Sturm-Liouville problems

This theory surrounds the ODE:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + q(x)y + \lambda\sigma(x)y = 0 \tag{1}$$

which is to be solved on the interval $a \le x \le b$ subject to the constraints that $p(x) \ge 0$ and $\sigma(x) \ge 0$ (over that interval), and certain boundary conditions. Those boundary conditions take one of three forms:

$$(i) \quad C_1 y(a) + C_2 y'(a) = 0, \ D_1 y(b) + D_2 y'(b) = 0, \tag{2}$$
$$(ii) \quad \text{Regularity, if } p(a) = 0 \text{ or } p(b) = 0. \tag{3}$$
$$(iii) \quad \text{Periodic conditions}: y(a) = y(b) \ \& \ y'(a) = y'(b). \tag{4}$$

Here, (C_1, C_2) and (D_1, D_2) correspond to suitably chosen constants. Sturm-Liouville (SL) problems can also combine boundary conditions of types (i) and (ii).

In the SL ODE (1), the parameter λ is not known, but must be found as part of the solution. In particular, unless λ takes one of a set of special values, the only solution is trivial; the non-trivial solutions at the special values of λ are those that are of interest. All this means that the SL problem is the differential equivalent of a matrix eigenvalue problem. *i.e.* a differential eigenvalue problem, in which λ is the eigenvalue and $y(x)$ is the "eigenfunction."

e.g. Solve

$$u_t + tu = u_{xx}, \quad u(0,t) = 0, \quad u_x(L,t) + u(L,t) = 0, \quad u(x,0) = f(x).$$

Separation of variables with $u(x,t) = X(x)T(t)$ gives

$$\frac{T' + tT}{T} = \frac{X''}{X} = -\lambda,$$

where λ is a separation constant. For the space part of the problem, we have $X \propto \sin kx$ with $k^2 = \lambda$, given that $X(0) = 0$. But the other boundary condition now implies that

$$k\cos kL + \sin kL = 0, \quad \text{or} \quad \tan kL = -k. \tag{5}$$

This is a trancendental algebraic equation for k (or λ). In view of the periodic nature of the tan function, there are an infinite number of solutions that can be constructed graphically; see figure 1. The solution at $k = 0$ can be discounted as it leads to a trivial solution for $X(x)$. There is also no need to consider the solutions with $k < 0$, as $\lambda = k^2$, and any change in sign of $\sin kx$ can be absorbed into the arbitrary constant in front of that function in $X(x)$. The positive solutions for k can be effortlessly computed in MATLAB (using fzero, for example). For the first few and $L = 1$, we find

$$k \approx 2.029, 4.913, 7.979, 11.086 \tag{6}$$
$$\tfrac{1}{2}(2n-1)\pi \approx 1.571, 4.712, 7.854, 10.996. \tag{7}$$

The solutions rapidly converge to $k \approx \frac{1}{2}(2n-1)\pi$ as n increases, for reasons that are obvious from figure 1.

The time part of the PDE problem now gives

$$T(t) = C\exp\left(-k_n^2 t - \tfrac{1}{2}t^2\right),$$

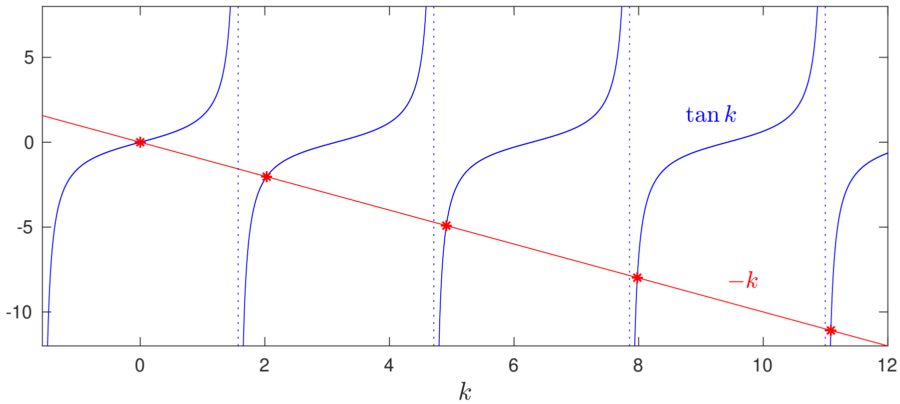

Figure 1: Graphical solution of (5) for $L = 1$.

for some arbitrary constant C, where k_n denotes the n^{th} possible value of k. Hence the general solution of the PDE is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\frac{1}{2}t^2 - k_n^2 t} \sin k_n x.$$

The constants c_n must be selected to enforce the initial condition,

$$f(x) = \sum_{n=1}^{\infty} c_n \sin k_n x. \tag{8}$$

But no amount of weird extensions or choice of period will reduce this to a genuine Fourier series. So our previous methodology, relying on Fourier series theory to justify the analogue of (8), is royally messed up. The space problem (for $X(x)$) is, however, a SL problem (we have $X(x) \equiv y(x)$, $p(x) = \sigma(x) = 1$ and $q(x) = 0$; $a = 0$ and $b = L$; both BCs are of type (i), with $C_2 = 0$ and $C_1 = D_1 = D_2 = 1$), and we can use SL theory for the justification instead.

Sturm-Liouville theory

SL theory establishes certain properties of the solutions to (1)-(4):

- The solution pairs $[\lambda, y(x)] = [\lambda_n, y_n(x)]$ form an infinite sequence, $n = 1, 2, ...$
- The eigenvalues can be ordered such that $\lambda_1 < \lambda_2 < \lambda_3 < ...$, with $\lambda_n \to \infty$ for $n \to \infty$ (as seen in the example above).
- $y_n(x)$ has exactly $n - 1$ zeros between $x = a$ and $x = b$ (we will not need this).
- The set $\{y_1(x), y_2(x), ...\}$ is **complete**, in the sense that any continuous function $f(x)$ can be represented as the series,

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x). \tag{9}$$

The last point here establishes that it is justified to write (8) in the previous example; no need for any Fourier series or painful extensions.

This might make you wonder why we bothered with Fourier series in previous lectures! It now turns out that if we can identify that we have a SL problem (by comparing our ODE with (1) and matching with the boundary conditions in (2)-(4)), then the SL theory immediately justifies our expansion in terms of the SL eigenfunctions. Indeed, this is far more straighforward than the Fourier series folderol.

Orthogonality and the Sturm-Liouville expansion formula

The SL eigenfunctions are **orthogonal** in the sense that

$$\int_a^b y_n(x) y_m(x) \sigma(x)\ \mathrm{d}x = 0, \text{ if } n \neq m. \tag{10}$$

Note the $\sigma(x)$ here!

To establish this orthogonality relation, we take the ODE for y_n, multiply by y_m and integrate:

$$\int_a^b y_m \frac{\mathrm{d}}{\mathrm{d}x}\left[p \frac{\mathrm{d}y_n}{\mathrm{d}x}\right]\ \mathrm{d}x + \int_a^b q y_m y_n\ \mathrm{d}x + \lambda_n \int_a^b y_m y_n \sigma\ \mathrm{d}x = 0.$$

Similarly, from the ODE for $y_m(x)$, we have

$$\int_a^b y_n \frac{\mathrm{d}}{\mathrm{d}x}\left[p \frac{\mathrm{d}y_m}{\mathrm{d}x}\right]\ \mathrm{d}x + \int_a^b q y_m y_n\ \mathrm{d}x + \lambda_m \int_a^b y_m y_n \sigma\ \mathrm{d}x = 0.$$

Subtracting these two relations, we end up with

$$\int_a^b \left\{ y_m \frac{\mathrm{d}}{\mathrm{d}x}\left[p \frac{\mathrm{d}y_n}{\mathrm{d}x}\right] - y_n \frac{\mathrm{d}}{\mathrm{d}x}\left[p \frac{\mathrm{d}y_m}{\mathrm{d}x}\right] \right\} \mathrm{d}x + (\lambda_n - \lambda_m) \int_a^b y_m y_n \sigma\ \mathrm{d}x = 0.$$

But a speedy integration by parts and use of the various SL boundary conditions immediately demonstrates that the first integral cancels identically. Hence, as long as $n \neq m$, (10) follows. And we now see why there must be a $\sigma(x)$ in this orthogonality relation.

As a final piece of the SL puzzle, it is all very well being able to write the series in (9), but it is absolutely useless if we cannot find a convenient way of computing the coefficients c_n. To do this, we use the same trick that we used for Fourier series: we multiply the series by one of the SL eigenfunctions and $\sigma(x)$, then integrate. The orthogonality relation ensures that we pick out one of the coefficients in this way:

$$\int_a^b f y_m \sigma\ \mathrm{d}x = \sum_{n=1}^{\infty} c_n \int_a^b y_m y_n \sigma\ \mathrm{d}x = c_m \int_a^b y_m^2 \sigma\ \mathrm{d}x.$$

Hence we arrive at the SL expansion formula,

$$c_m = \frac{\int_a^b f(x) y_m(x) \sigma(x)\ \mathrm{d}x}{\int_a^b [y_m(x)]^2 \sigma(x)\ \mathrm{d}x}. \tag{11}$$

Our previous examples with the heat equation or wave equation all led to spatial problems that had SL form, and our Fourier sine or cosine series were nothing more than examples of expansions using SL eigenfunctions. To see this, we note that all those problems descended to the spatial ODE

$$X'' + \lambda X = 0, \qquad 0 \leq x \leq \pi,$$

which is an example of (1) with $p = \sigma = 1$, $q = a = 0$ and $b = \pi$. The Dirichlet or Neumann conditions that we championed are both of the form (2) with either $C_2 = D_2 = 0$ or $C_1 = D_1 = 0$. The SL eigenvalues turned out to be $\lambda = n^2$, $n = 1, 2, ...$ For the Dirichlet conditions $u(0,t) = u(\pi,t) = 0$, the SL eigenfunctions were $\sin nx$. In the case of the Neuman conditions $u_x(0,t) = u_x(\pi,t) = 0$, we had $y_n(x) \equiv \cos nx$, but there is a minor awkward detail in that the first eigenvalue-eigenfunction pair is $(\lambda, y) = (0, \frac{1}{2})$. If we insist on always starting the SL sequence with $n = 1$, this means that $(\lambda_1, y_1) = (0, \frac{1}{2})$, $(\lambda_2, y_2) = (1, \cos x)$, $(\lambda_3, y_3) = (4, \cos 2x)$, *etc.* The $n^{th}-$member of this sequence is $(\lambda_n, y_n) = ((n-1)^2, \cos(n-1)x)$, which is perhaps a tad aesthically displeasing in the mismatch of the integers on the right and left. In any event, the expansion formula (11) corresponds precisely to the integrals giving the coefficients of the Fourier sine or cosine series, and there is no need to extend any functions outside the domain of the PDE when using SL theory to justify the solution.

Oscillations of a drum

Let's solve

$$u_{tt} = \nabla^2 u = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}, \qquad \begin{cases} u\ 2\pi-\text{periodic in } \theta, \\ u \text{ regular for } r \to 0, \\ u(1,\theta,t) = 0, \\ u(r,\theta,0) = f(r,\theta)\ \&\ u_t(r,\theta,0) = g(r,\theta). \end{cases}$$

With separation of variables, we set $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$, plug in, re-arrange, to find

$$\frac{(rR')'}{rR} + \frac{\Theta''}{r^2\Theta} = \frac{T''}{T}; \qquad i.e.\ funk(r,\theta) = funk(t).$$

Expecting oscillatory solutions in time (this is the wave equation for a drum with unit radius), we therefore set the right-hand side to a first separation constant $-\lambda = -\omega^2$, to obtain $T \propto \sin\omega t$ or $\cos\omega t$, as long as $\omega \neq 0$. Should the separation constant vanish ($\lambda = \omega = 0$), $T(t)$ is some linear function of t. Then,

$$\frac{r(rR')'}{R} + \lambda r^2 = -\frac{\Theta''}{\Theta}; \qquad i.e.\ funk(r) = funk(\theta),$$

so we need a second separation constant, m^2. The angular problem $\Theta'' + m^2\Theta = 0$ takes SL form, given that the periodic boundary conditions are of type (iii) in (4) (and $a = -\pi$, $b = \pi$). Obviously, $\Theta \propto \sin m\theta$ or $\cos m\theta$ with $m = 1, 2, ...$, or $\Theta = constant$ if $m = 0$, and we are headed to another full Fourier series in angle.

Last, the radial problem is

$$(rR')' + \lambda rR - \frac{m^2}{r}R = 0. \tag{12}$$

The signs of the last two terms indicate that we should take λ as the eigenvalue of this SL problem; $p(r) = \sigma(r) = r \geq 0$. Indeed, we already know what values m must take from the angular SL problem, so when solving (12), we should treat m as a given parameter. In other words, $q \equiv -m^2/r$. The boundary conditions, $R(r)$ regular for $r \to 0$ and $R(1) = 0$, correspond to the type (i) and (ii) conditions in (2) and (3) (given that $p(r) = r \to 0$ for $r \to 0$). We could now write the radial SL eigenfunctions as $R_n(r;m)$, $n = 1, 2, ...$, which highlights the dependence on the given parameter m. Alternatively, for those that like subscripts or superscripts, we could use the slightly more compact notation, $R = R_{mn}(r)$ or $R = R_n^m(r)$. Take your pick; I'll use the last one. We also have that $\lambda = \lambda_n^m$ or $\omega = \omega_n^m$, using this last notation (care must be taken so that we do not confuse the indicial superscripts with powers).

At this stage, we can formulate a general solution, which takes a form that is something like

$$u(r,\theta,t) = \sum_m \sum_{n=1}^{\infty} ☹_n^m \times R_n^m(r) \times \left\{ \begin{array}{c} \cos m\theta,\ m = 1, 2, ... \\ \sin m\theta,\ m = 1, 2, ... \\ 1,\ m = 0 \end{array} \right\} \times \left\{ \begin{array}{c} \cos\omega_n^m t,\ \omega_n^m \neq 0 \\ \sin\omega_n^m t,\ \omega_n^m \neq 0 \\ linear\ funk\ of\ t,\ \omega_n^m = 0 \end{array} \right\},$$

where $☹_n^m$ refers to the usual arbitrary constant. This is getting messy, though: we now have a double infinite sum, which is part Fourier series and part radial SL eigenfunction expansion. Somehow, we need to use the SL expansion formula to fix the constants $☹_n^m$ so that we can match up with the initial conditions. Before doing that, let's learn a little more about the radial SL problem.

One last general comment, however: when our PDE was of two dimensions (or one space, plus one time), we ended up with a single SL problem and a single sum. Now that we've buffed up in dimension (here, two space, one time), we arrived at two SL problems and two sums. As you might guess, adding more dimensions may well give yet more SL problems and sums. In other words, high dimensional PDEs solved with separation of variables or eigenfunctions expansions feature an excessive and likely impractical number of sums. Hopefully, we can handle one, two or even three.

Bessel's equation

Setting $z = \sqrt{\lambda}r = \omega r$, the radial SL problem above can be written as **Bessel's differential equation**,

$$z^2 y'' + zy' + (z^2 - \nu^2)y = 0.$$

This is an ODE with non-constant coefficients (oops) that, for general ν, has no solution that can be expressed in terms of simple elementary functions (such as powers or exponentials). Note that we have replaced m by a new parameter ν in the ODE here to emphasize how this coefficient need not be an integer. Despite the lack of a simple analytical solution, a large amount of information is known about the solutions, as any quick google search will turn up. The on-line "Digital Library of Mathematical Functions" or DLMF is a particularly useful resource in this regard. The two independent solutions are usually denoted by $J_\nu(z)$ and $Y_\nu(z)$. They are examples of special functions, and will become our friends.

To get a little more familiar with $J_\nu(z)$, here is a plot of three of them, for $\nu = 0$, 1 and 2. They

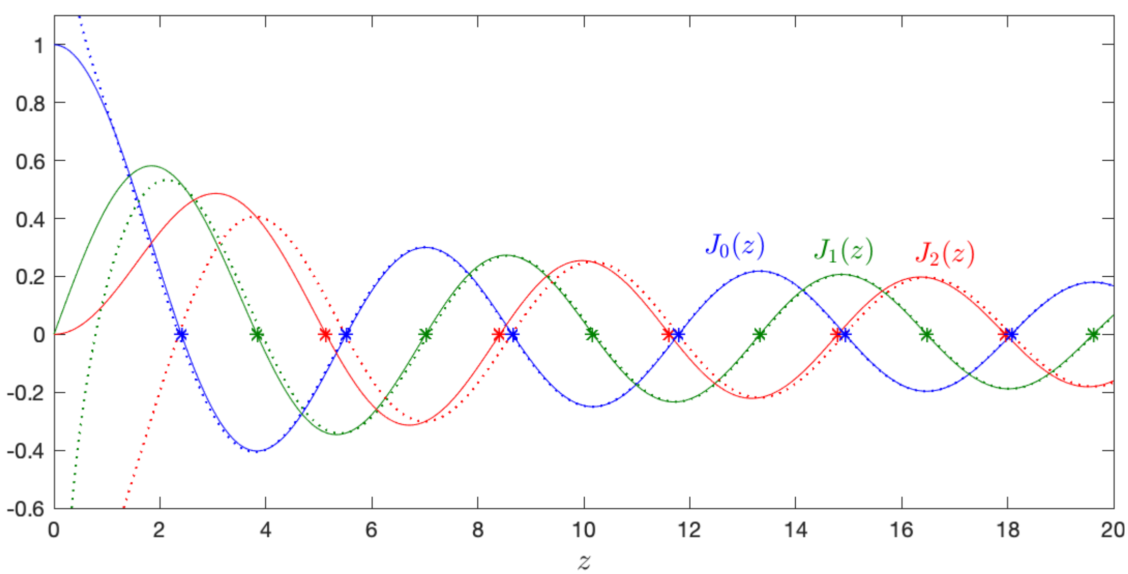

Figure 2: Bessel functions for $\nu = 0$ (blue), 1 (green) and 2 (red); the dotted lines show (13) and the stars are the (nontrivial) zeros.

are wiggly functions with lots of zeros. In fact, when $z \gg 1$, it is known that

$$J_\nu(z) \approx \sqrt{\frac{2}{\pi z}} \cos\left(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\right), \tag{13}$$

which certainly does not look so mysterious. It is also known that $J_\nu(z) \propto z^\nu$ for $z \ll 1$. Both behaviours can be seen in figure 2. This picture is generated in MATLAB with the in-built function besselj(nu,z), which performs much as it is written.

Similarly, $Y_\nu(z) \propto z^{-\nu}$ for $z \ll 1$, and there is another large-argument approximation like (13). In fact, the dependences for $z \to 0$ indicate why we usually prefer $J_\nu(z)$ to $Y_\nu(z)$: $z = 0$ is a singular point of Bessel's ODE. In its vicinity, there is a regular ($J_\nu(z)$) and a singular ($Y_\nu(z)$) solution. Obviously now, for our oscillating drum problem, the correct SL eigenfunction is $J_m(\omega r)$. Moreover, the other boundary condition ($R(1) = 0$, which corresponds to pinning the membrane of the drum around a ring of unit radius), implies that $J_m(\omega) = 0$. In other words, ω should be a zero of $J_m(z)$, for which there are an infinite number (this is demanded because there must be an infinite sequence of SL eigenvalues, and evident from the figure and the limiting form in (13)). *i.e.* $\omega = z_n^m$, where z_n^m is the n^{th} zero of $J_m(z)$. Note that $J_m(0) = 0$ for $m > 0$, but we are normally not interested in that particular zero.

The limiting form (13) actually indicates

$$z_n^m \approx \tfrac{1}{2}(2n - \tfrac{1}{2} + \nu)\pi, \quad n \gg 1.$$

More accurately, combining besselj(nu,z) and fzero in MATLAB (or looking at DLMF or other on-line resources), we may find

$$\begin{aligned}
z_n^0 &\approx 2.405, 5.520, 8.654, 11.792, 14.931, 18.071 \\
\tfrac{1}{2}(2n - \tfrac{1}{2})\pi &\approx 2.356, 5.498, 8.639, 11.781, 14.923, 18.064 \\
z_n^1 &\approx 3.832, 7.016, 10.174, 13.324, 16.471, 19.616 \\
\tfrac{1}{2}(2n + \tfrac{1}{2})\pi &\approx 3.927, 7.069, 10.210, 13.352, 16.493, 19.635 \\
z_n^2 &\approx 5.136, 8.417, 11.620, 14.796, 17.960, 21.117 \\
\tfrac{1}{2}(2n + \tfrac{3}{2})\pi &\approx 5.498, 8.639, 11.781, 14.923, 18.064, 21.206.
\end{aligned}$$

The limiting form (13) is evidently not that bad, even when z is not so big. It does, however, mess up for $z \ll 1$.

Note that the general solution to Bessel's equation is $AJ_\nu(z) + BY_\nu(z)$ for two arbitrary constants A and B. In order to remove any ambiguity in writing down such solution, and to provide definitive information about Bessel functions in general, those special functions are normalized to render them unique. The normalization of special functions overall is a bit of a mish-mash, with different special functions normalized in different ways. For Bessel functions, it is conventional to use the normalization,

$$\int_0^\infty J_\nu(z)\, \mathrm{d}z = 1,$$

which is not particularly helpful to us. It does imply, though, that $J_0(0) = 1$, as evident in figure 2.

Finally, there is an entire family of differential equations that have different versions of Bessel functions as the solutions. The family of ODEs can be written in the form

$$x^2 y'' + (1 - 2\alpha)xy' + (\omega^2\beta^2 x^{2\beta} + \alpha^2 - \nu^2\beta^2)y = 0,$$

with parameters α, ω, β and ν. The general solution is $y = Ax^\alpha J_\nu(\omega x^\beta) + Bx^\alpha Y_\nu(\omega x^\beta)$, with arbitrary constants A and B. This more general family of ODEs will prove useful to us later.

At this stage, you will hopefully be starting to gain a warm and fuzzy feeling about Bessel functions. Remember, they are our friends. Deep down, there's not that much difference with the more familiar exponential or trig function: if I presented you with some value for the argument z, the chances are that you could not quote to me the value of $\exp(z)$ or $\sin(z)$. However, you would know certain special values of the function for certain arguments (like $z = 0$), and some of its mathematical properties (such as what happens when it is differentiated). It is much like that with Bessel functions, although the mathematical properties are typically more convoluted. You can check many of them out at DLMF.

Back to the PDE

Returning to the PDE, we now use the fact that the solutions of the radial part of the problem must be Bessel functions. *i.e.* $R_n^m(r) \propto J_m(\sqrt{\lambda} r)$, given the regularity condition at $r = 0$. Moreover, the boundary condition $R_n^m(1) = 0$ demands that $J_m(\sqrt{\lambda}) = 0$. The fact that $J_m(z)$ is an oscillatory function with an infinite number of zeros now tells us how to pick λ: if $z = z_{mn}$ denotes the n^{th} zero of $J_m(z)$ then $\lambda = (z_{mn})^2$, or $\omega_n^m = z_{mn}$, and $R_n^m(r) \propto J_m(z_{mn} r)$. We then arrive at the general solution,

$$u = \sum_{n=1}^{\infty} \Bigg\{ \tfrac{1}{2}(c_{0n} \cos z_{0n} t + C_{0n} \sin z_{0n} t) J_0(z_{0n} r) + \\ \sum_{m=1}^{\infty} [(c_{mn} \cos z_{mn} t + C_{mn} \sin z_{mn} t) \cos m\theta + (d_{mn} \cos z_{mn} t + D_{mn} \sin z_{mn} t) \sin m\theta] J_m(z_{mn} r) \Bigg\},$$

given all the possible combinations of acceptable functions.

Simpler initial conditions

If the initial condition contains only a single term of the Fourier series in angle, things are easier. For example, if $f(r, \theta) = 0$ and $g(r, \theta) = G(r)$, then we can drop the sum over m and write

$$u(r, \theta, t) = \sum_{n=1}^{\infty} C_n J_0(z_n r) \sin(z_n t),$$

using the shorthand, $z_n = z_{0n}$ and $C_{0n} = C_n$. Then,

$$u_t(r, \theta, 0) = G(r) = \sum_{n=1}^{\infty} z_n C_n J_0(z_n r).$$

The fact that Bessel's equation and our boundary conditions form a SL problem with weight function $\sigma(r) = r$ justifies this representation of the solution and indicates that

$$C_n = \frac{\int_0^1 G(r) J_0(z_n r)\, r\mathrm{d}r}{z_n \int_0^1 [J_0(z_n r)]^2 r\mathrm{d}r},$$

from the expansion formula.

Similarly, if $f(r, \theta) = F(r) \sin m\theta$ and $g(r, \theta) = 0$, for some integer m, we have

$$u(r, \theta, t) = \sum_{n=1}^{\infty} c_n J_m(z_n r) \cos(z_n t) \sin m\theta, \quad c_n = \frac{\int_0^1 F(r) J_m(z_n r)\, r\mathrm{d}r}{\int_0^1 [J_m(z_n r)]^2 r\mathrm{d}r},$$

with $z_{mn} \to z_n$ and $c_{mn} \to c_n$.

The full initial-value problem

To satisfy $u(r, \theta, 0) = f(r, \theta)$ and $u_t(r, \theta, 0) = g(r, \theta)$ for less specific choices of f and g, we must suitably choose the coefficients, $\{c_{0n}, C_{0n}, c_{mn}, C_{mn}, d_{mn}, D_{mn}\}$. This can be accomplished using the SL expansion formulae, but is a big, bad, book-keeping exercise. One way to proceed is to first expand the initial conditions as Fourier series:

$$\left\{ \begin{array}{c} f(r,\theta) \\ g(r,\theta) \end{array} \right\} = \tfrac{1}{2} \left\{ \begin{array}{c} a_0(r) \\ A_0(r) \end{array} \right\} + \sum_{m=1}^{\infty} \left[\left\{ \begin{array}{c} a_m(r) \\ A_m(r) \end{array} \right\} \cos m\theta + \left\{ \begin{array}{c} b_m(r) \\ B_m(r) \end{array} \right\} \sin m\theta \right].$$

We can then match up terms to arrive at

$$\sum_{n=1}^{\infty} \left\{ \begin{array}{c} c_{mn} \\ z_{mn} C_{mn} \\ d_{mn} \\ z_{mn} D_{mn} \end{array} \right\} J_m(z_{mn} r) = \left\{ \begin{array}{c} a_m(r) \\ A_m(r) \\ b_m(r) \\ B_m(r) \end{array} \right\},$$

for both $m = 0$ and $m = 1, 2, ...$ The SL expansion formula now implies

$$\left\{ \begin{array}{c} c_{mn} \\ z_{mn} C_{mn} \\ d_{mn} \\ z_{mn} D_{mn} \end{array} \right\} = \left[\int_0^1 [J_m(z_{mn} r)]^2 r \mathrm{d}r \right]^{-1} \int_0^1 \left\{ \begin{array}{c} a_m(r) \\ A_m(r) \\ b_m(r) \\ B_m(r) \end{array} \right\} J_m(z_{mn} r) \; r \mathrm{d}r.$$

Finally, we can even use the definitions of $\{a_m(r), A_m(r), b_m(r), B_m(r)\}$ to write

$$\left\{ \begin{array}{c} c_{mn} \\ z_{mn} C_{mn} \\ d_{mn} \\ z_{mn} D_{mn} \end{array} \right\} = \frac{1}{\pi} \left[\int_0^1 [J_m(z_{mn} r)]^2 r \mathrm{d}r \right]^{-1} \int_{-\pi}^{\pi} \int_0^1 \left\{ \begin{array}{c} f(r,\theta) \cos m\theta \\ g(r,\theta) \cos m\theta \\ f(r,\theta) \sin m\theta \\ f(r,\theta) \sin m\theta \end{array} \right\} J_m(z_{mn} r) \; r \mathrm{d}r \; \mathrm{d}\theta.$$

This last result is exactly what one would find by multiplying the original general solution or its time derivative by the weight function r and pairs of the two sets of SL eigenfunctions (the Fourier series solutions and the Bessel functions), setting $t = 0$, and then performing a double integral over r and θ.