of longitude.

Looking ahead, separation of variables will split the PDE up into three ODEs, with two separation constants; the ODEs in ρ and θ will be non-constant-coefficient. Two of the ODEs should boil down to Sturm-Liouville problems so that we can find infinite sets of suitable eigenfunctions; in view of the various boundary conditions, these should correspond to the problems in the two angles. The general solution will then contain two infinite sums, with a bunch of constants to be set by demanding that we satisfy the more complicated, final condition in (2).

1 Solution by separation of variables

We set $u = R(\rho)\Theta(\theta)\Phi(\varphi)$. The PDE can then be re-arranged into

$$\sin^2\theta \left[\frac{(\rho^2 R')'}{R} + \frac{(\Theta' \sin\theta)'}{\Theta \sin\theta} \right] = -\frac{\Phi''}{\Phi}. \tag{3}$$

This function of (ρ, θ) or φ must equal our first separation constant. As we're heading to a problem that should furnish $2\pi-$periodic functions in φ, we'll set this constant to m^2, giving the ODE, $\Phi'' + m^2\Phi = 0$, with $m = 0, 1, ...$, and the usual functions ($\sin m\varphi$, $\cos m\varphi$ or a constant with $m = 0$) present in a Fourier series.

The other side of (3) can now be re-arranged into a function of θ equalling a function of ρ:

$$\frac{(\Theta' \sin\theta)'}{\Theta \sin\theta} - \frac{m^2}{\sin^2\theta} = -\frac{(\rho^2 R')'}{R}; \tag{4}$$

i.e. we need another separation constant, that we set to $-\lambda$.

At this stage, we might be a little stuck in deciding on which sign to to use for the second separation constant - there's no obvious physical argument, and the form of the solutions are also not apparent, so deciding what sign is needed to satisfy the boundary conditions is tricky. A minus sign is chosen above, and one can attribute this choice to the vast knowledge and infallibility of the professor. As someone pointed out in class, one could also be guided by the Sturm-Liouville problem that one needs to arrive at, which guarantees afterwards that everything works out happily.[1]

The $\theta-$part now gives the Sturm-Liouville problem,

$$(\Theta' \sin\theta)' + \lambda\Theta\sin\theta - \frac{m^2\Theta}{\sin\theta} = 0 \qquad \left([a,b]=[0,\pi],\ p(\theta)=\sigma(\theta)=\sin\theta \ \&\ q(\theta) = -\frac{m^2}{\sin\theta}\right), \tag{5}$$

given that the regularity conditions we must impose at the poles are of type (ii). Note that $0 \le \theta \le \pi$, ensuring that p and σ are, indeed, non-negative. The SL ODE looks a bit formidable, though. In fact, it is not quite as bad as it seems, because the transformation $x = \cos\theta$ and $y(x) = \Theta(\theta)$ (indicating $(d/d\theta) \to -\sin\theta(d/dx)$) turns it into the simpler-looking, but still non-constant-coefficient, ODE,

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \nu(\nu+1)y - \frac{m^2 y}{1-x^2} = 0, \quad \lambda = \nu(\nu+1). \tag{6}$$

The singular points (*i.e.* the poles) are now at $x = \pm 1$, and the corresponding SL problem in x has $[a,b] = [-1,1]$, $p(x) = 1-x^2$, $\sigma(x) = 1$ and $q(x) = -m^2/(1-x^2)$.

To motivate the last switch (from λ to ν), we look at the $\rho-$problem, which now can be seen to revolve around the ODE

$$(\rho^2 R')' - \nu(\nu+1)R = 0. \tag{7}$$

This is another Euler equation with solutions $R \propto \rho^\alpha$ where

$$\alpha(\alpha+1) = \nu(\nu+1) \quad \longrightarrow \quad \alpha = \nu \text{ or } -\nu-1;$$

the $\alpha-$values would have been a lot uglier expressed in terms of λ. In fact, to be definitive, we may take $\nu = \sqrt{\frac{1}{4}+\lambda} - \frac{1}{2} \ge 0$ (since λ is not negative, from SL theory). The solutions with $\rho^{-\nu-1}$ are therefore ruled out because they are not regular for $\rho \to 0$.

The form of the general solution is therefore

$$u(\rho,\theta,\varphi) = \sum_n \left[\tfrac{1}{2}a_{0n}\rho^{\nu_{0n}} y_{0n}(\cos\theta) + \sum_{m=1}^{\infty}(a_{mn}\cos m\theta + b_{mn}\sin m\theta)\rho^{\nu_{mn}} y_{mn}(\cos\theta)\right], \tag{8}$$

where $\{\lambda_{mn}, y_{mn}(x)\}$ denote the SL eigensolutions to (6), indexed by n, and we have added m as a subscript as a reminder that it appears as a known parameter in the ODE.

2 Legendre's ODE

To enjoy the solutions to (6), we divide and conquer. We kick off by considering axisymmetrical solutions without any dependence on φ. *i.e.* we put $m = 0$, corresponding to the boundary condition $u(1,\theta,\varphi) = f(\theta)$. Equation (6) reduces to

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \nu(\nu+1)y = 0, \tag{9}$$

which is Legendre's ODE.

[1]The most direct way to establish the sign of the separation constant is to multiply (5) by Θ and integrate: after an intergration by parts and a little re-arrangement, one obtains

$$\lambda\int_0^\pi \Theta^2 \sin\theta\, d\theta = \int_0^\pi \left[(\Theta')^2\sin\theta + \frac{m^2\Theta^2}{\sin\theta}\right] d\theta$$

($\sin\theta = 0$ at the limits of the integral and Θ must be regular). Since both integrals cannot be negative, neither can λ.

2.1 Lucky guesses

By inspecting this ODE, you might be able to come up with some solutions: *e.g.* if $\lambda = \nu = 0$, then

$$[(1-x^2)y']' = 0 \quad \longrightarrow \quad y' = \frac{A}{1-x^2} \quad \longrightarrow \quad y = A \ln\left(\frac{1+x}{1-x}\right) + B, \tag{10}$$

for two integration constants, A and B. The first of these solutions is not regular at $x = \pm 1$, but $(\lambda, y) = (0, B)$ looks perfectly reasonable. The eigenfunction $y(x) = B$ has no zeros over the interval $[-1, 1]$ and therefore looks as though it ought to be the first of the SL sequence (remember the SL oscillation theorem).

You might also be lucky and guess that a second solution is $y(x) = Cx$, for some constant C. Plugging this into (9) gives $\lambda = 2$ or $\nu = 1$. This solution has exactly one zero over $[-1, 1]$ and might well be the second of the SL sequence. As the ODE is second-order, there must be another independent solution for $\lambda = 2$. To see what this is, we may use reduction of order: set $y = xC(x)$. Then, plugging into (9) and clearing the smoke gives

$$x[(1-x^2)C']' + 2(1-x^2)C' = 0 \quad \Longrightarrow \quad C' = \frac{B}{x^2(1-x^2)} \quad \text{or } y = Cx = Ax + B\left[\tfrac{1}{2}x \log\left(\frac{1+x}{1-x}\right) - 1\right].$$

The first term returns our original guessed solution; the second term corresponds to the other solution, and is again singular at $x = \pm 1$.

On a role, you might try a third solution taking the form of a quadratic polynomial; this does indeed work (after plugging into (9)), provided that $y(x) = C(1 - 3x^2)$ and $\lambda = 6$ or $\nu = 2$. The solution has two zeros over $[-1, 1]$. Wow, said Jurgen. Could the SL eigenfunctions actually be polynomials with $\nu = n = 0, 1, 2, ...$?

2.2 Series solution and recurrence relation for coefficients

Armed with this insight, let's try a series solution,

$$y = \sum_{j=0}^{\infty} c_j x^j. \tag{11}$$

Plugging this into the ODE gives

$$\sum_{j=0}^{\infty} \{(j+2)(j+1)c_{j+2} + [\nu(\nu+1) - j(j+1)]c_j\} x^j = 0. \tag{12}$$

To arrive at this relation, we used the fact that

$$[(1-x^2)y']' = \sum_{j=0}^{\infty} [j(j-1)x^{j-2} - j(j+1)x^j]c_j \equiv \sum_{l=0}^{\infty} (l+2)(l+1)x^l c_{l+2} - \sum_{j=0}^{\infty} j(j+1)x^j c_j$$

after setting $j = l + 2$ in the first sum, then discarding $l = -2$ and $l = -1$ in view of the factor $(l + 2)(l + 1)$. Switching the integer over which we sum from l back to j in the first term then leads to (12).

Consequently, as (12) must hold for every value of x,

$$c_{j+2} = \frac{j(j+1) - \nu(\nu+1)}{(j+1)(j+2)} c_j. \tag{13}$$

This recurrence relation delivers the coefficient c_{j+2} in terms of c_j. Therefore, if we specify c_0, we may compute the sequence $\{c_0, c_2, c_4, ...\}$. Alternatively, we can specify c_1 and compute the sequence $\{c_1, c_3, c_5, ...\}$. This means that we can construct two independent series with either even or odd powers of x. *i.e.* the solutions must be either even or odd (something that could have been predicted from Legendre's equation, which is invariant under the reflection $x \to -x$). For the former, we take $c_0 \neq 0$ and $c_1 = 0$; for the latter, we set $c_1 \neq 0$ and $c_0 = 0$. More dramatically, if ν is an even integer n, the even series beginning with c_0 terminates at $c_n x^n$ (since $c_{n+2} = 0$ by (13)), leaving an even polynomial of degree n. Similarly, if $\nu = n$ is an odd integer, the odd series beginning with c_1 also terminates at $c_n x^n$, furnishing an odd polynomial.

In other words, the SL eigensolutions are

$$\lambda = n(n+1), \quad n = 0, 1, 2, ..., \quad y(x) = P_n(x) = c_0 + c_2 x^2 + ... + c_n x^n \quad \text{or} \quad c_1 x + c_3 x^3 + ... + c_n x^n, \tag{14}$$

along with (13). Note that, for this set and because of the first value for λ, it is convenient to begin our indexing of the SL sequence with $n = 0$, rather than $n = 1$, as done previously.

2.3 The first few Legendre polynomials

To make the polynomials into special functions, we need a normalization to eliminate the freedom in the choice of either c_0 or c_1. The tradition is to take $P_n(1) = 1$, and we then arrive at the Legendre polynomials. Given our earlier "guesses", we observe that the first three are

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \tfrac{3}{2}x^2 - \tfrac{1}{2}. \tag{15}$$

We can use the recurrence relation and normalization to compute $P_n(x)$ directly for each n. *e.g.* for $n = 3$, we have

$$P_3(x) = c_1 x + c_3 x^3 \;\&\; c_3 = \tfrac{1\times 2 - 3\times 4}{2\times 3} c_1 = -\tfrac{5}{3} c_1. \text{ Then } P_3(1) = c_1(1 - \tfrac{5}{3}) = 1 \implies c_1 = -\tfrac{3}{2},$$

giving

$$P_3(x) = \tfrac{5}{2}x^3 - \tfrac{3}{2}x.$$

For $n = 4$, similar calculations indicate that $c_4 = -\frac{7}{6}$, $c_2 = -10c_0$ and $c_0 = \frac{3}{8}$, giving

$$P_4(x) = \tfrac{35}{8}x^4 - \tfrac{15}{4}x^2 + \tfrac{3}{8}.$$

This obviously gets a little unwieldy for bigger n.

Figure 1 plots some of the Legendre polynomials. Each time n is raised by one, another zero appears in $P_n(x)$; the polynomials become wigglier. There is not a lot else worth saying about their spatial form.

For each $\lambda = n(n+1)$, the other independent solution is not regular at $x = \pm 1$ and is usually denoted as $Q_n(x)$. In view of the earlier guessed solutions, the first two must be

$$Q_0 = \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad \& \quad Q_1 = \tfrac{1}{2}x\log\left(\frac{1+x}{1-x}\right) - 1,$$

having imposed a different normalization (which can be viewed as insisting that Q_n diverges as $-\frac{1}{2}\ln(1-x)$ when $x \to 1$). In fact, regular solutions to (9) can only be found if λ takes the value $n(n+1)$ for $n = 0, 1, 2, ...$ Otherwise, with $\lambda \neq n(n+1)$, the best one can do is is to make the solution regular at one of the singular points, $x = \pm 1$, but it remains irregular at the other singular point.

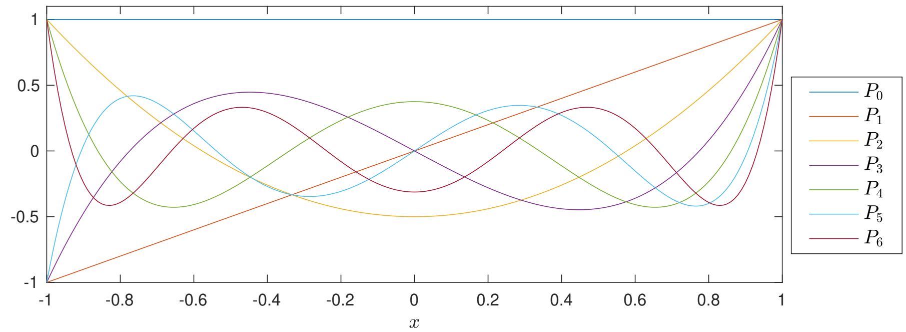


Figure 1: Legendre polynomials for $n = 0, 1, ..., 6$.

2.4 Orthogonality and expansion

From SL theory we know that the Legendre polynomials must satisfy the orthogonality relation,

$$\int_{-1}^{1} P_n P_m \, dx = 0 \quad \text{if } n \neq m \tag{16}$$

(the weight function for Legendre's equation is $\sigma(x) = 1$). In other words, they are examples of "orthogonal polynomials".

Any sensible function $F(x)$ can also be expanded in terms of the polynomnials, via the expansion formulae,

$$F(x) = \sum_{n=0}^{\infty} d_n P_n(x), \quad d_n = \frac{\int_{-1}^{1} F(x) P_n(x) \, dx}{\int_{-1}^{1} [P_n(x)]^2 \, dx}. \tag{17}$$

Had $F(x) = f(\theta)$, as in our original axisymmetrical PDE problem, this would have corresponded to

$$f(\theta) = \sum_{n=0}^{\infty} d_n P_n(\cos\theta), \quad d_n = \frac{\int_0^{\pi} f(\theta) P_n(\cos\theta) \ \sin\theta \, d\theta}{\int_0^{\pi} [P_n(\cos\theta)]^2 \ \sin\theta \, d\theta}. \tag{18}$$

In other words, the solution to (1)-(2) with $u(1, \theta) = f(\theta)$ must be

$$u(\rho, \theta) = \sum_{n=0}^{\infty} d_n P_n(\cos\theta) \rho^n \tag{19}$$

(given that we now see that $\nu_{mn} \to n$), with d_n prescribed as above.

2.5 Some other charming properties

Generating function:

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n. \tag{20}$$

Rodrigues' formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n. \tag{21}$$

A useful integral (appearing in the expansion formulae):

$$\int_{-1}^{1} [P_n(x)]^2 \, dx = \frac{2}{2n+1}. \tag{22}$$

Recurrence relation for the polynomials themselves:

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \tag{23}$$

All these, and much, more more, can be found in textbooks or online at Wikipedia or DLMF. The usual route to establishing them involves playing around with some combo of Legendre's ODE, (13) and the orthogonality condition, or one of the preceding properties. Although one can easily get lost admist all these charming details, we will avoid straying too far from our path of solving PDEs in Math 400.

3 Associated Legendre functions

Our original interest was in the SL ODE in (6) rather than the simpler one in (9). So coming up with Legendre polynomials to solve (9) doesn't look that helpful right now. In fact, this is not the case at all. Amazingly enough, the regular solutions to (6) are $CP_n^m(x)$, where C is arbitrary, λ is again $n(n+1)$ with $n = 0, 1, 2, \ldots$ and $P_n^m(x)$ is an associated Legendre function that is given by

$$P_n^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}[P_n(x)]. \tag{24}$$

In other words, ν_{mn} is again n, the solutions to (6) can be extracted from suitable derivatives of the Legendre polynomials, and we have another Jurgen Wow moment. Note that conventions are not always followed, and sometimes the $(-1)^m$ is omitted in (24). This can cause a bit of a headache at times.

We won't establish (24) generally, but let's at least consider $m = 1$: from Legendre's ODE we note

$$\frac{d}{dx}\left[(1-x)^2P_n'' - 2xP_n' + n(n+1)P_n\right] = \left[(1-x)^2\frac{d^2}{dx^2} - 4x\frac{d}{dx} - 2 + n(n+1)\right]P_n' = 0. \tag{25}$$

The first derivative term in the ODE for P_n' is different from that in the original Legendre ODE (we have $-4\frac{d}{dx}$ rather than $-2\frac{d}{dx}$), but this can be adjusted with the substitution, $P_n' = Y(x)/\sqrt{1-x^2}$. We then obtain

$$(1-x^2)Y'' - 2xY' + n(n+1)Y - \frac{Y}{1-x^2} = 0, \tag{26}$$

which is (6) with $m = 1$. *i.e.* we may take $P_n^1(x) = -\sqrt{1-x^2}P_n'$. In the same way, but with more effort, one can differentiate Legendre's equation m times, plug in $d^mP_n/dx^m = Y/(1-x^2)^{m/2}$, to arrive at (6), verifying (24). There is therefore nothing esoteric here, just repeated differentiation and a simple substitution (designed to adjust the first derivative term in the ODE for d^mP_n/dx^m).

As with the Legendre polynomials, there is another, independent solution to (6) for $\lambda = n(n+1)$. That solution is again irregular at $x = \pm 1$, and is normally denoted by $Q_n^m(x)$.

Finally, SL theory implies the orthogonality condition and expansion formulae:

$$\int_{-1}^{1} P_n^m P_j^m dx = \begin{cases} 0 & n \neq j, \\ 2(n+m)!/[(2n+1)(n-m)!] & n = j, \end{cases} \tag{27}$$

$$g(x) = \sum_{n=0}^{\infty} d_n P_n^m(x), \quad d_n = \frac{(2n+1)(n-m)!}{2(n+m)!}\int_{1}^{1} g(x)P_n^m(x)\,dx. \tag{28}$$

The result for $\int_{-1}^{1}[P_n^m]^2 dx$ is a generalization of the useful integral in (22) (and can be proved using Rodrigues' formula).

3.1 Some simple Legendre functions and negative order

The factor $(1-x^2)^{m/2}$ in (24) ensures that $P_n^m(x)$ is still a polynomial if m is even. Conversely, for m odd, $P_n^m(x)$ consists of a polynomial multiplied by the factor $\sqrt{1-x^2} \equiv \sin\theta$. For $n = 2$ or 3 and low m, we have

$$P_2^1 = -3x\sqrt{1-x^2}, \quad P_2^2 = 3(1-x^2), \quad P_3^1 = \tfrac{3}{2}(1-5x^2)\sqrt{1-x^2}, \quad P_3^2 = 15x(1-x^2), \tag{29}$$

Because $P_n(x)$ is a polynomial of degree n, d^nP_n/dx^n is a constant. Therefore when $n = m$, the associated Legendre function is just

$$P_n^n(x) = (-1)^n(1-x^2)^{n/2}\frac{(2n)!}{2^n n!}, \tag{30}$$

where the final constant factor follows from Rodrigues' formula in (21). For example, with $n = m = 1$,

$$P_1^1(x) = -\sqrt{1-x^2} \equiv -\sin\theta. \tag{31}$$

For our PDE problem, this implies that the boundary condition $f(\theta,\phi) = \sin\theta\sin\varphi$ leads to the solution $u(\rho,\theta,\varphi) = \rho\sin\theta\sin\varphi$.

It must also be true that

$$P_n^m(x) = 0 \text{ for } m > n. \tag{32}$$

Later, we see that this result allows a truncation of the eigenfunction expansion.

The parameter m is sometimes called the "order" of the associated Legendre function $P_n^m(x)$; n is the "degree". At the moment the order is purely positive and $P_n^m(x)$ only defined for $m > 0$. However, it proves useful to extend this definition so that the order can be negative. In particular, it proves convenient to do this by setting

$$P_n^{-m}(x) = \frac{(-1)^m(n-m)!}{(n+m)!}P_n^m(x). \tag{33}$$

4 General solution of the PDE; Spherical harmonics

At last, we use the preceding results to write the general solution to the PDE in (1):

$$u = \sum_{n=0}^{\infty}\left[\tfrac{1}{2}a_{0n}\rho^n P_n(\cos\theta) + \sum_{m=1}^{n}(a_{mn}\cos m\varphi + b_{mn}\sin m\varphi)\rho^n P_n^m(\cos\theta)\right], \tag{34}$$

where we have used (32). The boundary condition at $\rho = 1$ can be dealt with by first expanding $f(\theta,\varphi)$ as a Fourier series in φ:

$$f = \tfrac{1}{2}\alpha_0(\theta) + \sum_{m=1}^{\infty}[\alpha_m(\theta)\cos m\varphi + \beta_m(\theta)\sin m\varphi], \qquad \begin{bmatrix}\alpha_0\\ \alpha_m\\ \beta_m\end{bmatrix} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta,\varphi)\begin{bmatrix}1\\ \cos m\varphi\\ \sin m\varphi\end{bmatrix} d\varphi, \tag{35}$$

matching up the coefficients with those from (34), and then using the expansion formulae in (28):

$$\begin{bmatrix}a_{0n}\\ a_{mn}\\ b_{mn}\end{bmatrix} = \frac{(2n+1)(n-m)!}{2(n+m)!}\int_0^{\pi} P_m^n(\cos\theta)\begin{bmatrix}\alpha_0(\theta)\\ \alpha_m(\theta)\\ \beta_m(\theta)\end{bmatrix}\sin\theta\, d\theta. \tag{36}$$

This calculation can be cleaned up considerably by defining spherical harmonics. To begin, we note that, by Euler's formula the cosines and sines in (34) can be replaced by the exponentials $e^{\pm im\varphi}$. Therefore, by switching notation for the arbitrary constants (to A_n^m), we could just as well write the single double sum,

$$u = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_n^m \rho^n Y_n^m(\cos\theta, \varphi), \tag{37}$$

where we have used the extension to negative order in (33). and defined the expansion functions,

$$Y_n^m(\theta, \varphi) = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} e^{im\varphi} P_n^m(\cos\theta). \tag{38}$$

Because of (27) and

$$\int_{-\pi}^{\pi} e^{-im_1\varphi} e^{im_2\varphi} d\varphi = \begin{cases} 0 & m_1 \neq m_2, \\ 2\pi & m_1 = m_2, \end{cases} \tag{39}$$

we observe

$$\int_0^{\pi} \left[\int_{-\pi}^{\pi} (Y_{n_1}^{m_1})^* Y_{n_2}^{m_2} \, d\varphi \right] \sin\theta \, d\theta = \delta_{n_1 n_2} \delta_{m_1 m_2}, \tag{40}$$

where the star denotes complex conjugation and δ_{jk} denotes the Kronecker delta (the order of the double integral matters here and the φ integral needs to be performed first). Consequently, we can read off immediately the values of the coefficients in (37):

$$A_n^m = \int_{-\pi}^{\pi} \int_0^{\pi} (Y_n^m)^* f(\theta, \varphi) \, \sin\theta \, d\theta \, d\varphi \tag{41}$$

(where the integration order no longer matters).

The orthonormal eigenfunctions are the spherical harmonics. They form a complete set with which one can expand any sensible function of the two angles, θ and φ, and satisfy the PDE

$$\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left[\sin\theta \frac{\partial Y_n^m}{\partial\theta} \right] + \frac{1}{\sin^2\theta} \frac{\partial^2 Y_n^m}{\partial\varphi^2} = -n(n+1) Y_n^m. \tag{42}$$

5 The wave equation for a sphere

Now let's consider the wave equation for the ringing of a spherical fluid drop of unit radius. *i.e.,* find $u(\rho, \theta, \varphi, t)$, satisfying

$$u_{tt} = \frac{1}{\rho^2} (\rho^2 u_\rho)_\rho + \frac{1}{\rho^2 \sin\theta} (\sin\theta \, u_\theta)_\theta + \frac{1}{\rho^2 \sin^2\theta} u_{\varphi\varphi}, \qquad \begin{array}{c} u \text{ regular for } \rho \to 0, \ \theta \to 0, \ \theta \to \pi, \\ u \text{ is } 2\pi - \text{periodic in } \varphi, \\ u = 0 \text{ at } \rho = 1. \end{array} \tag{43}$$

We also need two initial conditions, but we won't be interested in any particular initial-value problem below.

We begin using the eigenfunction expansion,

$$u = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} R_n^m(\rho, t) Y_n^m(\cos\theta, \varphi). \tag{44}$$

Plugging in to the wave equation and using (42) furnishes a PDE for each coefficient:

$$\frac{\partial^2 R_n^m}{\partial t^2} = \frac{1}{\rho^2} \frac{\partial}{\partial\rho} \left(\rho^2 \frac{\partial R_n^m}{\partial\rho} \right) - \frac{n(n+1)}{\rho^2} R_n^m. \tag{45}$$

We can separate variables to solve this equation, which leads to solutions with time dependences of $\cos \omega t$ and $\sin \omega t$, for some separation constant $-\omega^2$. Imposing the boundary conditions in ρ then determines ω (from another SL problem). Following this route, the general solution for $u(\rho, \theta, \varphi, t)$ would involve a triple sum over n, m and a third integer indexing the SL eigensolutions in ρ.

Alternatively, if we are not interested in a specific initial-value problem, but simply the "normal-mode" frequencies at which the drop could ring, we can instead adopt the time dependence in $R_n^m = R_{mn}(\rho)e^{-\omega t}$, to arrive at the ODE

$$R''_{mn} + \frac{2}{\rho} R'_{mn} + \omega^2 R_{mn} - \frac{n(n+1)}{\rho^2} R_{mn} = 0. \tag{46}$$

This is Bessel's equation in disguise (*i.e.* it is an example of the more general ODE that leads to Bessel functions noted in Notes V; $\alpha = \frac{1}{2}$, $\beta = 1$, $\alpha^2 - \nu^2 = -n(n+1)$). The solutions that are regular for $\rho \to 0$ are

$$R_{mn} = C_{mn} \rho^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\omega \rho), \tag{47}$$

for some constant C_{mn}. The half-integer versions of $J_\nu(z)$ are called "spherical Bessel functions" (precisely because they commonly appear in PDE problems with spherical coordinates like (43)). The outer boundary condition now demand that the frequencies ω must be dictated by $J_{n+\frac{1}{2}}(\omega) = 0$.

Spherical Bessel functions can be written more explicitly in terms of trig functions (Bessel functions in general can not). We have already seen this with $J_{\frac{1}{2}}(z)$.

Note that ω depends only on n and not m. *i.e.* all normal modes with the same n, but different m have the same frequency of oscillation. There are $(2n+1)$ of these modes as m runs from $-n$ to $+n$. The frequencies are independent of m because the original drop was spherically symmetric, the PDE (43) has no $\varphi-$dependent coefficients, and any oscillations cannot tell the difference between east and west in the current coordinate system. Adding any additional effect that does distinguish east from west breaks this symmetry and splits up the frequencies of $(2n+1)-$modes that arise for each n. The removal of the $(2n+1)-$fold "degeneracy" of the frequencies corresponds to the origin of the Zeeman splitting of spectral lines by a magnetic field, and underlies how astronomers have searched for clues about the rotation rates and magnetic fields of stars by studying pulsation frequencies.